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SYSTEMS OF SETS WITH MULTIPLICATIVE ASYMPTOTIC DENSITY

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ABSTRACT. The authors define a notion of system of sets with multiplicative asymptotic density in this paper. A criterion and one necessary condition for a given system $\{A_i\}_{i=1}^{\infty}$ to be a system with multiplicative asymptotic density is given. Properties of certain special types of systems of sets with multiplicative asymptotic density are treated.

1. Introduction

In paper [3], we studied, among others, properties of the set of natural numbers which are expressible in the form of sum of two squares of integers (see [2], [4], [5]) and properties of a system of certain sets where this set is included.

We have denoted $p_1 < p_2 < \dots < p_i < \dots$ the sequence of prime numbers in the form $p_i = 4k+3$, where $k \in \mathbb{N}\cup\{0\}$ and $D_i = \{m\cdot p_i^{2j} : j \in \mathbb{N}\cup\{0\},\ m \in \mathbb{N},\ \gcd(m,p_i) = 1\}$. It means, that D_i is a set of those natural numbers which have not the prime number p_i with odd exponent in their canonical decompositions. It is well known (see [1]), that a set D containing all natural numbers which are expressible in the form of sum of two squares of integers is equal to the intersection of sets D_i, $i = 1, 2, \dots$, i.e. $D = \bigcap_{i=1}^{\infty} D_i$.

We have proved following two theorems in [3]:

Lemma 1.1. *Let us denote $p_1 < p_2 < \dots < p_i < \dots$ the sequence of prime numbers in the form $p_i = 4k+3$, where $k \in \mathbb{N}\cup\{0\}$ and $D_i = \{m\cdot p_i^{2j} : j \in \mathbb{N}\cup\{0\},\ m \in \mathbb{N},\ \gcd(m,p_i) = 1\}$. Then for asymptotic density of the set D_i it holds that*

$$d(D_i) = \frac{p_i}{p_i+1}.$$

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Lemma 1.2. *Let D_i be sets defined in the same way as in the previous lemma. Then for every $n \in \mathbb{N}$ it holds, that*

$$d\Big(\bigcap_{i=1}^{n} D_i\Big) = \prod_{i=1}^{n} d(D_i) = \prod_{i=1}^{n} \frac{p_i}{p_i+1}.$$

As a corollary of Lemma 1.2 we have had obtained:

Corollary 1.1. *Let D_i be sets defined in the same way as in* Lemma 1.1. *Then*

$$d\Big(\bigcap_{i=1}^{\infty} D_i\Big) = 0.$$

We can see that the asymptotic density of the intersection of sets $D_1, \dots, D_n$ is equal to the product of asymptotic densities of sets $D_1, \dots, D_n$. If we made minor alterations to the proof of 1.2, then we would find, that for an arbitrary finite subsystem of the system $\{D_i\}_{i=1}^{\infty}$ the asymptotic density of intersection of its sets is equal to the product of asymptotic densities of its sets.

2. Systems of sets with multiplicative asymptotic density

Inspired by properties of asymptotic densities of intersections of sets D_i which are described in Lemma 1.2, we define a notion of *system of sets with multiplicative asymptotic density.*

We will use following denotation.

Denotation 2.1. Let $\mathbb{N}$ be the set of natural numbers and A be a subset of $\mathbb{N}$. We denote by $A(n)$ the number of elements of the set A which are less or equal to n and by $d(A)$ the asymptotic density of A ($d(A) = \lim\limits_{n\to\infty} \frac{A(n)}{n}$). By $[x]$ we denote the integral part of number x. Finally $A - B$ is the difference of sets A and B.

Now we define the notion of system of sets with multiplicative asymptotic density.

Definition 2.1. Let Σ be the system of all sets $A \subseteq \mathbb{N}$ such that their asymptotic densities $d(A)$ exist. Then we say, that the system $\{A_i\}_{i=1}^{\infty}$, where $A_i \in \Sigma$, has *multiplicative asymptotic density* if and only if for every $n \in \mathbb{N}$ it holds that

$$d\Big(\bigcap_{j=1}^{n} A_{i_j}\Big) = \prod_{j=1}^{n} d(A_{i_j}), \tag{1}$$

where $i_j \in \mathbb{N}$, $i_j \neq i_k$ for $j \neq k$. Instead of "system $\{A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density" we can use denotation "$\{A_i\}_{i=1}^{\infty}$ is a m.a.d. system."

Verbal description of this definition could be the following. A system of sets of natural numbers (whose asymptotic densities exist) $\{A_i\}_{i=1}^{\infty}$, has multiplicative asymptotic density if and only if the asymptotic density of intersection of its arbitrary finite subsystem of distinct sets is equal to the product of asymptotic densities of sets from this subsystem.

Example 2.1. In [3], we have proved (see Lemma 1.2), that the system $\{D_i\}_{i=1}^{\infty}$, where D_i is a set of those natural numbers, which have not the prime number p_i with odd exponent in their canonical decompositions and $p_1 < p_2 < \dots < p_i < \dots$ is the sequence of prime numbers in the form $p_i = 4k+3$, fulfils the condition

$$d\Big(\bigcap_{i=1}^{n} D_i\Big) = \prod_{i=1}^{n} d(D_i) = \prod_{i=1}^{n} \frac{p_i}{p_i+1} \tag{2}$$

for every $n \in \mathbb{N}$.

We have not used the condition $p_i = 4k+3$ in the proof of Lemma 1.2. Therefore we can say, that equation (2) holds also for system of sets $\{D_i^*\}_{i=1}^{\infty}$, where D_i^* is a set of those natural numbers, which have not the prime number p_i with odd exponent in their canonical decompositions, $p_1 < p_2 < \cdots < p_i < \dots$ is the sequence of all prime numbers.

Also, we can say that in this case the proof of equation (1) could be done in the same way as the proof of equation (2), we just would have to change indexes of sets $D_1^*, \dots, D_n^*$ to $D_{j_1}^*, \dots, D_{j_n}^*$.

We can see, that the system $\{D_i^*\}_{i=1}^{\infty}$ is a system with multiplicative asymptotic density.

Trivial example of a system with multiplicative asymptotic density could be the system $\{A_i\}_{i=1}^{\infty}$, where $A_i = \mathbb{N} - F_i$, where F_i, $i = 1, \dots, n$, is finite set, $F_i \neq F_j$ for $i \neq j$.

There is a question, which systems of sets $\{A_i\}_{i=1}^{\infty}$ have multiplicative asymptotic density. Preliminary information about this problem gives the following criterion.

Theorem 2.1. *System of sets $\{A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density if and only if the system $\{\mathbb{N} - A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density.*

Proof. First we mention, that $A_i \in \Sigma$ if and only if $\mathbb{N} - A_i \in \Sigma$ (where Σ has the same sense as in Definition 2.1). Further let us suppose, that the system $\{A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density and consider its arbitrary subsystem $\{A_{i_j}\}_{j=1}^{n}$. We have to prove, that

$$d\Big(\bigcap_{j=1}^{n} (\mathbb{N} - A_{i_j})\Big) = \prod_{j=1}^{n} d(\mathbb{N} - A_{i_j}).$$

First we analyze the case when there is a set A_{i_k}, $k \in \{1, \dots, n\}$, such that $d(A_{i_k}) = 0$. Since in this case it holds that $d(\mathbb{N} - A_{i_k}) = 1$, we can write

$$d\Big(\bigcap_{j=1}^{n} (\mathbb{N} - A_{i_j})\Big) = d\Big(\bigcap_{\substack{j=1 \\ j \neq k}}^{n} (\mathbb{N} - A_{i_j})\Big).$$

Exploiting this fact we can see, that we can consider just arbitrary subsystem $\{A_{i_j}\}_{j=1}^{n}$, where $d(A_{i_j}) \neq 0$ for every $j \in \{1, \dots, n\}$.

For simplicity of denotation we change the indexes of the sets of this system $\{A_{i_j}\}_{j=1}^{n}$ and we will denote it as $\{A_i\}_{i=1}^{n}$. With regard to this change of indexes, we are going to prove the relation

$$d\Big(\bigcap_{i=1}^{n} (\mathbb{N} - A_i)\Big) = \prod_{i=1}^{n} d(\mathbb{N} - A_i).$$

If we denote $A(m)$ the number of elements of a set A which are less or equal to m, then using simple combinatoric consideration we conclude, that

$$\Big(\bigcup_{i=1}^{n} A_i\Big)(m) = \sum_{i_1=1}^{n} A_{i_1}(m) - \sum_{i_1=1}^{n} \sum_{\substack{i_2=1 \\ i_1 > i_2}}^{n} (A_{i_1} \cap A_{i_2})(m) + \dots$$
$$\dots + (-1)^{n-1} \sum_{i_1=1}^{n} \dots \sum_{\substack{i_n=1 \\ i_1 > \dots > i_n}}^{n} (A_{i_1} \cap \dots \cap A_{i_n})(m).$$

Hence we obtain the relation

$$d\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i_1=1}^{n} d(A_{i_1}) - \sum_{i_1=1}^{n} \sum_{\substack{i_2=1 \\ i_1 > i_2}}^{n} d(A_{i_1} \cap A_{i_2}) + \dots$$
$$\dots + (-1)^{n-1} \sum_{i_1=1}^{n} \dots \sum_{\substack{i_n=1 \\ i_1 > \dots > i_n}}^{n} d(A_{i_1} \cap \dots \cap A_{i_n}).$$

As we have said before, we consider a system of sets $\{A_i\}_{i=1}^{n}$, where $d(A_i) \neq 0$ for every $i \in \{1, \dots, n\}$. Hence there are real numbers $k_i \in \langle 1, \infty)$, $i \in \{1, \dots, n\}$, such that

$$d(A_i) = \frac{1}{k_i}.$$

By assumption, the system $\{A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density. Thanks to this fact we can write

$$d\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i_1=1}^{n} \frac{1}{k_{i_1}} - \sum_{i_1=1}^{n} \sum_{\substack{i_2=1 \\ i_1 > i_2}}^{n} \frac{1}{k_{i_1} k_{i_2}} + \dots + (-1)^{n-1} \sum_{i_1=1}^{n} \dots \sum_{\substack{i_n=1 \\ i_1 > \dots > i_n}}^{n} \frac{1}{k_{i_1} \dots k_{i_n}}. \tag{3}$$

We are going to prove by induction that

$$d\Big(\bigcup_{i=1}^{n} A_i\Big) = 1 - \prod_{i=1}^{n} d(\mathbb{N} - A_i).$$

From (3) we obtain for $n = 2$ the equation

$$d(A_1 \cup A_2) = \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_1 k_2} = 1 - \left(1 - \frac{1}{k_1}\right)\left(1 - \frac{1}{k_2}\right) = 1 - \prod_{i=1}^{2} d(\mathbb{N} - A_i).$$

Now we are going to make the induction step. Let us suppose, that $d\Big(\bigcup_{i=1}^{r-1} A_i\Big) = 1 - \prod_{i=1}^{r-1} d(\mathbb{N} - A_i)$. Let us denote

$$S_m = \sum_{i_1=1}^{m} \frac{1}{k_{i_1}} - \sum_{i_1=1}^{m} \sum_{\substack{i_2=1 \\ i_1 > i_2}}^{m} \frac{1}{k_{i_1} k_{i_2}} + \dots + (-1)^{m-1} \sum_{i_1=1}^{m} \dots \sum_{\substack{i_m=1 \\ i_1 > \dots > i_m}}^{m} \frac{1}{k_{i_1} \dots k_{i_m}}$$

for arbitrary $m \in \mathbb{N}$, $m \geq 2$. From equation (3) it follows that $S_r = d\Big(\bigcup_{i=1}^{r} A_i\Big)$. Moreover, we can see that

$$d\Big(\bigcup_{i=1}^{r} A_i\Big) = S_{r-1} + \frac{1}{k_r} - S_{r-1}\frac{1}{k_r},$$

hence

$$d\Big(\bigcup_{i=1}^{r} A_i\Big) = S_{r-1}\left(1 - \frac{1}{k_r}\right) + \frac{1}{k_r}.$$

By induction assumption, that $d\Big(\bigcup_{i=1}^{r-1} A_i\Big) = 1 - \prod_{i=1}^{r-1} d(\mathbb{N} - A_i)$, we obtain

$$d\Big(\bigcup_{i=1}^{r} A_i\Big) = \left(1 - \prod_{i=1}^{r-1}\Big(1 - \frac{1}{k_i}\Big)\right)\left(1 - \frac{1}{k_r}\right) + \frac{1}{k_r},$$

$$d\Big(\bigcup_{i=1}^{r} A_i\Big) = 1 - \prod_{i=1}^{r}\Big(1 - \frac{1}{k_i}\Big). \tag{4}$$

Using known relation $\bigcup_{i=1}^{r} A_i = \mathbb{N} - \bigcap_{i=1}^{r} (\mathbb{N} - A_i)$ we obtain

$$d\Big(\bigcup_{i=1}^{r} A_i\Big) = 1 - d\Big(\bigcap_{i=1}^{r} (\mathbb{N} - A_i)\Big). \tag{5}$$

By comparison of (4) and (5) we conclude

$$d\Big(\bigcap_{i=1}^{r} (\mathbb{N} - A_i)\Big) = \prod_{i=1}^{r}\left(1 - \frac{1}{k_i}\right) = \prod_{i=1}^{r} d(\mathbb{N} - A_i).$$

We have proved that the system $\{\mathbb{N} - A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density. The proof of reversal implication is trivial now. $\square$

Example 2.2. As we know (see Example 2.1), the system $\{D_i^*\}_{i=1}^{\infty}$, where D_i^* is a set of those natural numbers which have not the prime number p_i with even exponent (or equal to zero) in their canonical decompositions, is a system with multiplicative asymptotic density.

We define the system of sets $\{B_i\}_{i=1}^{\infty}$, where

$$B_i = \Big\{ m \cdot p_i^{2j+1} : \ j \in \mathbb{N} \cup \{0\}, \ \ m \in \mathbb{N}, \ \ \gcd(m, p_i) = 1 \Big\},$$

$p_1 < p_2 < \dots < p_i < \dots$ is the sequence of all prime numbers.

It means that B_i is a set of those natural numbers which have the prime number p_i with odd exponent in their canonical decompositions.

We can see that $B_i = \mathbb{N} - D_i^*$, so

$$\{B_i\}_{i=1}^{\infty} = \{\mathbb{N} - D_i^*\}_{i=1}^{\infty}.$$

From Theorem 2.1 it follows that the system $\{B_i\}_{i=1}^{\infty}$ is a system with multiplicative asymptotic density. It means, that for arbitrary $n \in \mathbb{N}$ it holds that

$$d\Big(\bigcap_{j=1}^{n} B_{i_j}\Big) = \prod_{j=1}^{n} d(B_{i_j}) = \prod_{j=1}^{n} d(\mathbb{N} - D_{i_j}^*) = \prod_{j=1}^{n} \Big(1 - \frac{p_{i_j}}{1 + p_{i_j}}\Big), \tag{6}$$

where $i_j \in \mathbb{N}$, $i_j \neq i_k$ for $j \neq k$.

Corollary 2.1. *Let $\{A_i\}_{i=1}^{\infty}$ be a system with multiplicative asymptotic density. Then for every $n \in \mathbb{N}$ and for every $i_1, \dots, i_n \in \mathbb{N}$ it holds that*

$$d\Big(\bigcup_{j=1}^{n} A_{i_j}\Big) = 1 - \prod_{j=1}^{n} d(\mathbb{N} - A_{i_j}).$$

Proof. We just have to realize, that $\bigcup_{j=1}^{n} A_{i_j} = \mathbb{N} - \bigcap_{j=1}^{n} (\mathbb{N} - A_{i_j})$. The system of sets $\{\mathbb{N} - A_i\}_{i=1}^{\infty}$ is, with regard to Theorem 2.1, a system of sets with multiplicative asymptotic density. The statement of Corollary 2.1 is then obvious. $\square$

In addition to the criterion from Theorem 2.1, we introduce a necessary condition which every system of sets with multiplicative asymptotic density fulfils.

Point is that sets of such a system which asymptotic density belongs to the open interval $(0, 1)$ must differ each other in infinitely many elements.

Theorem 2.2. *Let $\{A_i\}_{i=1}^{\infty}$ be a system of sets with multiplicative asymptotic density. Then for arbitrary $i, j \in \mathbb{N}$, $i \neq j$, such that $0 < d(A_i) < 1$, $0 < d(A_j) < 1$ it holds that $A_i - A_j$ and $A_j - A_i$ are infinite sets and in particular $d(A_i - A_j) > 0$, $d(A_j - A_i) > 0$.*

P r o o f. We have to realize that

$$A_i \cap A_j = A_i - (A_i - A_j)$$

and

$$A_i \cap A_j = A_j - (A_j - A_i).$$

$A_i - A_j \subseteq A_i$, hence

$$(A_i \cap A_j)(n) = A_i(n) - (A_i - A_j)(n).$$

Asymptotic densities $d(A_i \cap A_j)$ and $d(A_i)$ exist, thus the asymptotic density $d(A_i - A_j)$ exists, too, and it is easy to see that

$$d(A_i \cap A_j) = d(A_i) - d(A_i - A_j).$$

If the set $A_i - A_j$ was finite, then $d(A_i - A_j) = 0$ and it would mean that

$$d(A_i \cap A_j) = d(A_i) - d(A_i - A_j) = d(A_i).$$

By assumption, the system $\{A_i\}_{i=1}^{\infty}$ has multiplicative asymptotic density. Hence

$$d(A_i \cap A_j) = d(A_i)d(A_j).$$

But this would mean that $d(A_j) = 1$ or $d(A_i) = 0$, what is in contradiction with the assumption $0 < d(A_j) < 1$ or with $0 < d(A_i) < 1$.

Likewise, finiteness of the set $A_j - A_i$ leads to contradiction with the assumption $0 < d(A_i) < 1$ or with $0 < d(A_j) < 1$.

From

$$d(A_i \cap A_j) = d(A_i) - d(A_i - A_j) \neq d(A_i)$$

it also follows that $d(A_i - A_j)$ must be greater than zero. Similarly we could obtain $d(A_j - A_i) > 0$. □

We generalize Lemma 1.1 and Lemma 1.2 now.

Theorem 2.3. *Let $p_1 < p_2 < \dots < p_i < \dots$ be a sequence of natural numbers, such that $p_i > 1$ for $i \in \mathbb{N}$ and for each $i \neq j$, $\gcd(p_i, p_j) = 1$ holds.*

Further let $a \in \mathbb{N}$, $b \in \mathbb{N} \cup \{0\}$ be given. We define, for every $i \in \mathbb{N}$, the sets

$$A_i = \left\{p_i^{ka+b} \cdot m : \ m \in \mathbb{N}; \ \ m \neq p_i \cdot m_1, \ \ m_1 \in \mathbb{N}; \ \ k \in \mathbb{N} \cup \{0\}\right\}.$$

Then

1) *For every $i \in \mathbb{N}$ it holds that*

$$d(A_i) = \frac{p_i^a}{p_i^b}\Big(\frac{1}{p_i^a - 1}\Big)\Big(1 - \frac{1}{p_i}\Big).$$

2) *The system of sets $\{A_i\}_{i=1}^{\infty}$ is a system of sets with multiplicative asymptotic density, i.e. for every $n \in \mathbb{N}$ and every $\{i_1, \dots, i_n\} \subset \mathbb{N}$ the following equation holds*

$$d\Big(\bigcap_{j=1}^{n} A_{i_j}\Big) = \prod_{j=1}^{n} d(A_{i_j}) = \prod_{j=1}^{n} \frac{p_{i_j}^a}{p_{i_j}^b}\Big(\frac{1}{p_{i_j}^a - 1}\Big)\Big(1 - \frac{1}{p_{i_j}}\Big).$$

P r o o f.

ad 1) We can see that

$$A_i = \{p_i^{ka+b} \cdot m:\ m \in \mathbb{N};\ \ m \neq p_i \cdot m_1,\ \ m_1 \in \mathbb{N};\ \ k \in \mathbb{N} \cup \{0\}\}.$$

Let us define, for arbitrary fixed $k \in \mathbb{N} \cup \{0\}$, the sets

$$B_{ka+b} = \{p_i^{ka+b} \cdot m:\ m \in \mathbb{N}\}$$

and

$$C_{ka+b+1} = \{p_i^{ka+b+1} \cdot m:\ m \in \mathbb{N}\}.$$

Asymptotic densities of these sets fulfill, for every $k \in \mathbb{N} \cup \{0\}$, the conditions

$$d(B_{ka+b}) = \frac{1}{p_i^{ka+b}} \qquad \text{and} \qquad d(C_{ka+b+1}) = \frac{1}{p_i^{ka+b+1}}. \tag{7}$$

It is obvious that

$$\begin{gathered} B_b - C_{b+1} \subseteq A_i \subseteq B_b, \\ (B_b - C_{b+1}) \cup (B_{a+b} - C_{a+b+1}) \subseteq A_i \subseteq (B_b - C_{b+1}) \cup B_{a+b} \\ (B_b - C_{b+1}) \cup (B_{a+b} - C_{a+b+1}) \cup (B_{2a+b} - C_{2a+b+1}) \subseteq A_i \\ A_i \subseteq (B_b - C_{b+1}) \cup (B_{a+b} - C_{a+b+1}) \cup B_{2a+b} \end{gathered}$$

and so on. Therefore, using equation (7), we can write for every $k \in \mathbb{N} \cup \{0\}$

$$\sum_{j=0}^{k} \frac{1}{p_i^{ja+b}} - \sum_{j=0}^{k} \frac{1}{p_i^{ja+b+1}} \leq \underline{d}(A_i) \leq \bar{d}(A_i) \leq \sum_{j=0}^{k} \frac{1}{p_i^{ja+b}} - \sum_{j=0}^{k-1} \frac{1}{p_i^{ja+b+1}}.$$

It means that

$$d(A_i) = \sum_{j=0}^{\infty} \frac{1}{p_i^{ja+b}} - \sum_{j=0}^{\infty} \frac{1}{p_i^{ja+b+1}} = \frac{1}{p_i^b} \frac{1}{1-\frac{1}{p_i^a}} - \frac{1}{p_i^{b+1}} \frac{1}{1-\frac{1}{p_i^a}} = \frac{p_i^a}{p_i^b} \frac{1}{p_i^a - 1} \Big(1 - \frac{1}{p_i}\Big).$$

ad 2) By induction. The case of $n = 1$ was shown in part ad 1). Now, consider sets $A_{i_1}, \dots, A_{i_n}$. Let us suppose that

$$d\Big(\bigcap_{j=1}^{n-1} A_{i_j}\Big) = \prod_{j=1}^{n-1} d(A_{i_j}) = \prod_{j=1}^{n-1} \frac{p_{i_j}^a}{p_{i_j}^b} \Big(\frac{1}{p_{i_j}^a - 1}\Big)\Big(1 - \frac{1}{p_{i_j}}\Big).$$

We have to realize that $\bigcap_{j=1}^{n} A_{i_j} = \{p_{i_n}^{k_n a+b} p_{i_{n-1}}^{k_{n-1} a+b} \dots p_{i_1}^{k_1 a+b} \cdot m:\ m \in \mathbb{N};\ m \neq p_{i_j} \cdot m_{i_j},\ \ m_{i_j} \in \mathbb{N};\ \ k_j \in \mathbb{N} \cup \{0\},\ \ j = 1, \dots, n\}$ and define, for arbitrary $k \in \mathbb{N} \cup \{0\}$, the sets

$$B_{ka+b} = \Big\{p_{i_n}^{ka+b} \cdot m:\ m \in \bigcap_{j=1}^{n-1} A_{i_j}\Big\}$$

and

$$C_{ka+b+1} = \Big\{p_{i_n}^{ka+b+1} \cdot m:\ m \in \bigcap_{j=1}^{n-1} A_{i_j}\Big\}.$$

If we denote $A^* = \bigcap_{j=1}^{n-1} A_{i_j}$, then asymptotic densities of these sets fulfil for every $k \in \mathbb{N} \cup \{0\}$ the conditions

$$d(B_{ka+b}) = \frac{1}{p_{i_n}^{ka+b}} d(A^*) \qquad \text{and} \qquad d(C_{ka+b+1}) = \frac{1}{p_{i_n}^{ka+b+1}} d(A^*). \tag{8}$$

It is obvious, that

$$B_b - C_{b+1} \subseteq \bigcap_{j=1}^{n} A_{i_j} \subseteq B_b,$$

$$(B_b - C_{b+1}) \cup (B_{a+b} - C_{a+b+1}) \subseteq \bigcap_{j=1}^{n} A_{i_j} \subseteq (B_b - C_{b+1}) \cup B_{a+b}$$

$$(B_b - C_{b+1}) \cup (B_{a+b} - C_{a+b+1}) \cup (B_{2a+b} - C_{2a+b+1}) \subseteq \bigcap_{j=1}^{n} A_{i_j}$$

$$\bigcap_{j=1}^{n} A_{i_j} \subseteq (B_b - C_{b+1}) \cup (B_{a+b} - C_{a+b+1}) \cup B_{2a+b}$$

and so on. Therefore, using equation (8) we can write for every $k \in \mathbb{N} \cup \{0\}$

$$d(A^*)\left(\sum_{j=0}^{k} \frac{1}{p_{i_n}^{ja+b}} - \sum_{j=0}^{k} \frac{1}{p_{i_n}^{ja+b+1}}\right) \leq \underline{d}\Big(\bigcap_{j=1}^{n} A_{i_j}\Big)$$
$$\leq \bar{d}\Big(\bigcap_{j=1}^{n} A_{i_j}\Big) \leq d(A^*)\left(\sum_{j=0}^{k} \frac{1}{p_{i_n}^{ja+b}} - \sum_{j=0}^{k-1} \frac{1}{p_{i_n}^{ja+b+1}}\right).$$

It means that

$$\begin{aligned}
d\Big(\bigcap_{j=1}^{n} A_{i_j}\Big) &= d(A^*)\left(\sum_{j=0}^{\infty} \frac{1}{p_{i_n}^{ja+b}} - \sum_{j=0}^{\infty} \frac{1}{p_{i_n}^{ja+b+1}}\right) \\
&= d(A^*)\left(\frac{1}{p_{i_n}^{b}} \frac{1}{1 - \frac{1}{p_{i_n}^{a}}} - \frac{1}{p_{i_n}^{b+1}} \frac{1}{1 - \frac{1}{p_{i_n}^{a}}}\right) \\
&= d(A^*)\left(\frac{p_{i_n}^{a}}{p_{i_n}^{b}} \frac{1}{p_{i_n}^{a} - 1}\left(1 - \frac{1}{p_{i_n}}\right)\right) \\
&= \left(\prod_{j=1}^{n-1} d(A_{i_j})\right) d(A_{i_n}) = \prod_{j=1}^{n} d(A_{i_j}).
\end{aligned}$$

□

Theorem 2.4. *Let* $1 < p_1 < p_2 < \dots < p_k < \dots$ *and* $1 < q_1 < q_2 < \dots$ $\dots < q_k < \dots$ *are sequences of natural numbers such that for every natural numbers* $i \neq j$ *it holds that* $\gcd(p_i, p_j) = \gcd(q_i, q_j) = 1$ *and for every natural* i, j, $\gcd(p_i, q_j) = 1$ *holds. If we denote for arbitrary given* $a_1, a_2 \in \mathbb{N}$, $b_1, b_2 \in \mathbb{N} \cup \{0\}$ *and for arbitrary given* $i_r, i_s \in \mathbb{N}$:

$$A_{i_r} = \left\{p_{i_r}^{ka_1+b_1} \cdot m : \ m \in \mathbb{N}; \ \ m \neq p_{i_r} \cdot l, \ \ l \in \mathbb{N}; \ \ k \in \mathbb{N} \cup \{0\}\right\}$$

and

$$B_{i_s} = \left\{q_{i_s}^{ka_2+b_2} \cdot m : \ m \in \mathbb{N}; \ \ m \neq q_{i_s} \cdot l, \ \ l \in \mathbb{N}; \ \ k \in \mathbb{N} \cup \{0\}\right\},$$

then for every $m, n \in \mathbb{N}$ *it holds that*

$$d\Big(\Big(\bigcap_{r=1}^{m} A_{i_r}\Big) \cap \Big(\bigcap_{s=1}^{n} B_{i_s}\Big)\Big) = \prod_{r=1}^{m} d(A_{i_r}) \prod_{s=1}^{n} d(B_{i_s}).$$

P r o o f. We prove this theorem by induction. First let $m = 1$. We have to prove that

$$d\Big((A_{i_1}) \cap \Big(\bigcap_{s=1}^{n} B_{i_s}\Big)\Big) = d(A_{i_r}) \prod_{s=1}^{n} d(B_{i_s}).$$

From the conditions:

$\gcd(p_i, p_j) = \gcd(q_i, q_j) = 1$ for every natural numbers $i \neq j$ and
for every natural i, j it holds that $\gcd(p_i, q_j) = 1$

it follows that the set $(A_{i_1}) \cap \Big(\bigcap_{s=1}^{n} B_{i_s}\Big)$ is formed from natural numbers in the form $p_{i_1}^{k_1 a_1+b_1} q_{j_1}^{l_1 a_2+b_2} \dots q_{j_n}^{l_n a_2+b_2} \cdot m$, where $k_1, l_1, \dots, l_n \in \mathbb{N}$ and $p_{i_1}, q_{j_1}, \dots, q_{j_n}$ does not divide the number m.

Let us denote $B^* = \bigcap_{s=1}^{n} B_{i_s}$. We can see that $B^* = \left\{q_{j_1}^{l_1 a_2+b_2} \dots q_{j_n}^{l_n a_2+b_2} \cdot m : m \in \mathbb{N}; \ \ l_i \in \mathbb{N} \cup \{0\}; \ \ m \neq q_{j_i} m_i, \ \ m_i \in \mathbb{N}; \ \ i = 1, \dots, n\right\}$.

Now, we define, for arbitrary $k \in \mathbb{N} \cup \{0\}$, the sets

$$C_{ka_1+b_1} = \left\{p_{i_1}^{ka_1+b_1} \cdot m : \ m \in B^*\right\}$$

and

$$D_{ka_1+b_1+1} = \left\{p_{i_1}^{ka_1+b_1+1} \cdot m : \ m \in B^*\right\}.$$

Asymptotic densities of these sets fulfil, for every $k \in \mathbb{N} \cup \{0\}$, the conditions

$$d(C_{ka_1+b_1}) = \frac{1}{p_{i_1}^{ka_1+b_1}} d(B^*) \qquad \text{and} \qquad d(D_{ka_1+b_1+1}) = \frac{1}{p_{i_1}^{ka_1+b_1+1}} d(B^*). \quad (9)$$

It is obvious that

$$C_{b_1} - D_{b_1+1} \subseteq (A_{i_j} \cap B^*) \subseteq C_{b_1},$$

$$(C_{b_1} - D_{b_1+1}) \cup (C_{a_1+b_1} - D_{a_1+b_1+1}) \subseteq (A_{i_j} \cap B^*) \subseteq (C_{b_1} - D_{b_1+1}) \cup C_{a_1+b-1},$$

$$(C_{b_1} - D_{b_1+1}) \cup (C_{a_1+b_1} - D_{a_1+b_1+1}) \cup (C_{2a_1+b_1} - D_{2a_1+b_1+1}) \subseteq (A_{i_j} \cap B^*),$$

$$(A_{i_j} \cup B^*) \subseteq (C_{b_1} - D_{b_1+1}) \cup (C_{a_1+b_1} - D_{a_1+b_1+1}) \cap C_{2a_1+b_1}$$

and so on. Therefore, using equation (9) we can write, for every $k \in \mathbb{N} \cup \{0\}$,

$$d(B^*)\left(\sum_{j=0}^{k}\frac{1}{p_{i_1}^{ja_1+b_1}} - \sum_{j=0}^{k}\frac{1}{p_{i_1}^{ja_1+b_1+1}}\right) \le \underline{d}(A_{i_j} \cap B^*)$$
$$\le \bar{d}(A_{i_j} \cap B^*) \le d(B^*)\left(\sum_{j=0}^{k}\frac{1}{p_{i_1}^{ja_1+b_1}} - \sum_{j=0}^{k-1}\frac{1}{p_{i_1}^{ja_1+b_1+1}}\right).$$

□

As an obvious corollary of previous theorem we obtain:

Corollary 2.2. *If we have systems of sets $\{A_i\}_{i=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$, where sets A_i and B_j are defined in the same way as in* Theorem 2.4, *then both of them are systems of sets with multiplicative asymptotic density. If we denote $\{A_i, B_j\}_{i,j=1}^{\infty}$ the system of sets which contains sets from $\{A_i\}_{i=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$, then $\{A_i, B_j\}_{i,j=1}^{\infty}$ is also a system of sets with multiplicative asymptotic density.*

Theorem 2.5. *Let $\{p_{i1}\}_{i=1}^{\infty}$, $\{p_{i2}\}_{i=1}^{\infty}$, ..., $\{p_{ir}\}_{i=1}^{\infty}$, $r \in \mathbb{N}$, be sequences of natural numbers, such that $p_{ik} > 2$ for $i \in \mathbb{N}$, $k \in \{1,\dots,r\}$, $\gcd(p_{ik}, p_{jl}) = 1$ for arbitrary $(i,j,k,l) \in (\mathbb{N}^2 \times \{1,\dots,r\}^2) - \big\{(a,a,a,a) : a \in \mathbb{N}\big\}$. If we define for arbitrary $i \in \mathbb{N}$, $k \in \{1,\dots,r\}$ and for given $a_k \in \mathbb{N}$ and $b_k \in \mathbb{N}\cup\{0\}$:*

$$A_{i,k} = \big\{p_{ik}^{sa_k+b_k}\cdot m : \ m \in \mathbb{N}; \ \ m \ne p_{ik}m_k, \ \ m_k \in \mathbb{N}; \ \ s \in \mathbb{N}\cup\{0\}\big\},$$

then

1) *For every $i \in \mathbb{N}$, $k \in \{1,\dots,r\}$ it holds that*
$$d(A_{i,k}) = \frac{p_{ik}^a}{p_{ik}^b}\Big(\frac{1}{p_{ik}^a - 1}\Big)\Big(1 - \frac{1}{p_{ik}}\Big).$$
2) *The system of sets $\Delta = \big\{A_{i,k} : \ i \in \mathbb{N}, \ k \in \{1,\dots,r\}\big\}$ is a system of sets with multiplicative asymptotic density, i.e. for every $r \in \mathbb{N}$, $m, k_1, \dots, k_m \in \{1,\dots,r\}$ and $\{n_1,\dots,n_m\} \subset \mathbb{N}$ there holds the equation*

$$d\Big(\bigcap_{t=1}^{m}\bigcap_{u=1}^{n_t} A_{i_u,k_t}\Big) = \prod_{t=1}^{m}\prod_{u=1}^{n_t} d(A_{i_u,k_t}) = \prod_{t=1}^{m}\prod_{u=1}^{n_t}\frac{p_{i_u,k_t}^a}{p_{i_u,k_t}^b}\Big(\frac{1}{p_{i_u,k_t}^a - 1}\Big)\Big(1 - \frac{1}{p_{i_u,k_t}}\Big).$$

Proof.

ad 1) This statement follows from Theorem 2.3.

ad 2) We can prove this statement by induction. We have proved Theorem 2.4, which is the case of $m = 2$. Now, we suppose that equations

$$d\Big(\bigcap_{t=1}^{m-1}\bigcap_{u=1}^{n_t} A_{i_u,k_t}\Big) = \prod_{t=1}^{m-1}\prod_{u=1}^{n_t} d(A_{i_u,k_t})$$
$$= \prod_{t=1}^{m-1}\prod_{u=1}^{n_t}\frac{p_{i_u,k_t}^a}{p_{i_u,k_t}^b}\Big(\frac{1}{p_{i_u,k_t}^a - 1}\Big)\Big(1 - \frac{1}{p_{i_u,k_t}}\Big).$$

are true. Let us denote $B^* = \bigcap_{t=1}^{m-1} \bigcap_{u=1}^{n_t} A_{i_u,k_t}$. We have to prove, that

$$d\Big(\Big(\bigcap_{u=1}^{n_m} A_{i_u,k_m}\Big) \cap B^*\Big) = d\Big(\bigcap_{u=1}^{n_m} A_{i_u,k_m}\Big) d(B^*).$$

If we realize that the set $\bigcap_{u=1}^{n_m} A_{i_u,k_m} \cap B^*$ consists of elements in the form:

$$p_{i_1,k_m}^{s_1 a_{k_m}+b_{k_m}} p_{i_2,k_m}^{s_2 a_{k_m}+b_{k_m}} \cdot \dots \cdot p_{i_{n_m},k_m}^{s_{n_m} a_{k_m}+b_{k_m}} \cdot m^*$$

where $m^* \in B^*$, p_{i_u,k_m} does not divide m^* and $s_u \in \mathbb{N} \cup \{0\}$ for $u = 1, \dots, n_m$, then we can finish this proof using the same techniques as in the proof of Theorem 2.3. □

REFERENCES

[1] HARDY, G. H. — WRIGHT, E. M.: *An Introduction to the Theory of Numbers*, Oxford, Clarendon Press, 1954.

[2] HOOLEY, C.: *On the intervals between numbers that are sums of two squares*, J. Reine Angew. Math. **267** (1974), 207–218.

[3] JAHODA, P.: *Notes on the expression of natural numbers as sum of powers*, Tatra Mt. Math. Publ. **34** (2005), 1–11.

[4] LANDAU, E.: *Über die Einteilung der … Zahlen in 4 Klassen …*, Arch. Math. Phys. **13** (1908), 305–312.

[5] RIEGER, G. J.: *Zum Satz von Landau über die Summe aus zwei Quadraten*, J. Reine Angew. Math. **244** (1970), 198–200.

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