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A GENERALIZED HENSTOCK-STIELTJES INTEGRAL INVOLVING DIVISION FUNCTIONS

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ABSTRACT. We can consider the Riemann-Stieltjes integral $\int\limits_a^b f\,\mathrm{d}g$ as an integral of a point function f with respect to an interval function g. We could extend it to the Henstock-Stieltjes integral. In this paper, we extend it to a generalized Stieltjes integral $\int\limits_a^b f\,\mathrm{d}g$ of a point function f with respect to a function g of divisions of an interval. Then we prove for this integral the standard results in the theory of integration, including the controlled convergence theorem.

1. Introduction

The Riemann-Stieltjes integral is well-known. It can be extended to the Henstock-Stieltjes integral ([3]). Das et al [9] extended it further to include the case when g in $\int\limits_a^b f\,\mathrm{d}g$ is a second difference function $g(u,v,w) = g(w) - 2g(v) + g(u)$ or other similar functions. To unify the approach, we defined in the language of Henstock the GR_k integral ([6]) and the modified GR_k integral ([7], [8]) and proved some properties for both the integrals. So far, we have proved among other results the Saks-Henstock lemma, one version of the fundamental theorem of calculus and the equi-integrability convergence theorem.

The GR_k integral is in fact, a Stieltjes integral $\int\limits_a^b f\,\mathrm{d}g$ of a point function f with respect to a function g of divisions of an interval. We considered g as a

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division function so that the Saks-Henstock lemma holds. We modified the GR_k integral in [7] so that further properties of the integral can be proved. As we proceed to develop the full theory, we realize that we need a second function δ for the tagging of the subintervals in addition to the first δ function for the division of each subinterval. Hence we define in this paper a generalized Henstock-Stieltjes integral or in symbol GS_k integral which is an extension of the GR_k integral and its modified version; the controlled convergence theorem is proved for the GS_k integral. For similar integrals existing in the literature, see also Das and Kundu [10], [11].

2. Preliminaries

Definition 2.1. Given $\delta\colon [a,b] \to \mathbb{R}_+$, we call a division D given by $a = a_1 < a_2 < \dots < a_{p+1} = b$ and $\{x_1, x_2, \dots, x_p\}$ satisfying $x_i \in [a_i, a_{i+1}] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ for $i = 1, 2, \dots, p$ a *δ-fine division of* $[a,b]$.

We write $b_i = a_{i+1}$, $i = 1, 2, \dots, p$, and denote a δ-fine division by $D = \big\{[a_i, b_i]; x_i]\big\}_{i=1}^{p}$.

A δ-fine division $D = \{[a_i, b_i]; x_i\}_{i=1}^{p}$ is called a *strictly δ-fine* division of $[a,b]$ if either $x_i = a_i$ or $x_i = b_i$.

We can make a δ-fine division $D = \big\{[a_i, b_i]; x_i\big\}_{i=1}^{p}$ of $[a,b]$ a strictly δ-fine division of $[a,b]$ by splitting $[a_i, b_i]$ at x_i as $([a_i, b_i]; x_i) = ([a_i, x_i]; x_i) \cup ([x_i, b_i]; x_i)$ when $a_i < x_i < b_i$.

So, given $\delta\colon [a,b] \to \mathbb{R}_+$, there always exists a strictly δ-fine division of $[a,b]$.

Definition 2.2. Let k be a fixed positive integer and δ be a positive function defined on $[a,b]$. We shall call a division D of $[a,b]$ given by $a = x_0 < x_1 < \dots < x_n = b$ with associated points $\{\xi_0, \xi_1, \dots, \xi_{n-k}\}$ satisfying

$$\xi_i \in [x_i, x_{i+k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \qquad \text{for} \quad i = 0, 1, \dots, n-k$$

a *δ^k-fine division of* $[a,b]$.

For a given positive function δ, we denote a δ^k-fine division D by $\big\{[x_i, x_{i+k}], \xi_i\big\}_{i=0,1,\dots,n-k}$. Using the compactness of $[a,b]$ it is easy to verify that such a δ^k-fine division exists. When $k = 1$, it coincides with the usual definition of δ-fine division.

Let g be a real-valued function defined on closed interval $[a,b]^{k+1}$ in the $(k+1)$-dimensional space, and f be a real-valued function defined on $[a,b]$. Given a δ-fine division $D = \big\{([x_i, x_{i+k}], \xi_i)\big\}_{i=0,1,\dots,n-k}$ we call $\sum\limits_{i=0}^{n-k} f(\xi_i) g(x_i, \dots$ $\dots, x_{i+k})$, the *Riemann sum of f with respect to g* and denote it by $s(f, g; D)$.

Next, let $[a_i, b_i]$, $i = 1, 2, \dots, p$, be pairwise non-overlapping, and $\bigcup_{i=1}^{p} [a_i, b_i] \subset [a, b]$. Then $\{D_i\}_{i=1,2,\dots,p}$ is said to be a δ^k-fine *partial division* of $[a, b]$ if each D_i is a δ^k-fine division of $[a_i, b_i]$. Its corresponding partial Riemann sum is given by $\sum_{i=1}^{p} s(f, g; D_i)$.

Let g be a real-valued function defined on a closed interval $[a, b]^{k+1}$ in the $(k+1)$-dimensional space.

Now corresponding to the division $x_i < x_{i+1} < \dots < x_{i+k}$ of $[x_i, x_{i+k}]$ we can associate a real-valued function $g(x_i, x_{i+1}, \dots, x_{i+k})$. In this sense we regard g as a division function.

Let $x \in [x_i, x_{i+k}]$ where $x_i < x_{i+1} < \dots < x_{i+k}$. The jump of g at x, denoted by $J(g; x)$, is defined by

$$J(g; x) = \lim_{\substack{x_i \to x \\ x_{i+k} \to x}} g(x_i, \dots, x_{i+k}),$$

if the limit exists and is finite.

In what follows we assume that $J(g; x)$ exists for all $x \in [a, b]$.

Definition 2.3. Let $f\colon [a, b] \to \mathbb{R}$, $g\colon [a, b]^{k+1} \to \mathbb{R}$. We say that f is GS_k *integrable with respect to* g *on* $[a, b]$ *to* I if for any $\epsilon > 0$, there is $\delta_1\colon [a, b] \to \mathbb{R}_+$ such that for every strictly δ_1-fine division $D = \{[a_i, b_i]; x_i\}_{i=1}^{p}$ of $[a, b]$ there exists $\delta_2\colon [a, b] \to \mathbb{R}_+$, depending on D such that for any δ_2^k-fine division D_i of $[a_i, b_i]$, $i = 1, \dots, p$, we have

$$\left| \sum_{i=1}^{p} s(f, g; D_i) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g; b_i) - I \right| < \epsilon.$$

If f is GS_k integrable with respect to g on $[a, b]$, we write $(f, g) \in GS_k[a, b]$ and denote the integral by $\int_a^b f \,\mathrm{d}g$.

Notation. Henceforth for convenience we shall write $\{x_i, [a_i, b_i], D_i\}_{i=1}^{p}$ is a (δ_1, δ_2^k)*-fine division of* $[a, b]$ to mean that $\{[a_i, b_i]; x_i\}_{i=1}^{p}$ is a strictly δ_1-fine division of $[a, b]$ and depending on which there exists $\delta_2\colon [a, b] \to \mathbb{R}_+$ such that D_i is a δ_2^k-fine division of $[a_i, b_i]$, $i = 1, 2, \dots, p$.

We shall also say that $D = \{x_i, [a_i, b_i], D_i\}_{i=1}^{p}$ is a (δ_1, δ_2^k)*-fine partial division of* $[a, b]$ if $\{[a_i, b_i]; x_i\}_{i=1}^{p}$ is a strictly δ_1-fine partial division of $[a, b]$ i.e. $\bigcup_{i=1}^{p} [a_i, b_i] \subset [a, b]$ and depending on which there exists $\delta_2\colon [a, b] \to \mathbb{R}_+$ such that D_i is a δ_2^k-fine division of $[a_i, b_i]$.

Theorem 2.4. *The* GS_k *integral is uniquely defined.*

P r o o f. Let us assume that for $\epsilon > 0$, there exist positive functions $\delta_1(x)$, $\delta_2(x)$ defined on $[a,b]$ such that for every (δ_1, δ_2^k)-fine $D = \{x_i, [a_i, b_i], D_i\}_{i=1}^p$ of $[a,b]$ we have

$$\left| \sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g;b_i) - I_1 \right| < \epsilon$$

and also that there exist $\delta_3(x), \delta_4(x) \colon [a,b] \to \mathbb{R}_+$ such that for every (δ_3, δ_4^k)-fine $P = \{y_j, [c_j, d_j]; P_j\}_{j=1}^q$ of $[a,b]$ we have

$$\left| \sum_{j=1}^{q} s(f,g;P_j) + \sum_{j=1}^{q-1} (k-1) f(d_j) J(g;d_j) - I_2 \right| < \epsilon.$$

Let $\delta_5(x) = \min\{\delta_1(x), \delta_3(x)\}$.

We fix a strictly δ_5-fine division $\{[a_l, b_l]; x_l\}_{l=1}^r$ for which there exist $\delta(x)$ and $\delta'(x)$ such that for any δ^k-fine D_l and δ'^k-fine P_l of $[a_l, b_l]$ we have

$$\left| \sum_{l=1}^{r} s(f,g;D_l) + \sum_{l=1}^{r-1} (k-1) f(b_i) J(g;b_i) - I_1 \right| < \epsilon \tag{i}$$

$$\left| \sum_{l=1}^{r} s(f,g;P_l) + \sum_{l=1}^{r-1} (k-1) f(b_l) J(g;b_l) - I_2 \right| < \epsilon. \tag{ii}$$

We take $\delta_6(x) = \min\{\delta(x), \delta'(x)\}$ and fix a δ_6^k-fine division of $[a_l, b_l]$ for which both (i) and (ii) hold. Hence $|I_1 - I_2| < 2\epsilon$. Therefore, $I_1 = I_2$.

In Section 6, we shall give examples of the GS_k integral. □

3. Simple properties

The following theorem follows directly from the definition of the GS_k integral.

Theorem 3.1. *Let $(f_i, g) \in GS_k[a,b]$ and $(f, g_i) \in GS_k[a,b]$ for $i = 1, 2, \dots, n$. Then for real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ we have*

(i) $\left(\sum\limits_{i=1}^{n} \lambda_i f_i, g \right) \in GS_k[a,b]$ *and* $\int\limits_a^b \sum\limits_{i=1}^{n} (\lambda_i f_i) \, \mathrm{d}g = \sum\limits_{i=1}^{n} \lambda_i \left(\int\limits_a^b f_i \, \mathrm{d}g \right)$.

(ii) $\left(f, \sum\limits_{i=1}^{n} \lambda_i g_i \right) \in GS_k[a,b]$ *and* $\int\limits_a^b f \, \mathrm{d}\left(\sum\limits_{i=1}^{n} \lambda_i g_i \right) = \sum\limits_{i=1}^{n} \lambda_i \int\limits_a^b f \, \mathrm{d}g_i$.

(iii) *If $f_1(x) \le f_2(x)$ for all $x \in [a,b]$ and $g \colon [a,b]^{k+1} \to [0, \infty)$, then* $\int\limits_a^b f_1 \, \mathrm{d}g \le \int\limits_a^b f_2 \, \mathrm{d}g$.

Theorem 3.2. *Let $a < c < b$. If $(f,g) \in GS_k[a,c]$ and $(f,g) \in GS_k[c,b]$ then $(f,g) \in GS_k[a,b]$ and*

$$\int_a^b f\,\mathrm{d}g = \int_a^c f\,\mathrm{d}g + \int_c^b f\,\mathrm{d}g + (k-1)f(c)J(g;c).$$

Proof. Since $(f,g) \in GS_k[a,c] \cap GS_k[c,b]$, for $\epsilon > 0$, there exist $\delta_1(x), \delta_2(x) > 0$ defined on $[a,c]$ and $\delta_3(x), \delta_4(x) > 0$ defined on $[c,b]$ respectively such that

$$\Big| \sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1}(k-1)f(b_i)J(g;b_i) - \int_a^c f\,\mathrm{d}g \Big| < \epsilon$$

and

$$\Big| \sum_{j=1}^{q} s(f,g;P_j) + \sum_{j=1}^{q-1}(k-1)f(d_j)J(g;d_j) - \int_c^b f\,\mathrm{d}g \Big| < \epsilon$$

for every (δ_1, δ_2^k)-fine division $\{x_i, [a_i,b_i], D_i\}_{i=1}^p$ of $[a,c]$ and (δ_3, δ_4^k)-fine division $\{y_j, [c_j,d_j], P_j\}_{j=1}^q$ of $[c,b]$ respectively.

We define $\delta_5(x) = \min\{\delta_1(x), c-x\}$ when $x \in [a,c)$; $\min\{\delta_3(x), x-c\}$ when $x \in (c,b]$, and $\min\{\delta_1(c), \delta_3(c)\}$ when $x = c$.

We note that with the above definition of δ_5, c is always a division point of any strictly δ_5-fine division of $[a,b]$.

Let $a = u_1 < v_1 = u_2 < v_2 = u_3 < \cdots < v_s = c = u_{s+1} < v_{s+1} < \cdots < v_r = b$ be a strictly δ_5-fine division of $[a,b]$ with z_l being a tag point of $[u_l,v_l]$. Then $\{[u_l,v_l];z_l\}_{l=1}^s$ and $\{[u_l,v_l];z_l\}_{l=s+1}^r$ are a strictly δ_1-fine division of $[a,c]$ and a strictly δ_3-fine division of $[c,b]$ respectively. So, there exist δ_6' defined on $[a,c]$ and δ_6'' defined on $[c,b]$ such that for every $\delta_6'^k$-fine division D_l' of $[u_l,v_l]$, $l = 1,2,\ldots,s$, and for every $\delta_6''^k$-fine division D_l'' of $[u_l,v_l]$, $l = s+1, s+2, \ldots, r$, we have

$$\Big| \sum_{l=1}^{s} s(f,g;D_l') + \sum_{l=1}^{s-1}(k-1)f(v_l)J(g;v_l) - \int_a^c f\,\mathrm{d}g \Big| < \epsilon$$

and

$$\Big| \sum_{l=s+1}^{r} s(f,g;D_l'') + \sum_{l=s+1}^{r-1}(k-1)f(v_l)J(g;v_l) - \int_c^b f\,\mathrm{d}g \Big| < \epsilon$$

respectively.

We define $\delta_6(x) = \delta_6'(x)$ when $x \in [a,c)$, $\min\{\delta_6'(c), \delta_6''(c)\}$ when $x = c$ and $\delta_6''(x)$ when $x \in (c,b]$.

Let us take any δ_6^k-fine division D_l of $[u_l, v_l]$, $l = 1, 2, \dots, r$. Then,

$$\begin{aligned}
&\left| \sum_{l=1}^{r} s(f,g;D_l) + \sum_{l=1}^{r-1}(k-1)f(v_l)J(g;v_l) \right. \\
&\left. - \left\{ \int_a^c f \, \mathrm{d}g + \int_c^b f \, \mathrm{d}g + (k-1)f(c)J(g;c) \right\} \right| \\
&\le \left| \sum_{l=1}^{s} s(f,g;D_l) + \sum_{l=1}^{s-1}(k-1)f(v_l)J(g;v_l) - \int_a^c f \, \mathrm{d}g \right| \\
&+ \left| \sum_{l=s+1}^{r} s(f,g;D_l) + \sum_{l=s+1}^{r-1}(k-1)f(v_l)J(g;v_l) - \int_c^b f \, \mathrm{d}g \right| < 2\epsilon.
\end{aligned}$$

So, $(f,g) \in GS_k[a,b]$ and the equality holds. □

Remark 3.3. We here note that if we define $F(u,v) = \int_u^v f \, \mathrm{d}g$ for $[u,v] \subset [a,b]$ then in general F is not an additive function on the closed subintervals of $[a,b]$ for $k > 1$. But for $k = 1$ it is additive because the extra term vanishes.

Definition 3.4. Let the domain of F be $\big\{[u,v] \subset [a,b] : \ u \le v\big\}$. We call F to be *nearly additive* if for $a < c < b$, $F(a,b) = F(a,c) + F(c,c) + F(c,b)$.

Further, F is called *g-nearly additive with respect to f* if $F(x,x) = (k-1) \cdot \cdot f(x)J(g;x)$ for all $x \in (a,b)$. So, the integral function F of the $G\mathbb{R}_k$-integral is g-nearly additive with respect to f in $[a,b]$.

The following two theorems can be easily verified and so the proofs are omitted.

Theorem 3.5 (Cauchy Condition). *$(f,g) \in GS_k[a,b]$ if and only if for every $\epsilon > 0$ there exist positive functions $\delta_1, \delta_2 \colon [a,b] \to \mathbb{R}_+$ such that for any (δ_1, δ_2^k)-fine division $\{x_i, [a_i,b_i], D_i\}_{i=1}^p$ and $\{y_j, [c_j,d_j], P_j\}_{j=1}^q$ of $[a,b]$, we have*

$$\begin{aligned}
&\left| \left(\sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1}(k-1)f(b_i)J(g;b_i) \right) \right. \\
&\left. - \left(\sum_{j=1}^{q} s(f,g;P_j) + \sum_{j=1}^{q-1}(k-1)f(d_j)J(g;d_j) \right) \right| < \epsilon.
\end{aligned}$$

Theorem 3.6. *If $(f,g) \in GS_k[a,b]$ and $a \le c < d \le b$, then $(f,g) \in GS_k[c,d]$.*

We now prove the Saks-Henstock Lemma analogue for the GS_k integral.

Theorem 3.7 (**Saks-Henstock Lemma**). *Let $f\colon[a,b]\to\mathbb{R}$ and $g\colon[a,b]^{k+1}\to\mathbb{R}$ be such that $J(g;x)$ exists for all $x\in[a,b]$. Then $(f,g)\in GS_k[a,b]$ if and only if there exists a function F, g-nearly additive with respect to f, satisfying the condition that for all $\epsilon>0$ there exist $\delta_1,\delta_2\colon[a,b]\to\mathbb{R}_+$ such that for any (δ_1,δ_2^k)-fine partial division $D=\{x_i,[a_i,b_i],D_i\}_{i=1}^p$ of $[a,b]$ we have*

$$\left|\sum_{i=1}^{p}\{s(f,g;D_i)-F(a_i,b_i)\}\right|<\epsilon.$$

P r o o f. Let $(f,g)\in GS_k[a,b]$. So for $\epsilon>0$ there exist $\delta_1,\delta_2'\colon[a,b]\to\mathbb{R}_+$ such that for any $(\delta_1,\delta_2'^k)$-fine division $\{x_i,[u_r,v_r];Q_r\}_{r=1}^t$ of $[a,b]$ we have

$$\left|\sum_{r=1}^{t}s(f,g;Q_r)+\sum_{r=1}^{t-1}(k-1)f(v_r)J(g;v_r)-F(a,b)\right|<\epsilon, \tag{i}$$

where $F(u,v)=\int_u^v f\,\mathrm{d}g$. We define $F(u,v)=(k-1)f(u)J(g;u)$ when $u=v$. Let $\{[a_i,b_i];x_i\}_{i=1}^p$ be a strictly δ_1-fine partial division of $[a,b]$, and $\bigcup_{j=1}^{q}[c_j,d_j]$ be the closure of the complement of $\bigcup_{i=1}^{p}[a_i,b_i]$ in $[a,b]$. By Theorem 3.6, $(f,g)\in GS_k[c_j,d_j]$, $j=1,2,\ldots,q$, and so we can find $\delta_{1j}(x),\delta_{2j}(x)>0$, $j=1,2,\ldots,q$, defined on $[c_j,d_j]$ such that for all $(\delta_{1j},\delta_{2j}^k)$-fine $\{y_{js},[c_{js},d_{js}],D_{js}\}_{s=1}^{m_j}$ of $[c_j,d_j]$, $j=1,2,\ldots,q$, we have

$$\left|\sum_{s=1}^{m_j}s(f,g;D_{js})+\sum_{s=1}^{m_j-1}f(d_{js})J(g;d_{js})-F(c_{js},d_{js})\right|<\frac{\epsilon}{q},$$

where we may assume that $\delta_{1j}(x)\le\delta_1(x)$ for $x\in[c_j,d_j]$, $j=1,2,\ldots,q$.

So, $\{[a_i,b_i];x_i\}_{i=1}^p$ and $\{[c_{js},d_{js}];y_{js}\}_{s=1}^{m_j}$, $j=1,2,\ldots,q$, together form a strictly δ_1-fine division of $[a,b]$.

Let Λ be the set of common end points of $[a_i,b_i]$ and $[c_j,d_j]$. Hence in view of (i) above there exists $\delta_2\colon[a,b]\to\mathbb{R}_+$ (we may assume that $\delta_2(x)\le\delta_{2j}(x)$ for $x\in[c_j,d_j]$, $j=1,2,\ldots,q$) such that for any δ_2^k-fine division D_i of $[a_i,b_i]$, $i=1,2,\ldots,p$, and D_{js}, $s=1,2,\ldots,m_j$, $j=1,2,\ldots,q$, of $[c_{js},d_{js}]$ we have

$$\begin{aligned}\Bigg|&\sum_{j=1}^{q}\sum_{s=1}^{m_j}s(f,g;D_{js})+\sum_{i=1}^{p}s(f,g;D_i)\\&+\sum_{j=1}^{q}\sum_{s=1}^{m_j-1}(k-1)f(d_{js})J(g;d_{js})+\sum_{x\in\Lambda}(k-1)f(x)J(g;x)-F(a,b)\Bigg|<\epsilon.\end{aligned}$$

By Theorem 3.2, $F(a,b) = \sum_{i=1}^{p} F(a_i,b_i) + (k-1)\sum_{x\in\Lambda} f(x)J(g;x) + \sum_{j=1}^{q} F(c_j,d_j)$.
So, we have,

$$\begin{aligned}
&\left|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}\right| \\
&\le \left|\sum_{j=1}^{q}\sum_{s=1}^{m_j} s(f,g;D_{js}) + \sum_{i=1}^{p} s(f,g;D_i) + (k-1)\sum_{j=1}^{q}\sum_{s=1}^{m_j-1} f(d_{js})J(g;d_{js})\right. \\
&\qquad\qquad\qquad\qquad \left. + (k-1)\sum_{x\in\Lambda} f(x)J(g;x) - F(a,b)\right| \\
&\quad + \left|(k-1)\sum_{j=1}^{q}\sum_{s=1}^{m_j-1} f(d_{js})J(g;d_{js}) + \sum_{j=1}^{q}\sum_{s=1}^{m_j} s(f,g;D_{js}) - \sum_{j=1}^{q} F(c_j,d_j)\right| < 2\epsilon.
\end{aligned}$$

Conversely, let the condition hold.

We take a (δ_1,δ_2^k)-fine division $\{x_i,[a_i,b_i],D_i\}_{i=1}^{p}$ of $[a,b]$. Then

$$\left|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}\right| < \epsilon.$$

Since F is g-nearly additive, then $F(a,b) = \sum_{i=1}^{p} F(a_i,b_i) + \sum_{i=1}^{p-1}(k-1)f(b_i)J(g;b_i)$.
So,

$$\left|\sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1}(k-1)f(b_i)J(g;b_i) - F(a,b)\right| < \epsilon.$$

Hence $(f,g)\in GS_k[a,b]$. □

4. Some Results

Definition 4.1. Let $g\colon [a,b]^{k+1} \to \mathbb{R}$. For $X \subset [a,b]$ we define the *slope variation*

$$SV_g^k(X) = \inf_{\delta_1}\sup_{D}\inf_{\delta_2}\sup_{D_i}\sum_{i=1}^{p} |s(1,g;D_i)|,$$

where the first supremum is taken over all δ_2^k-fine divisions D_i of $[a_i,b_i]$ and then infimum over all δ_2 keeping a strictly δ_1-fine partial division $D = \{[a_i,b_i];x_i\}_{i=1}^{p}$, $x_i \in X$ of $[a,b]$ fixed at present and then supremum over all D and then infimum over all δ_1.

If $SV_g^k(X) < \infty$, we say that $g \in SV^k(X)$(of bounded slope variation).

It follows from the above definition that if $g \in SV^k(X)$ then there exist $\delta_1, \delta_2\colon [a,b] \to \mathbb{R}_+$ such that for any (δ_1, δ_2^k)-fine partial division $\{x_i, [a_i,b_i], D_i\}_{i=1}^p$, with $x_i \in X$ of $[a,b]$ we have $\sum\limits_{i=1}^{p} |s(1,g;D_i)| \le SV_g^k(X)$.

We now give an example of a function g which belongs to $SV^2[a,b]$.

Example 4.2. Let G be of bounded slope variation on $[a,b]$ and $g\colon [a,b]^3 \to \mathbb{R}$ be defined as $g(u,v,w) = \frac{G(w)-G(v)}{w-v} - \frac{G(v)-G(u)}{v-u}$ when $u < v < w$ and $= 0$ otherwise. Then $g \in SV^2[a,b]$.

P r o o f. Since G is of bounded slope variation on $[a,b]$ there exists [4, p. 147], $M > 0$ such that

$$\sum_{i=0}^{n-2} \left| \frac{G(x_{i+2}) - G(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{G(x_{i+1}) - G(x_i)}{x_{i+1} - x_i} \right| + \left| \frac{G(x_n) - G(x_{n-1})}{x_n - x_{n-1}} \right| < M,$$

for any division $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a,b]$.

Let $\delta_1, \delta_2\colon [a,b] \to \mathbb{R}_+$ be any two functions, $D = \{[a_i,b_i]; x_i\}_{i=1}^p$ be any δ_1-fine partial division of $[a,b]$ and $D_i = \{[z_r, z_{r+k}]; \xi_r\}_{r=1}^l$ be any δ_2^k-fine division of $[a_i,b_i], i = 1,2,\ldots,p$. Then, $|s(1,g;D_i)| = \left| \frac{G(z_{l+k})-G(z_{l+k-1})}{z_{l+k}-z_{l+k-1}} - \frac{G(z_2)-G(z_1)}{z_2-z_1} \right|$. Since G is of bounded slope variation, we have

$$\inf_{\delta_1} \sup_{D} \inf_{\delta_2} \sup_{D_i} \sum_{i=1}^{p} |s(1,g;D_i)| < \infty.$$

□

Definition 4.3. Let F be a function g-nearly additive with respect to f on $[a,b]$. F is said to be *weakly g-regular* with respect to f at $x \in [a,b]$ if for all $\epsilon > 0$, there exists $\delta_1(x) > 0$ such that for every $[u,v]$ with $x = u$ or $x = v$ and $[u,v] \subset (x - \delta_1(x), x + \delta_1(x))$ there exists $\delta\colon [a,b] \to \mathbb{R}_+$ such that for every δ^k-fine division $D = \{[x_i, x_{i+k}]; \xi_i\}_{i=0}^{n-k}$ of $[u,v]$ we have

$$\left| \sum_{i=0}^{n-k} f(\xi_i) g(x_i,\ldots,x_{i+k}) - F(u,v) \right| < \epsilon \sum_{i=0}^{n-k} |g(x_i,\ldots,x_{i+k})|.$$

Theorem 4.4. *Let F be a g-nearly additive function defined on $[a,b]$ and $g \in SV^k[a,b]$. If F is weakly g-regular at all $x \in [a,b]$, then $(f,g) \in GS_k[a,b]$ with primitive F.*

P r o o f. Since $g \in SV^k[a,b]$, we can find $M > 0$ and $\delta_1'\colon [a,b] \to \mathbb{R}_+$ such that for any strictly δ_1'-fine partial division $\{[a_i,b_i];x_i\}_{i=1}^p$ of $[a,b]$, there exists $\delta_2'\colon [a,b] \to \mathbb{R}_+$ such that for all $\delta_2'^k$-fine divisions D_i of $[a_i,b_i]$ we have $\sum\limits_{i=1}^p |s(1,g;D_i)| < M$.

Now, F being weakly g-regular at all $x \in [a,b]$, for $\epsilon > 0$ there exists $\delta_3\colon [a,b] \to \mathbb{R}_+$ such that for all strictly δ_3-fine divisions $\{[c_i,d_i];y_i\}_{i=1}^q$ there exists $\delta_4\colon [a,b] \to \mathbb{R}_+$ such that for any δ_4^k-fine division $P_i = \{[x_j^i, x_{j+k}^i];\xi_j^i\}_{j=0}^{n_i-k}$ of $[c_i,d_i]$ we have

$$\Big| \sum_{j=0}^{n_i-k} f(\xi_j^i) g(x_j^i,\dots,x_{j+k}^i) - F(c_i,d_i) \Big| < \frac{\epsilon}{M} \sum_{j=0}^{n_i-k} |g(x_j^i,\dots,x_{j+k}^i)|.$$

We define $\delta_1(x) = \min\{\delta_1'(x), \delta_3(x)\}$, $x \in [a,b]$.

Let $D = \{[a_i,b_i];x_i\}_{i=1}^p$ be a strictly δ_1-fine partial division of $[a,b]$. Then there exists $\delta_2\colon [a,b] \to \mathbb{R}_+$ with $\delta_2(x) \le \min\{\delta_2'(x), \delta_4(x)\}$ for $x \in [a,b]$ such that for all δ_2^k-fine divisions D_i of $[a_i,b_i]$, $i = 1,2,\dots,p$, we have

$$\begin{aligned}\Big| \sum_{i=1}^p \{s(f,g;D_i) - F(a_i,b_i)\} \Big| &\le \sum_{i=1}^p |s(f,g;D_i) - F(a_i,b_i)| \\ &< \frac{\epsilon}{M} \sum_{i=1}^p |s(1,g;D_i)| < \epsilon.\end{aligned}$$

So, by Theorem 3.7, $(f,g) \in GS_k[a,b]$ with primitive F. □

In [1], the authors obtained a characterization of the Henstock integral in $\Re^m$. Keeping in view of this we shall now give a characterization of the GS_k integral.

Definition 4.5. Let $f\colon [a,b] \to \mathbb{R}$, $g\colon [a,b]^{k+1} \to \mathbb{R}$. Further, let F be g-nearly additive with respect to f. Given $\epsilon > 0$ and $\delta_1, \delta_2\colon [a,b] \to \mathbb{R}_+$ we define the set

$$\begin{aligned}\Gamma_{\epsilon,\delta_1,\delta_2} = \Big\{ D : D = \{x_i, [a_i,b_i], D_i\}_{i=1}^p \\ \text{is a } (\delta_1,\delta_2^k)\text{-fine partial division of } [a,b] \text{ such that} \\ \Big| \sum_{i=1}^p \{s(f,g;D_i) - F(a_i,b_i)\} \Big| \ge \epsilon \sum_{i=1}^p |s(1,g;D_i)| \Big\}.\end{aligned}$$

Let $X(\epsilon,\delta_1,\delta_2) = \big\{ x_i : \ D = \{x_i, [a_i,b_i], D_i\}_{i=1}^p \in \Gamma_{\epsilon,\delta_1,\delta_2} \big\}$.

Theorem 4.6. *Let F be a function g-nearly additive with respect to f on $[a,b]$ and $g \in SV^k[a,b]$. Then $(f,g) \in GS_k[a,b]$ with primitive F if for all $\epsilon > 0$ there exist $\delta_1, \delta_2\colon [a,b] \to \mathbb{R}_+$ such that for all (δ_1,δ_2^k)-fine partial divisions*

$D = \{x_i, [a_i, b_i], D_i\}_{i=1}^p \in \Gamma_{\epsilon,\delta_1,\delta_2}$ *of* $[a, b]$ *we have*

$$\sum_{i=1}^{p} |s(f, g; D_i)| < \epsilon \qquad \textit{and} \qquad \sum_{i=1}^{p} |F(a_i, b_i)| < \epsilon.$$

The converse also holds if $[a, b] = \bigcup_{l=1}^{\infty} X_l$ *where the* X_l*'s are such that for each* l *there exist* $\delta_{1,l}\colon [a, b] \to \mathbb{R}_+$ *and* $M_l > 0$ *such that for any strictly* $\delta_{1,l}$*-fine partial division* $\{[a_i, b_i]; x_i\}_{i=1}^p$ *of* $[a, b]$ *with* $x_i \in X_l$*, there exists* $\delta_{2,l}\colon [a, b] \to \mathbb{R}_+$ *such that for any* $\delta_{2,l}^k$*-fine division* D_i *of* $[a_i, b_i]$ *we have*

$$\Big| \sum_{i=1}^{p} s(f, g; D_i) \Big| \le M_l \sum_{i=1}^{p} |s(1, g; D_i)|.$$

P r o o f. Since $g \in SV^k[a, b]$, there exist $\delta_1', \delta_2'\colon [a, b] \to \mathbb{R}_+$ and $M > 0$ such that for any $(\delta_1', \delta_2'^k)$-fine partial division $\{y_j, [c_j, d_j]; P_j\}_{j=1}^q$ of $[a, b]$ we have

$$\sum_{j=1}^{q} |s(1, g; P_j)| < M.$$

From the given condition, for all $\epsilon > 0$ there exist $\delta_1, \delta_2\colon [a, b] \to \mathbb{R}_+$ such that for any (δ_1, δ_2^k)-fine partial division $D = \{t_r, [u_r, v_r], Q_r\}_{r=1}^s \in \Gamma_{\epsilon,\delta_1,\delta_2}$ we have

$$\sum_{r=1}^{s} |s(f, g; Q_r)| < \epsilon \quad \text{and} \quad \sum_{r=1}^{s} |F(u_r, v_r)| < \epsilon.$$

We may assume that $\delta_1(x) \le \delta_1'(x)$ and $\delta_2(x) \le \delta_2'(x)$ for all $x \in [a, b]$.

Let $\{x_i, [a_i, b_i], D_i\}_{i=1}^p$ be any (δ_1, δ_2^k)-fine partial division of $[a, b]$. Then

$$\begin{aligned} &\Big| \sum_{i=1}^{p} \{s(f, g; D_i) - F(a_i, b_i)\} \Big| \\ &\le \sum_{x_i \in X(\epsilon,\delta_1,\delta_2)} |s(f, g; D_i) - F(a_i, b_i)| + \sum_{x_i \in X'(\epsilon,\delta_1,\delta_2)} |s(f, g; D_i) - F(a_i, b_i)|, \end{aligned}$$

where X' denotes the complement of X

$$\begin{aligned} &\le \sum_{x_i \in X(\epsilon,\delta_1,\delta_2)} |s(f, g; D_i)| + \sum_{x_i \in X(\epsilon,\delta_1,\delta_2)} |F(a_i, b_i)| + \sum_{x_i \in X'(\epsilon,\delta_1,\delta_2)} |s(f, g; D_i) - F(a_i, b_i)| \\ &< 2\epsilon + \sum_{x_i \in X'(\epsilon,\delta_1,\delta_2)} \epsilon |s(1, g; D_i)| < 2\epsilon + \epsilon M = (2 + M)\epsilon. \end{aligned}$$

So, by Theorem 3.7, $(f, g) \in GS_k[a, b]$.

Conversely, let $(f, g) \in GS_k[a, b]$ with primitive F and f satisfy the given condition. We may assume that the X_l's are disjoint. So, there exist $M_l > 0$ and $\delta_{1,l}(x)\colon [a, b] \to \mathbb{R}_+$ such that for any strictly $\delta_{1,l}$-fine partial division

$\{[a_i, b_i]; x_i\}_{i=1}^p$ of $[a, b]$ with $x_i \in X_l$, there exists $\delta_{2,l} \colon [a, b] \to \mathbb{R}_+$ such that for any $\delta_{2,l}^k$-fine division D_i of $[a_i, b_i]$ we have

$$\left| \sum_{i=1}^{p} s(f, g; D_i) \right| \le M_l \sum_{i=1}^{p} |s(1, g; D_i)|. \tag{1}$$

Now, by the Henstock lemma for $\epsilon > 0$ there exists $\delta_{3,l}, \delta_{4,l} \colon [a, b] \to \mathbb{R}_+$ such that for every $(\delta_{3,l}, \delta_{4,l}^k)$-fine partial division $\{x_i, [a_i, b_i], D_i\}_{i=1}^p$ of $[a, b]$ we have

$$\left| \sum_{i=1}^{p} \{s(f, g; D_i) - F(a_i, b_i)\} \right| < \frac{\epsilon^2}{2^{l+1} M_l}. \tag{2}$$

We define $\delta_1(x) = \min\bigl\{\delta_{1,l}(x), \delta_{3,l}(x)\bigr\}$, for $x \in X_l$. Let $\{[c_j, d_j]; y_j\}_{j=1}^q$ be a strictly δ_1-fine partial division of $[a, b]$. Then there exists $\delta_2 \colon [a, b] \to \mathbb{R}_+$ with $\delta_2(x) \le \min\{\delta_{2,l}(x), \delta_{4,l}(x)\}$ for $x \in X_l$ such that both (1) and (2) above hold for any δ_2^k-fine division P_j of $[c_j, d_j]$, $j = 1, 2, \dots, q$.

We take a (δ_1, δ_2^k)-fine partial division $\{x_i, [a_i, b_i], D_i\}_{i=1}^p \in \Gamma_{\epsilon, \delta_1, \delta_2}$ of $[a, b]$. Then

$$\begin{aligned}\sum_{i=1}^{p} |s(f, g; D_i)| &\le \sum_{l=1}^{\infty} \sum_{x_i \in X_l} |s(f, g; D_i)| \le \sum_{l=1}^{\infty} M_l \sum_{x_i \in X_l} |s(1, g; D_i)| \\ &\le \sum_{l=1}^{\infty} \frac{M_l}{\epsilon} \sum_{x_i \in X_l} |s(f, g; D_i) - F(a_i, b_i)| \le \sum_{l=1}^{\infty} \frac{M_l 2\epsilon^2}{\epsilon 2^{l+1} M_l} = \epsilon.\end{aligned}$$

Also, $\sum\limits_{i=1}^{p} |F(a_i, b_i)| \le \sum\limits_{i=1}^{p} |s(f, g; D_i) - F(a_i, b_i)| + \sum\limits_{i=1}^{p} |s(f, g; D_i)| < 2\epsilon.$ □

5. Convergence

In this section we prove some convergence results for the GS_k integral.

Theorem 5.1 (Uniform Convergence Theorem). *Let $g \in SV^k[a, b]$ and $\{f_n\}$ be a sequence of functions defined on $[a, b]$ such that $(f_n, g) \in GS_k[a, b]$ for all $n = 1, 2, \dots$. If f_n is uniformly convergent to f on $[a, b]$ as $n \to \infty$, then $\int\limits_a^b f \, \mathrm{d}g$ exists and $\lim\limits_{n \to \infty} \int\limits_a^b f_n \, \mathrm{d}g = \int\limits_a^b f \, \mathrm{d}g$.*

Proof. Since $g \in SV^k[a, b]$ there exist $\delta_1' \colon [a, b] \to \mathbb{R}_+$ and $M > 0$ such that for any strictly δ_1'-fine partial division $D' = \{[c_i, d_i]; \xi_i\}_{i=1}^q$ of $[a, b]$ there exists $\delta_2' \colon [a, b] \to \mathbb{R}_+$ such that for any $\delta_2'^k$-fine division D_i' of $[c_i, d_i]$, $i = 1, 2, \dots, q$, we have $\sum\limits_{i=1}^{q} |s(1, g; D_i')| < M$.

Let $A_n = \int\limits_a^b f_n \,\mathrm{d}g$. By the Saks-Henstock lemma, for $\epsilon > 0$ there exists $\delta_{1,n}\colon [a,b] \to \mathbb{R}_+$, $n = 1,2,\dots$, where $\delta_{1,n} \leq \delta_1'$ such that for every strictly $\delta_{1,n}$-fine partial division $D_n = \{[a_i^n, b_i^n]; x_i^n\}_{i=1}^{p_n}$ of $[a,b]$ there exists $\delta_{2,n}'\colon [a,b] \to \mathbb{R}_+$ such that for every $\delta_{2,n}'^k$-fine division $D_i'^n$ of $[a_i^n, b_i^n]$ we have

$$\left| \sum_{i=1}^{p_n} s(f_n, g; D_i'^n) - A_n \right| < \epsilon.$$

We choose $\delta_{1,n+1}$ such that $\delta_{1,n+1} \leq \delta_{1,n}$, $n = 1,2,\dots$. For $n,m \in N$ and $n > m$ we fix a strictly $\delta_{1,n}$-fine partial division $\{[u_l^n, v_l^n]; t_l^n\}_{l=1}^r$ of $[a,b]$. So, there exists $\delta_{2n}\colon [a,b] \to \mathbb{R}_+$ with $\delta_{2n} \leq \delta_2'$ and $\delta_{2,n+1} \leq \delta_{2,n}$, $n = 1,2,\dots$, such that for any δ_{2n}^k-fine division D_l^n of $[u_l^n, v_l^n]$, $l = 1,2,\dots,r$, we have

$$|A_n - A_m| < 2\epsilon + \sum_{l=1}^r |s(f_n, g; D_i^n) - s(f_m, g; D_i^m)| \leq 2\epsilon + ||f_n - f_m||(SV_g^k[a,b]),$$

where $\|f_n - f_m\| = \sup\limits_{a\leq x\leq b} |f_n(x) - f_m(x)|$. As f_n is uniformly convergent to f, we have $\|f_n - f_m\| \to 0$ as $n,m \to \infty$. So, there exists a positive integer N_1 such that for $n,m > N_1$, $\|f_n - f_m\| < \frac{\epsilon}{M}$. Thus $\{A_n\}$ is a Cauchy sequence in $\mathbb{R}$ and let $A = \lim\limits_{n\to\infty} A_n$. We can find a positive integer $N_2 > N_1$ such that for $n \geq N_2$ we have $|A_n - A| < \epsilon$. Let $\delta_1(x) = \delta_{1,N_2}(x)$ for $x \in [a,b]$. Then for any strictly δ_1-fine division $\{[a_i, b_i]; x_i\}_{i=1}^p$ of $[a,b]$ there exists $\delta_2\colon [a,b] \to \mathbb{R}_+$, $\delta_2 \leq \delta_{2,N_2}$, such that for any δ_2^k-fine division D_i of $[a,b]$ we have

$$\begin{aligned}\left| \sum_{i=1}^p s(f, g; D_i) - A \right| &\leq \left| \sum_{i=1}^p \{s(f, g; D_i) - s(f_{N_2}, g; D_i)\} \right| \\ &\quad + \left| \sum_{i=1}^p s(f_{N_2}, g; D_i) - A_{N_2} \right| + |A_{N_2} - A| < 3\epsilon.\end{aligned}$$

So, $(f,g) \in GS_k[a,b]$ and $\int\limits_a^b f \,\mathrm{d}g = \lim\limits_{n\to\infty} \int\limits_a^b f_n \,\mathrm{d}g$. □

Theorem 5.2 (**Monotone Convergence Theorem**). *If*

(i) *the sequence $\{f_n\}$ is monotonic everywhere in $[a,b]$,*

(ii) *g is a non-negative function defined on $[a,b]^{k+1}$ such that $(f_n, g) \in GS_k[a,b]$ for all n and the sequence $\left\{ \int\limits_a^b f_n \,\mathrm{d}g \right\}$ is bounded,*

(iii) *$\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all x in $[a,b]$,*

then $(f,g) \in GS_k[a,b]$ and $\int\limits_a^b f \,\mathrm{d}g = \lim\limits_{n\to\infty} \int\limits_a^b f_n \,\mathrm{d}g$.

Proof. We proceed similarly as in the proof of [4, Theorem 3.5.2]. By considering $-f_n$ or $f_n - f_1$ instead of f_n, if need be, we can achieve that the sequence $\{f_n\}$ is increasing and $f_n \geq 0$. Since $g \geq 0$, $\left\{ \int\limits_a^b f_n \,\mathrm{d}g \right\}$ is also monotonic and bounded. So, $\lim\limits_{n\to\infty} \int\limits_a^b f_n \,\mathrm{d}g$ exists. Let us denote it by L. Given $\epsilon > 0$ we can find N such that $\int\limits_a^b f_N \,\mathrm{d}g > L - \frac{\epsilon}{3}$. Next we find $n(x) \geq N$ such that, for $n \geq n(x)$,

$$\frac{3L + 3\epsilon}{3L + \epsilon} f_n(x) \geq f(x).$$

If $f(x) > 0$ this is possible because the left-hand side has a limit strictly larger than the right-hand side; if $f(x) = 0$ we can take $n(x) = N$. By the Saks-Henstock Lemma, there is $\delta_{1,n}\colon [a,b] \to \mathbb{R}_+$ such that for any strictly $\delta_{1,n}$-fine partial division $\{[c_i, d_i]; \xi_i\}_{i=1}^q$ there exists $\delta'_{2,n}\colon [a,b] \to \mathbb{R}_+$ such that for every $\delta'^k_{2,n}$-fine division P_i^n of $[c_i, d_i]$ we have

$$\sum_{i=1}^{q} |s(f_n, g; P_i^n) - \int\limits_{c_i}^{d_i} f_n \,\mathrm{d}g| < \frac{\epsilon}{3 \cdot 2^n}. \tag{i}$$

We define $\delta_1(x) = \delta_{1,n(x)}(x)$. Let $\{[a_i, b_i]; x_i\}_{i=1}^p$ be a strictly δ_1-fine division of $[a,b]$. So, there exists $\delta_{2,n}\colon [a,b] \to \mathbb{R}$ so that (i) holds for any $\delta_{2,n}^k$-fine D_i^n of $[a_i, b_i]$.

We define $\delta_2\colon [a,b] \to \mathbb{R}_+$ by $\delta_2(x) = \delta_{2,n(x)}(x)$ for $x \in [a,b]$ and take any δ_2^k-fine division D_i of $[a_i, b_i]$.

The proof will be complete if we show that

$$\left| \sum_{i=1}^{p} s(f, g; D_i) + (k-1) \sum_{i=1}^{p-1} f(b_i) J(g; b_i) - L \right| < \epsilon.$$

Now,

$$\begin{aligned} &\sum_{i=1}^{p} \int\limits_{a_i}^{b_i} f_{n(x_i)} \,\mathrm{d}g + (k-1) \sum_{i=1}^{p-1} f_{n(x_i)}(b_i) J(g; b_i) \\ &\geq \sum_{i=1}^{p} \int\limits_{a_i}^{b_i} f_N \,\mathrm{d}g + (k-1) \sum_{i=1}^{p-1} f_N(b_i) J(g; b_i) = \int\limits_a^b f_N \,\mathrm{d}g > L - \frac{\epsilon}{3}. \end{aligned}$$

Denoting by $\widehat{N}$ the largest $n(x_i)$ we also have

$$\sum_{i=1}^{p} \int_{a_i}^{b_i} f_{n(x_i)}\,\mathrm{d}g + (k-1)\sum_{i=1}^{p-1} f_{n(x_i)}(b_i)J(g;b_i)$$

$$\leq \sum_{i=1}^{p} \int_{a_i}^{b_i} f_{\widehat{N}}\,\mathrm{d}g + (k-1)\sum_{i=1}^{p-1} f_{\widehat{N}}(b_i)J(g;b_i) = \int_{a}^{b} f_{\widehat{N}}\,\mathrm{d}g \leq L.$$

Now

$$\Big| \sum_{i=1}^{p} \Big\{ s(f_{n(x_i)}, g; D_i) - \int_{a_i}^{b_i} f_{n(x_i)}\,\mathrm{d}g \Big\} \Big|$$

$$\leq \sum_{l=1}^{\infty} \sum_{n(x_i)=l} \Big| s(f_{n(x_i)}, g; D_i) - \int_{a_i}^{b_i} f_{n(x_i)}\,\mathrm{d}g \Big| < \sum_{l=1}^{\infty} \frac{\epsilon}{3.2^l} = \frac{\epsilon}{3}.$$

Again,

$$\sum_{i=1}^{p} s(f, g; D_i) + (k-1)\sum_{i=1}^{p-1} f(b_i)J(g;b_i)$$

$$\geq \sum_{i=1}^{p} s(f_{n(x_i)}, g; D_i) + (k-1)\sum_{i=1}^{p-1} f_{n(x_i)}(b_i)J(g;b_i)$$

$$> \sum_{i=1}^{p} \int_{a_i}^{b_i} f_{n(x_i)}\,\mathrm{d}g + (k-1)\sum_{i=1}^{p-1} f_{n(x_i)}(b_i)J(g;b_i) - \frac{\epsilon}{3} > L - \frac{2\epsilon}{3}$$

and on the other hand

$$\frac{(3L+\epsilon)}{3(L+\epsilon)} \Big[\sum_{i=1}^{p} s(f, g; D_i) + (k-1)\sum_{i=1}^{p-1} f(b_i)J(g;b_i) \Big]$$

$$\leq \sum_{i=1}^{p} s(f_{n(x_i)}, g; D_i) + (k-1)\sum_{i=1}^{p-1} f_{n(x_i)}(b_i)J(g;b_i)$$

$$< \sum_{i=1}^{p} \int_{a_i}^{b_i} f_{n(x_i)}\,\mathrm{d}g + \frac{\epsilon}{3} + (k-1)\sum_{i=1}^{p-1} f_{n(x_i)}(b_i)J(g;b_i) \leq L + \frac{\epsilon}{3}.$$

So,

$$\Big| \sum_{i=1}^{p} s(f, g; D_i) + (k-1)\sum_{i=1}^{p-1} f(b_i)J(g;b_i) - L \Big| < \epsilon.$$

This completes the proof. $\square$

Definition 5.3. Let F be an interval function defined on $\Im = \{[u,v]: u,v \in [a,b],\ u \leq v\}$. For $X \subseteq [a,b]$ we say that F is $AC^k(X)$ if for all $\epsilon > 0$ there exist $\delta_1 \colon [a,b] \to \mathbb{R}_+$ and $\eta > 0$ such that for every strictly δ_1-fine partial division $D = \{[a_i,b_i]; x_i\}_{i=1}^p$ of $[a,b]$ with $x_i \in X$ and $\sum\limits_{i=1}^{p}(b_i - a_i) < \eta$ we have

$$\sum_{i=1}^{p} |F(a_i,b_i)| < \epsilon.$$

A sequence of interval functions $\{F_n\}$, each defined on $\Im$ is said to be *uniformly* $AC^k(X)$, and we write $F_n \in UAC^k(X)$ if the above inequality holds with F replaced by F_n for all n and where δ_1, η are independent of n.

$\{F_n\}$ is said to be $UAC^kG[a,b]$ if $[a,b] = \bigcup\limits_{j=1}^{\infty} Y_j$, where each Y_j is closed and $\{F_n\}$ is $UAC^k(Y_j)$ for all j.

Definition 5.4. Let $X \subseteq [a,b]$ be closed and $f_n \colon [a,b] \to \mathbb{R}$. We say that the sequence $\{f_n\}$ has *uniformly locally broken small Riemann sum with respect to g on X* to be denoted by $f_n \in ULBRS_g^k(X)$ if for $\epsilon > 0$ there are $\delta_1 \colon [a,b] \to \mathbb{R}_+$ and $\eta > 0$ such that for every open set G with $|G| < \eta$ and for every strictly δ_1-fine partial division $\{[a_i,b_i]; x_i\}_{i=1}^p$, $x_i \in X$, of $[a,b]$ there exists $\delta_2 \colon [a,b] \to \mathbb{R}_+$ such that for all δ_2^k-fine division D_i of $[a_i,b_i]$ we have

$$\sum_{i=1}^{p} |s(f_n, g; D_i|G)| < \epsilon \qquad \text{for all} \quad n$$

where δ_1, δ_2, η are independent of n and $s(f_n,g;D_i|G)$ denote the part of the Riemann sum $s(f_n,g;D_i)$ for which the associated points of D_i are in G and $|G|$ denotes the measure of G.

$\{f_n\}$ is said to be $ULBS_g^kG(X)$ if $X = \bigcup\limits_{j=1}^{\infty} Y_j$ with each Y_j closed and such that $\{f_n\}$ is $ULBS_g^k(Y_j)$ for all j.

Now we prove a version of the controlled convergence theorem for the Henstock-Stieltjes integral. We first prove the absolute version in Theorem 5.5 below and then the non-absolute version in Theorem 5.6. Although Theorem 5.5 is a particular case of Theorem 5.6, the proof of it is presented here for the better understanding of the new technique used, specially in the use of the concept in Definition 5.4.

Theorem 5.5. *Let*

(i) $g \in SV^k[a,b]$ *and* $J(g;x) = 0$ *for all* $x \in [a,b]$,

(ii) $(f_n, g) \in GS_k[a,b]$ *for all* n *with primitive* F_n,

(iii) $f_n \to f$ *as* $n \to \infty$ *everywhere in* $[a,b]$,

(iv) $\{F_n\}$ *is* $UAC^k[a,b]$,

(v) $\{f_n\}$ *is* $ULBRS_g^k[a,b]$,

(vi) $\{F_n(a,b)\}$ *converges.*

Then $(f,g) \in GS_k[a,b]$ *and* $\lim\limits_{n\to\infty} \int\limits_a^b f_n \,\mathrm{d}g = \int\limits_a^b f \,\mathrm{d}g$.

P r o o f. Let $\lim\limits_{n\to\infty} F_n(a,b) = A$ and $|F_n(a,b) - A| < \epsilon$ for $n \geq N_1$. Since $g \in SV^k[a,b]$, for $\epsilon > 0$ we can find $M > 0$ and $\delta_1', \delta_2'\colon [a,b] \to \mathbb{R}_+$ such that for any $(\delta_1', \delta_2'^k)$-fine partial division $\{y_j, [c_j, d_j], P_j\}_{j=1}^q$ of $[a,b]$ we have

$$\sum_{j=1}^{q} |s(1, g; P_j)| < M. \tag{1}$$

Again as $\{F_n\}$ is $UAC^k[a,b]$ there exist $\delta_1''\colon [a,b] \to \mathbb{R}_+$ and $\eta_1 > 0$ such that for every strictly δ_1''-fine partial division $\{[u_r, v_r]; t_r\}_{r=1}^s$ of $[a,b]$ with $\sum\limits_{r=1}^{s} (v_r - u_r) < \eta_1$ we have

$$\sum_{r=1}^{s} |F_n(u_r, v_r)| < \epsilon. \tag{2}$$

Here δ_1'', δ_2'' and η_1 are independent of n.

Also since $\{f_n\}$ is $ULBRS_g^k[a,b]$ there exist $\delta_3\colon [a,b] \to \mathbb{R}_+$ and $\eta_2 > 0$ such that for every open set G with $|G| < \eta_2$ and for every strictly δ_3-fine division $\{[\alpha_s, \beta_s]; \gamma_s\}_{s=1}^r$ of $[a,b]$ there exists $\delta_4\colon [a,b] \to \mathbb{R}_+$ such that for all δ_4^k-fine divisions D_s of $[\alpha_s, \beta_s]$ we have

$$\sum_{s=1}^{r} |s(f_n, g; D_s|G)| < \epsilon \tag{3}$$

for all n and δ_3, δ_4, η_2 are independent of n. Now since $(f_n, g) \in GS_k[a,b]$, $n = 1, 2, \ldots$, by the Saks-Henstock lemma there exist $\delta_{1,n}, \delta_{2,n}\colon [a,b] \to \mathbb{R}_+$ such that for every $(\delta_{1,n}, \delta_{2,n}^k)$-fine partial division $\{x_i^n, [a_i^n, b_i^n], D_i^n\}_{i=1}^{p_n}$ of $[a,b]$ we have

$$\left| \sum_{i=1}^{p_n} \{s(f_n, g; D_i^n) - F_n(a_i^n, b_i^n)\} \right| < \epsilon. \tag{4}$$

Also by condition (iii), using the Egoroff theorem we can find ([5]) an open set G and a positive integer $N > N_1$ with $|G| < \eta = \min\{\eta_1, \eta_2\}$ such that for all $n \geq N$ we have

$$\sup_{x \in [a,b]-G} |f_n(x) - f(x)| < \frac{\epsilon}{M}. \tag{5}$$

Let us write $[a,b] - G = X$.

We choose an open set $H \supset X$ such that $|H - X| < \eta$. Now, for $n > N$ we choose $\delta_1 \colon [a, b] \to \mathbb{R}_+$ such that $\delta_1(x) < \min\{\delta_1'(x), \delta_1''(x), \delta_3(x), \delta_{1,n}(x)\}$ and also $(x - \delta_1(x), x + \delta_1(x)) \subset H$ when $x \in X$; $(x - \delta_1(x), x + \delta_1(x)) \subset G$ when $x \in G - H$; and $(x - \delta_1(x), x + \delta_1(x)) \subset H - X$ when $x \in (H - X) \cap G$.

Let $\{[a_i, b_i]; x_i\}_{i=1}^{p}$ be a strictly δ_1-fine division of $[a, b]$. Then we can find $\delta_2 \colon [a, b] \to \mathbb{R}_+$ such that (1), (2), (3), (4) above hold for any δ_2^k-fine division D_i of $[a_i, b_i]$. Now

$$\begin{aligned}
&\Big| \sum_{i=1}^{p} s(f, g; D_i) - A \Big| \\
&\leq |A - F_n(a, b)| + \Big| \sum_{1} \{F_n(a_i, b_i) - s(f_n, g; D_i)\} \Big| \\
&\quad + \Big| \sum_{1} \{s(f_n, g; D_i) - s(f, g; D_i)\} \Big| + \sum_{2} |F_n(a_i, b_i)| + \Big| \sum_{2} s(f, g; D_i) \Big|
\end{aligned}$$

where $\sum_1$, $\sum_2$ denote the partial sums for which $x_i \notin G$ and $x_i \in G$ respectively.

By the definition of δ_1, it follows that

$$\sum_{2} |s(f_n, g; D_i|G)| = \sum_{2} |s(f_n, g; D_i)| < \epsilon,$$

independent of n.

Taking the limit as $n \to \infty$, we get

$$\sum_{2} |s(f, g; D_i)| \leq \epsilon.$$

Now, we split $\sum_1 \{s(f_n, g; D_i) - s(f, g; D_i)\}$ into two partial sums $\sum_3$, $\sum_4$ where $\sum_3$, $\sum_4$ contain those terms for which the associated points of D_i are in X and $(H - X) \cap G$ respectively.

Since $|H - X|$ is open and $|H - X| < \eta$ by (3) we have

$$\Big| \sum_{4} \{s(f_n, g; D_i) - s(f, g; D_i)\} \Big| \leq \sum_{4} |s(f_n, g; D_i)| + \sum_{4} |s(f, g; D_i)| < 2\epsilon.$$

Also using (4) we get

$$\Big| \sum_{3} \{s(f_n, g; D_i) - s(f, g; D_i)\} \Big| < \epsilon.$$

So, applying all the above inequalities we get

$$\Big| \sum_{i=1}^{p} s(f, g; D_i) - A \Big| < 7\epsilon.$$

Hence, the proof is complete. □

Remark 5.6. We note that in order to prove the integrability of (f,g) in Theorem 4.6 we need to impose conditions on F and also on the Riemann sum of (f,g). It seems unavoidable that we need to impose both conditions in order to carry through the proof of the controlled convergence. Further complications occur due to the use of two δ functions. When we take a Riemann sum $s(f_n,g;D_i)$ with D_i being a δ_2^k-fine division of $[a_i,b_i]$ and $\{[a_i,b_i];x_i\}$ being strictly δ_1-fine, there is no way to ensure that the associated points of D_i will always belong to a given set. The condition in Definition 5.4 is to ensure that those broken pieces of the Riemann sum with associated points not belonging to the given set will still be small. The broken pieces are broken with respect to δ_1 though not with δ_2. This condition is crucial in the proof of the controlled convergence theorem.

Theorem 5.7. *Let*

(i) $g \in SV^k[a,b]$ *and* $J(g;x)=0$ *for all* $x\in[a,b]$,

(ii) $(f_n,g)\in GS_k[a,b]$ *for all* n *with primitive* F_n,

(iii) $f_n\to f$ *as* $n\to\infty$ *everywhere in* $[a,b]$,

(iv) $\{F_n\}$ *is* $UAC^kG[a,b]$,

(v) $\{f_n\}$ *is* $ULBRS_g^kG[a,b]$,

(vi) $\{F_n(a,b)\}$ *converges.*

Then $(f,g)\in GS_k[a,b]$ *and* $\lim\limits_{n\to\infty}\int_a^b f_n\,\mathrm{d}g=\int_a^b f\,\mathrm{d}g$.

Proof. Let $\lim\limits_{n\to\infty}F_n(a,b)=A$ and N_0 be an integer such that for $n\ge N_0$

$$|F_n(a,b)-A|<\epsilon. \tag{1}$$

Using the conditions (iv) and (v) and without any loss of generality we can find a sequence of pairwise disjoint sets Y_q, closed in $[a,b]$, with $\bigcup\limits_{q=1}^{\infty}Y_q=[a,b]$ such that $\{F_n\}$ is $UAC^k(Y_q)$ and $\{f_n\}$ is $ULBRS_g^k(Y_q)$ for each $q=1,2,\dots$. Now, since $g\in SV^k[a,b]$ there exist $M>0$ and δ_1', δ_2' such that for all $(\delta_1',\delta_2'^k)$-fine partial divisions $\{y_j,[c_j,d_j],P_j\}_{j=1}^r$ of $[a,b]$ we have

$$\sum_{j=1}^{r}|s(1,g;P_j)|<M. \tag{2}$$

Since $\{f_n\}$ is $ULBRS_g^k(Y_q)$, for each $q=1,2,\dots$, there exist $\delta_{3q}\colon[a,b]\to\mathbb{R}_+$ and $\eta_{1,q}>0$ such that for every open set G_q' with $|G_q'|<\eta_{1,q}$ and for every strictly δ_{3q}-fine partial division $\{[\alpha_{sq},\beta_{sq}];\gamma_{sq}\}_{s=1}^t$ of $[a,b]$ with $\gamma_{sq}\in Y_q$, there exists $\delta_{4q}\colon [a,b] \to \mathbb{R}_+$ such that for all δ_{4q}^k-fine divisions D_{sq} of $[\alpha_{sq}, \beta_{sq}]$ we have

$$\sum_{s=1}^{t} |s(f_n, g; D_{sq}|G'_q)| < \frac{\epsilon}{2^q} \tag{3}$$

for all n where $\delta_{3q}, \delta_{4q}, \eta_{1q}$ are independent of n.

Since $\{F_n\}$ is $UAC^k(Y_q)$, for each $q = 1, 2, \ldots$ there exist $\delta'_{3q}\colon [a,b] \to \mathbb{R}_+$ and $\eta_{2q} > 0$ such that for every strictly δ'_{3q}-fine partial division $\{[u_{rq}, v_{rq}]; t_{rq}\}_{r=1}^{w}$ of $[a,b]$, $t_{rq} \in Y_q$, with $\sum_{r=1}^{w} (v_{rq} - u_{rq}) < \eta_{2q}$ we have

$$\sum_{r=1}^{w} |F_n(u_{rq}, v_{rq})| < \frac{\epsilon}{2^q} \tag{4}$$

for all n where δ'_{3q}, η_{2q} are independent of n.

Again as $(f_n, g) \in GS_k[a,b]$, $n = 1, 2, \ldots$, by the Saks-Henstock lemma there exist $\delta_{1,n}, \delta_{2,n}\colon [a,b] \to \mathbb{R}_+$ such that for every $(\delta_{1,n}, \delta_{2,n}^k)$-fine partial division $\{x_i^n, [a_i^n, b_i^n], D_i^n\}_{i=1}^{p_n}$ of $[a,b]$ we have

$$\sum_{i=1}^{p_n} |\{s(f_n, g; D_i^n) - F_n(a_i^n, b_i^n)\}| < \frac{\epsilon}{2^n}. \tag{5}$$

Also by condition (iii) and applying Egoroff's theorem, for each $q = 1, 2, \ldots$ there exist G_q open in Y_q with $|G_q| < \eta_q = \min\{\eta_{1q}, \eta_{2q}\}$ and a positive integer $N_q > N_0$ such that for all $n \geq N_q$ we have

$$\sup_{x \in Y_q - G_q} |f_n(x) - f(x)| < \frac{\epsilon}{M2^q}. \tag{6}$$

Let us write $Y_q - G_q = X_q$, $q = 1, 2, \ldots$. We choose an open set $H_q \supset X_q$ such that $|H_q - X_q| < \eta_q$. For $x \in Y_q$, $q = 1, 2, \ldots$, we define

$$\delta_1(x) < \min\{\delta'_1(x), \delta_{3q}(x), \delta'_{3q}(x), \delta_{1,n}(x)\}$$

and also we take $\delta_1(x)$ so that $(x - \delta_1(x), x + \delta_1(x)) \subset H_q$ when $x \in X_q$, $(x-\delta_1(x), x+\delta_1(x)) \subset G_q$ when $x \in G_q - H_q$, and $(x-\delta_1(x), x+\delta_1(x)) \subset H_q - X_q$ when $x \in (H_q - X_q) \cap Y_q$.

Let $\{[a_i, b_i]; x_i\}_{i=1}^{p}$ be a strictly δ_1-fine division of $[a,b]$. Then we can find $\delta_2\colon [a,b] \to \mathbb{R}_+$ such that (2), (3), (5) above hold for any δ_2^k-fine division D_i of $[a_i, b_i]$.

Now,

$$\begin{aligned}&\Big|\sum_{i=1}^{p} s(f,g;D_i) - A\Big| \\ &\le |F_N(a,b) - A| + \Big|\sum_{q=1}^{\infty}\sum_{1q}\{s(f_{N_q},g;D_i) - F_{N_q}(a_i,b_i)\}\Big| \\ &\quad + \Big|\sum_{q=1}^{\infty}\Big[\sum_{1q}\{s(f_{N_q},g;D_i) - s(f,g;D_i)\}\Big]\Big| \\ &\quad + \Big|\sum_{q=1}^{\infty}\Big[\sum_{1q}\{F_{N_q}(a_i,b_i) - F_N(a_i,b_i)\}\Big]\Big| \\ &\quad + \sum_{q=1}^{\infty}\sum_{2q}|F_N(a_i,b_i)| + \sum_{q=1}^{\infty}\sum_{2q}|s(f,g;D_i)|,\end{aligned}$$

where $\sum\limits_{1q}, \sum\limits_{2q}$ denote the partial sums for which $x_i \in X_q$ and $x_i \in G_q$ respectively. Since there can be at most p distinct N_q's, let $N = \max\{N_q\}$. Now, $|A - F_N(a,b)| < \epsilon$, by (1).

$$\begin{aligned}\Big|\sum_{q=1}^{\infty}\sum_{1q}\{s(f_{N_q},g;D_i) - F_{N_q}(a_i,b_i)\}\Big| &\le \sum_{n=1}^{\infty}\Big|\sum_{N_q=n}\{s(f_{N_q},g;D_i) - F_{N_q}(a_i,b_i)\}\Big| \\ &\le \sum_{n=1}^{\infty}\frac{\epsilon}{2^n} = \epsilon,\end{aligned}$$

by (5). Now,

$$\begin{aligned}&\Big|\sum_{q=1}^{\infty}\Big[\sum_{1q}\{s(f_{N_q},g;D_i) - s(f,g;D_i)\}\Big]\Big| \\ &\le \sum_{q=1}^{\infty}\Big|{\sum_{1q}}'\{s(f_{N_q},g;D_i) - s(f,g;D_i)\}\Big| + \sum_{q=1}^{\infty}\Big|{\sum_{1q}}''\{s(f_{N_q},g;D_i) - s(f,g;D_i)\}\Big|\end{aligned}$$

where ${\sum}'_{1q}$, ${\sum}''_{1q}$ denote the partial sums of $\sum\limits_{1q}$ for which the associated points of D_i are in X_q and $H_q - X_q$ respectively.

Now by (6), $|\sum\limits_{1q}'\{s(f_{N_q},g;D_i) - s(f,g;D_i)| \le \frac{\epsilon}{2^q}$. Also by the way we defined δ_1, if an associated point of D_i is in $(H_q - X_q) \cap Y_q$ then the corresponding subinterval is contained in $(H_q - X_q) \cap Y_q$, $q = 1, 2, \ldots$. So, by (3),

$${\sum_{1q}}''|s(f_n,g;D_i)| < \frac{\epsilon}{2^q}, \qquad \text{for all} \quad n.$$

Taking the limit as $n \to \infty$ we get

$$\sum_{1q}'' |s(f, g; D_i)| \leq \frac{\epsilon}{2^q}.$$

So,

$$\sum_{1q}'' |\{s(f_{N_q}, g; D_i) - s(f, g; D_i)\}| < \frac{2\epsilon}{2^q}$$

and

$$\Big| \sum_{q=1}^{\infty} \{s(f_{N_q}, g; D_i) - s(f, g; D_i)\} \Big| < \sum_{q=1}^{\infty} \frac{3\epsilon}{2^q} = 3\epsilon.$$

Again,

$$\begin{aligned}
&\Big| \sum_{q=1}^{\infty} \sum_{1q} \{F_{N_q}(a_i, b_i) - F_N(a_i, b_i)\} \Big| \\
&\leq \Big| \sum_{q=1}^{\infty} \sum_{1q} \{F_{N_q}(a_i, b_i) - s(f_{N_q}, g; D_i)\} \Big| \\
&\quad + \Big| \sum_{q=1}^{\infty} \sum_{1q} \{F_N(a_i, b_i) - s(f_N, g; D_i)\} \Big| + \sum_{q=1}^{\infty} \sum_{1q}'' |s(f_{N_q}, g; D_i)| \\
&\quad + \sum_{q=1}^{\infty} \sum_{1q}'' |s(f_N, g; D_i)| + \sum_{q=1}^{\infty} \sum_{1q}' |s(f_{N_q}, g; D_i) - s(f_N, g; D_i)| \\
&\leq \Big| \sum_{n \geq N_0} \sum_{N_q = n} \{F_{N_q}(a_i, b_i) - s(f_{N_q}, g; D_i)\} \Big| \\
&\quad + \Big| \sum_{n=1}^{\infty} \sum_{N=n} \{F_N(a_i, b_i) - s(f_N, g; D_i)\} \Big| + 3 \sum_{q=1}^{\infty} \frac{\epsilon}{2^q} \\
&\leq 2 \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} + 3 \sum_{q=1}^{\infty} \frac{\epsilon}{2^q} = 5\epsilon.
\end{aligned}$$

Also by (3), (4) we have

$$\sum_{q=1}^{\infty} \sum_{1q}'' |s(f_{N_q}, g; D_i)| < \epsilon$$

and

$$\sum_{q=1}^{\infty} \sum_{1q}'' |s(f_N, g; D_i)| < \epsilon.$$

Hence combining all the above inequalities we have

$$\Big|\sum_{i=1}^{p} s(f,g;D_i) - A\Big| < 11\epsilon.$$

$\square$

6. Examples

Definition 6.1. A function f defined on $[a,b]$ is said to be a *regulated function* if f has one-sided limits at every point of $[a,b]$, see [2, p. 139].

Also it is known that a function f is a regulated function on $[a,b]$, if and only if there is a sequence of step functions $\{f_n\}$, uniformly convergent to f on $[a,b]$.

In this section we give an example of g so that if f is regulated then $(f,g) \in GS_k[a,b]$. We also derive an integration by parts formula for suitable choices of g. For simplicity, we write the example for $k=2$.

Example 6.2. Let us define $g\colon [a,b]^3 \to \mathbb{R}$ by $g(u,v,w) = \frac{G(w)-G(v)}{w-v} - \frac{G(v)-G(u)}{v-u}$ for $u<v<w$ and $=0$ otherwise, where G is the Henstock primitive of a function g^* which is continuous and of bounded variation on $[a,b]$. Then $(f,g) \in GS_2[a,b]$ for any regulated function f on $[a,b]$.

Proof. Since G is derivable on $[a,b]$ we have $J(g;c)=0$ for all $c \in [a,b]$. Let $\epsilon > 0$. For any function $\delta_1\colon [a,b] \to \mathbb{R}_+$ we take a strictly δ_1-fine division $\{[a_i,b_i];x_i\}_{i=1}^{p}$ of $[a,b]$. Then using continuity of g^* we choose $\delta_2\colon [a,b] \to \mathbb{R}_+$ such that for any δ_2^2-fine division $D_i = \{[x_j^i, x_{j+2}^i]; \xi_j^i\}_{j=0}^{n_i-2}$ of $[a_i,b_i]$ we have

$$|g^*(a_i) - g^*(x)| < \frac{\epsilon}{p}$$

whenever $x \in [x_0^i, x_1^i)$ and

$$|g^*(b_i) - g^*(y)| < \frac{\epsilon}{p}$$

whenever $y \in (x_{n_i-1}^i, x_{n_i}^i]$. Then

$$\begin{aligned}
&\Big|\sum_{i=1}^{p} s(1,g;D_i) - (g^*(b) - g^*(a))\Big| \\
&= \Big|\sum_{i=1}^{p} \big\{ s(1,g;D_i) - \big(g^*(b_i) - g^*(a_i)\big)\big\}\Big| \\
&\le \sum_{i=1}^{p} |\{g^*(\beta_i) - g^*(b_i)\} + \{g^*(\alpha_i) - g^*(a_i)\}|
\end{aligned}$$

for some $\alpha_i \in (x_0^i, x_1^i)$, $\beta_i \in (x_{n_i-1}^i, x_{n_i}^i)$, applying Lagrange's Mean Value Theorem on G in $[x_0^i, x_1^i]$ and in $[x_{n_i-1}^i, x_{n_i}^i]$

$$\leq \sum_{i=1}^{p} (|g^*(b_i) - g^*(\beta_i)| + |g^*(a_i) - g^*(\alpha_i)|) < 2\epsilon.$$

Thus $(1, g) \in GS_2[a,b]$. Using the elementary properties of the GS_k integral (Theorem 3.1 and Theorem 3.5) we can easily verify that if f is a step function on $[a,b]$, then $(f,g) \in GS_2[a,b]$. Since f is regulated, there is a sequence of step functions $\{f_n\}$ such that f_n converges to f uniformly on $[a,b]$. Also since G is the primitive of a function of bounded variation, it is of bounded slope variation on $[a,b]$. So, by Theorem 5.1 we can say that if f is regulated then $(f,g) \in GS_2[a,b]$.

We now give an integration by parts formula for $k = 2$ and for particular functions. □

Theorem 6.3 (**Integration by parts**). *Let $f, g \colon [a,b] \to \mathbb{R}$ be continuous and of bounded variation on $[a,b]$ and F, G be the Henstock primitive of f, g respectively. We define $f_1, g_1 \colon [a,b]^3 \to \mathbb{R}$ by*

$$f_1(u,v,w) = \frac{F(w) - F(v)}{w - v} - \frac{F(v) - F(u)}{v - u}$$

when $u < v < w$ and 0 otherwise,

$$g_1(u,v,w) = \frac{G(w) - G(v)}{w - v} - \frac{G(v) - G(u)}{v - u}$$

when $u < v < w$ and 0 otherwise.

Then

$$\int_a^b f \, \mathrm{d}g_1 + \int_a^b g \, \mathrm{d}f_1 = 2\{f(b)g(b) - f(a)g(a)\}.$$

Proof. Here F, G, being primitive of continuous functions, are of bounded slope variation on $[a,b]$. So f_1, g_1 are $SV^2[a,b]$ by Example 4.2 and thus $(f, g_1), (g, f_1) \in GS_2[a,b]$. Also there exists $M_1 > 0$ such that

$$\sum_{i=0}^{n-2} \left| \frac{G(x_{i+2}) - G(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{G(x_{i+1}) - G(x_i)}{x_{i+1} - x_i} \right| < M_1,$$

and

$$\sum_{i=0}^{n-2} \left| \frac{F(x_{i+2}) - F(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} \right| < M_1,$$

for any division $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a,b]$.

As F, G are derivable on $[a,b]$ we have $J(f_1;x) = 0 = J(g_1;x)$ for all $x \in (a,b)$. Since $(f,g_1),(g,f_1) \in GS_2[a,b]$ there exist $\delta_1,\delta_2\colon [a,b] \to \mathbb{R}_+$ such that for any (δ_1,δ_2^2)-fine division $\{x_i,[a_i,b_i],D_i\}_{i=1}^p$ of $[a,b]$ where $D_i = \{[x_j^i,x_{j+2}^i];\xi_j^i\}_{j=0}^{n_i-2}$ we have

$$\left|\sum_{i=1}^{p} s(f,g_1;D_i) - \int_a^b f\,\mathrm{d}g_1\right| < \epsilon$$

and

$$\left|\sum_{i=1}^{p} s(g,f_1;D_i) - \int_a^b g\,\mathrm{d}f_1\right| < \epsilon.$$

Let $\{[a_i,b_i];x_i\}_{i=1}^p$ be a strictly δ_1-fine division of $[a,b]$ and

$$M = \left\{2\sum_{i=1}^{p}|f(b_i)| + 2|f(a_1)| + 2\sum_{i=1}^{p}|g(b_i)| + 2|g(a_1)| + M_1\right\}.$$

Since f, g are continuous on $[a,b]$, for $\epsilon > 0$ there exists $\eta > 0$ such that $|f(x_1) - f(x_2)| < \frac{\epsilon}{2M}$ and $|g(x_1) - g(x_2)| < \frac{\epsilon}{2M}$ whenever $|x_1 - x_2| < \eta$.

We modify δ_2 in such a way that $\delta_2(x) < \frac{\eta}{2}$ for all x and let $D_i = \{[x_j^i,x_{j+2}^i];\xi_j^i\}_{j=0}^{n_i-2}$ be a δ_2^2-fine division of $[a_i,b_i]$. Now by some routine calculations we can show that

$$\left|\sum_{i=1}^{p} s(f,g_1;D_i) + \sum_{i=1}^{p} s(g,f_1;D_i) - 2\big(f(b)g(b) - f(a)g(a)\big)\right| < \epsilon.$$

Hence,

$$\int_a^b f\,\mathrm{d}g_1 + \int_a^b g\,\mathrm{d}f_1 = 2\{f(b)g(b) - f(a)g(a)\}.$$

$\square$

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