

TRINOMIAL RANDOM WALK WITH ONE OR TWO IMPERFECT ABSORBING BARRIERS

M. A. EL-SHEHAWEY

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ABSTRACT. Trinomial random walk, with one or two barriers, on the non-negative integers is discussed. At the barriers, the particle is either annihilated or reflects back to the system with respective probabilities $1 - \rho$, ρ at the origin and $1 - \omega$, ω at L , $0 \leq \rho, \omega \leq 1$. Theoretical formulae are given for the probability distribution, its generating function as well as the mean of the time taken before absorption. In the one-boundary case, very qualitatively different asymptotic forms for the result, depending on the relationship between transition probabilities and the annihilation probability, are obtained.

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1. Introduction

Relatively random walk problems with absorbing or reflecting barriers, often serve as reduced examples of much more complicated many-body phenomena. They play an essential role in various fields such as physics, chemistry, biology, meteorology, geography, mathematics, computer science and others. For comprehensive treatments of random walk and their applications, cf. Kac (1954), Weesakul (1961), Parzen (1962), Cox and Miller (1965), Feller (1968), Srinivasan and Mehata (1976), Isoifescu (1980), Berg (1983), Percus (1985) and Murthy and Kehr (1989). Despite its long history, novel aspects continue to surface, cf., Weiss (1994), El-Shehawey (1994), Durrett (1995), Hughes (1996), Norris (1997), Canjar (2000), El-Shehawey (2000), (2002) and Ethier and Khoshnevisan (2002).

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In this paper, a trinomial random walk (R.W.), on the set of integers $S = \{0, 1, \dots, L\}$, governed by unsymmetric imperfect absorbing barriers at 0 and L is investigated. A particle, starting from the initial position x_0 , $x_0 \in S$, moves one unit to the right or to the left with probabilities α and β , respectively. The probability that it does not move will be taken as $\gamma = 1 - (\alpha + \beta)$. When the barriers 0 or L are reached, it is annihilated or reflectes (to the points 1 and $L-1$, respectively) with respective probabilities $1 - \rho$ and ρ at 0 and $1 - \omega$ and ω at L , $0 \leq \rho, \omega \leq 1$. This corresponds to the situation when, at the barriers 0 or L , the particle is either lost from the system with respective probabilities $1 - \rho$ and $1 - \omega$ or turned back with probabilities ρ and ω , and reduces to the classical problems of R.W. with absorbing, reflecting and combination of absorbing and reflecting barriers. Such model is a generalization of many particular R.W. problems, depending on appropriate choices of the reflect boundary probabilities ρ and ω .

The definition of reflecting barrier in Feller [10, p. 343] is modified to be at the points 0 and L , and the particle is allowed to start from (or reach to) the barriers. Instead, whenever the particle is at point 1, it has probability p of moving to position 2 and probability q to stay at 1.

Generating functions for the probability of being in a certain position after a given number of transitions have been extensively studied (cf. Feller (1968), Srinivasan and Mehata (1976), Percus (1985), and El-Shehawey (2000)). However, determination of explicit expressions for the probability distribution from the generating functions is generally quit difficult. The purpose of this paper is to find closed form expressions for the $L+1$ probabilities, $p_n(x | x_0)$ for all $x, x_0 \in S$, that the particle starting from $x_0 \in S$ is at $x \in S$ at step n . These are easily derived from their generating functions. This work is a generalization to previous works, it is apparently not covered by the literature. In Section 2, the difference equation, with its associated initial and boundary conditions, and the corresponding system of generating functions are presented. Exact solution of this system is collected in a simple general form in Theorem 2.1, with proof given in the Appendix. In Section 3, some theoretical different formulae of generating functions for R.W. models as special cases, for completeness, are introduced explicitly through appropriate choices of the reflection probabilities ρ and ω . Explicit expressions for the simple two cases $p_n(x | x_0)$, $x = 0, L$ are discussed in Section 4. The mean occupation total and the absorption probabilities are obtained in Section 5. In Section 6 the one boundary case is investigated; explicit formula for the n -step probability $v_n(x | x_0)$, as well as, the survival probability and the mean number of steps taken until annihilated are also presented.

2. Solution for the difference equations

Let $p_n(x | x_0)$ be the n -step transition probability that the particle reaches the position x at time n given that x_0 was its initial position, $x, x_0 \in S$. Then $p_n(x | x_0)$ satisfies the following difference equation:

For $n \geq 1$, $x \in \{2, 3, \dots, L-2\}$

$$p_n(x | x_0) = \alpha p_{n-1}(x-1 | x_0) + \gamma p_{n-1}(x | x_0) + \beta p_{n-1}(x+1 | x_0). \quad (2.1)$$

Subject to the following “initial and boundary” conditions

$$p_0(x | x_0) = \delta_{x, x_0}; \quad x, x_0 \in S, \quad (2.2)$$

$$\begin{aligned} p_n(1 | x_0) &= \rho p_{n-1}(0 | x_0) + \gamma p_{n-1}(1 | x_0) + \beta p_{n-1}(2 | x_0), \\ p_n(0 | x_0) &= \beta p_{n-1}(1 | x_0), \\ p_n(L-1 | x_0) &= \alpha p_{n-1}(L-2 | x_0) + \gamma p_{n-1}(L-1 | x_0) + \omega p_{n-1}(L | x_0), \\ p_n(L | x_0) &= \alpha p_{n-1}(L-1 | x_0), \end{aligned} \quad (2.3)$$

where δ_{x, x_0} denotes the Kronecker delta.

To solve (2.1)–(2.3), we first calculate the probability generating function (pgf)

$$G(z; x | x_0) = \sum_{n=0}^{\infty} z^n p_n(x | x_0), \quad |z| < 1, \quad (2.4)$$

and then the probability

$$p_n(x | x_0) = \frac{1}{2\pi i} \oint \left(\sum_{k=0}^{\infty} z^k p_k(x | x_0) \right) \frac{dz}{z^{n+1}}, \quad (2.4a)$$

where the integration path is the circle around $z = 0$ in the complex plane.

Multiplying the equations (2.1) and (2.3) by z^n and summing over $n = 1, 2, \dots$, we obtain the recursion

$$\begin{aligned} G(z; x | x_0) &= \delta_{x, x_0}^{\circ}(z) + \lambda(z)G(z; x-1 | x_0) + \mu(z)G(z; x+1 | x_0), \\ x &\in \{2, 3, \dots, L-2\}, \end{aligned} \quad (2.5)$$

subject to the boundary conditions

$$\begin{aligned} G(z; 0 | x_0) &= \delta_{0, x_0}(z) + \mu_0(z)G(z; 1 | x_0), \\ G(z; 1 | x_0) &= \delta_{1, x_0}^{\circ}(z) + \rho^{\circ}(z)G(z; 0 | x_0) + \mu(z)G(z; 2 | x_0), \\ G(z; L-1 | x_0) &= \delta_{L-1, x_0}^{\circ}(z) + \lambda(z)G(z; L-2 | x_0) + \omega^{\circ}(z)G(z; L | x_0), \\ G(z; L | x_0) &= \delta_{L, x_0}(z) + \lambda_0(z)G(z; L-1 | x_0) \end{aligned} \quad (2.6)$$

with the denotations

$$\begin{aligned}\mu(z) &= \frac{\beta z}{1 - \gamma z}, & \lambda(z) &= \frac{\alpha z}{1 - \gamma z}, & \mu_0(z) &= \beta z, & \lambda_0(z) &= \alpha z, \\ \rho^\circ(z) &= \frac{\rho z}{1 - \gamma z}, & \omega^\circ(z) &= \frac{\omega z}{1 - \gamma z} & \text{and} & \delta_{x,x_0}^\circ(z) &= \frac{\delta_{x,x_0}}{1 - \gamma z}.\end{aligned}\quad (2.7)$$

Solving (2.5) and (2.6) systematically we get the next general theorem (for details of the calculations see Appendix):

THEOREM 2.1. *In the presence of the unsymmetric imperfect absorbing boundaries at 0 and L, the pgf of the n-step occupation probabilities $p_n(x | x_0)$ for all deferent values of $x, x_0 \in S$ are given by*

$$\begin{aligned}G(z; x | 0) &= \frac{1}{\Lambda} \begin{cases} \Lambda + z\rho\sqrt{\frac{\beta}{\alpha}}\Psi_{L-1}(z; \beta, \alpha, \omega), & x = 0 \\ \frac{\rho}{\alpha} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}x} \Psi_{L-x}(z; \beta, \alpha, \omega), & x \in S, \ x \neq 0, L \\ z\rho(\theta_2 - \theta_1) \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-1)}, & x = L, \end{cases} \quad (2.8) \\ G(z; x | x_0) &= \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{(x_0-x)}{2}}}{\Lambda} \begin{cases} \Psi_{L-x_0}(z; \beta, \alpha, \omega), & x = 0, \ x_0 \in S, \ x_0 \neq 0, L \\ \frac{1}{z\sqrt{\alpha\beta}(\theta_2-\theta_1)} \Psi_x(z; \alpha, \beta, \rho) \Psi_{L-x_0}(z; \beta, \alpha, \omega), & x \in \{1, 2, \dots, x_0\}, \ x_0 \in S, \ x_0 \neq 0, L \\ \frac{1}{z\sqrt{\alpha\beta}(\theta_2-\theta_1)} \Psi_{x_0}(z; \alpha, \beta, \rho) \Psi_{L-x}(z; \beta, \alpha, \omega), & x \in \{x_0, x_0 + 1, \dots, L-1\}, \ x_0 \in S, \ x_0 \neq 0, L \\ \Psi_{x_0}(z; \alpha, \beta, \rho), & x = L, \ x_0 \in S, \ x_0 \neq 0, L. \end{cases} \quad (2.9)\end{aligned}$$

$G(z; x | L)$ can easily be obtained from formula (2.8) on replacing $\alpha, \beta, \rho, \omega$ and x by $\beta, \alpha, \omega, \rho$ and $L - x$, respectively, where

$$\Psi_k(z; \alpha, \beta, j) = jz\sqrt{\frac{\beta}{\alpha}}(\theta_1^{k-1} - \theta_2^{k-1}) - (\theta_1^k - \theta_2^k), \quad j = \rho, \omega, \quad (2.10)$$

$$\Lambda = \Psi_L(z; \alpha, \beta, \rho) - z\omega\sqrt{\frac{\alpha}{\beta}}\Psi_{L-1}(z; \alpha, \beta, \rho), \quad (2.11)$$

$$\begin{aligned}\theta_1 \equiv \theta_1(z) &= \frac{1}{2z\sqrt{\alpha\beta}} \left[(1 - z\gamma) + \sqrt{(1 - z\gamma)^2 - 4z^2\alpha\beta} \right], \\ \theta_2 \equiv \theta_2(z) &= \frac{1}{2z\sqrt{\alpha\beta}} \left[(1 - z\gamma) - \sqrt{(1 - z\gamma)^2 - 4z^2\alpha\beta} \right],\end{aligned}\quad (2.12)$$

with the assumption that z is real and positive and that the function inside the square root (2.12) is positive, i.e.

$$0 < z < \frac{1}{1 - (\sqrt{\alpha} - \sqrt{\beta})^2}.$$

Formulae (2.9)–(2.12) include generalizations of previous works (cf. Cox and Miller (1965), Feller (1968), Srinivasan and Mehata (1976), Percus (1985), and El-Shehawey (2000) among others.

Note that, most of earlier work on the subject has mainly been confined to cases where the particle does not allow to start from or reach the end points. Here no such assumption exists (cf. formula (2.8)).

3. R.W. for some special cases

Many interesting pgf as special cases can be derived from Theorem 2.1 through an appropriate choice of the reflection probabilities ρ and ω . In the following several special cases are listed:

Case (i). $\rho = \omega$. When $\rho = \omega$, the pgf $G(z; x | x_0)$, for all $x, x_0 \in S$, becomes

$$\sum_{n=0}^{\infty} z^n p_n(x | x_0) = \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{x_0-x}{2}}}{z\sqrt{\alpha\beta}(\theta_2 - \theta_1)\Delta} \times$$

$$\times \begin{cases} \left[\theta_1^x - \theta_2^x - z\rho\sqrt{\frac{\beta}{\alpha}}(\theta_1^{x-1} - \theta_2^{x-1}) \right] \left[\theta_1^{L-x_0} - \theta_2^{L-x_0} - z\rho\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1}) \right], \\ x_0 \neq 0, \quad x \in \{1, 2, \dots, x_0\}, \\ \left[\theta_1^{x_0} - \theta_2^{x_0} - z\rho\sqrt{\frac{\beta}{\alpha}}(\theta_1^{x_0-1} - \theta_2^{x_0-1}) \right] \left[\theta_1^{L-x} - \theta_2^{L-x} - z\rho\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-x-1} - \theta_2^{L-x-1}) \right] \left(\frac{\alpha}{\beta}\right)^{x-x_0}, \\ x_0 \neq L, \quad x \in \{x_0, x_0 + 1, \dots, L-1\}, \end{cases} \quad (3.1)$$

$$\sum_{n=0}^{\infty} z^n p_n(0 | x_0) =$$

$$= \begin{cases} \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0}}{\Delta} \left[z\rho\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1}) - (\theta_1^{L-x_0} - \theta_2^{L-x_0}) \right], & x_0 \neq 0, L \\ 1 + \frac{z\rho\sqrt{\frac{\beta}{\alpha}}}{\Delta} \left[z\rho\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-2} - \theta_2^{L-2}) - (\theta_1^{L-1} - \theta_2^{L-1}) \right], & x_0 = 0 \\ \frac{z\rho(\theta_2 - \theta_1)}{\Delta} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(L-1)}, & x_0 = L \end{cases} \quad (3.2)$$

$$\sum_{n=0}^{\infty} z^n p_n(L | x_0) =$$

$$= \begin{cases} \frac{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-x_0)}}{\Delta} \left[z\rho\sqrt{\frac{\beta}{\alpha}}(\theta_1^{x_0-1} - \theta_2^{x_0-1}) - (\theta_1^{x_0} - \theta_2^{x_0}) \right], & x_0 \neq 0, L \\ \frac{z\rho(\theta_2 - \theta_1)}{\Delta} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-1)}, & x_0 = 0 \\ 1 + \frac{z\rho\sqrt{\frac{\beta}{\alpha}}}{\Delta} \left[z\rho\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-2} - \theta_2^{L-2}) - (\theta_1^{L-1} - \theta_2^{L-1}) \right], & x_0 = L \end{cases} \quad (3.3)$$

where θ_1 and θ_2 are given in formula (2.12), and

$$\Delta = - \left[\theta_1^L - \theta_2^L - \frac{z\rho}{\sqrt{\alpha\beta}}(\alpha + \beta) (\theta_1^{L-1} - \theta_2^{L-1}) + \rho^2 z^2 (\theta_1^{L-2} - \theta_2^{L-2}) \right]. \quad (3.4)$$

This pgf arises from the R.W. running between symmetry imperfect absorbing barriers at 0 and L and starting at x_0 , $x_0 \in S$.

Case (ii). $\rho = \omega = 0$. When $\rho = \omega = 0$ the pgf becomes

$$\sum_{n=0}^{\infty} z^n p_n(0 | x_0) = \frac{\theta_1^{L-x_0} - \theta_2^{L-x_0}}{\theta_1^L - \theta_2^L} \left(\sqrt{\frac{\beta}{\alpha}} \right)^{x_0}, \quad x_0 \neq 0, L, \quad (3.5)$$

$$\begin{aligned} \sum_{n=0}^{\infty} z^n p_n(x | x_0) = \\ = \begin{cases} \frac{(\theta_1^x - \theta_2^x)(\theta_1^{L-x_0} - \theta_2^{L-x_0})}{z\sqrt{\alpha\beta}(\theta_1 - \theta_2)(\theta_1^L - \theta_2^L)} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}(x_0-x)}, & x_0 \neq 0, \quad x \in \{1, 2, \dots, x_0\} \\ \frac{(\theta_1^{x_0} - \theta_2^{x_0})(\theta_1^{L-x} - \theta_2^{L-x})}{z\sqrt{\alpha\beta}(\theta_1 - \theta_2)(\theta_1^L - \theta_2^L)} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}(x_0-x)}, & x_0 \neq L, \quad x \in \{x_0, x_0+1, x_0+2, \dots, L-1\}, \end{cases} \end{aligned} \quad (3.6)$$

$$\sum_{n=0}^{\infty} z^n p_n(L | x_0) = \frac{(\theta_1^{x_0} - \theta_2^{x_0}) \left(\sqrt{\frac{\alpha}{\beta}} \right)^{L-x_0}}{\theta_1^L - \theta_2^L}, \quad x_0 \neq 0, L, \quad (3.7)$$

where θ_1 and θ_2 are given in formula (2.12). This pgf arises from the R.W. between perfect absorbing barriers at 0 and L and starting at x_0 , $x_0 \in S$.

Formula (3.6) represents the pgf of the n -step occupation probabilities $p_n(x | x_0)$, that the particle, starting from $x_0 \in S$, $x_0 \neq 0, L$, arrives at location $x \in S$, $x_0 \neq 0, L$ after n steps, while formulae (3.5) and (3.7) represent a well known results for the pgf of the absorption probabilities at 0 and L at the n th step (cf. Feller (1968) and El-Shehawey (2002)).

Case (iii). $\rho = \omega = 1$. When $\rho = \omega = 1$, the pgf $G(z; x | x_0)$, for all $x, x_0 \in S$, becomes

$$\begin{aligned} \sum_{n=0}^{\infty} z^n p_n(x | x_0) = \frac{\left(\frac{\beta}{\alpha} \right)^{\frac{x_0-x}{2}}}{z\sqrt{\alpha\beta}(\theta_2 - \theta_1)\Delta} \times \\ \times \begin{cases} \left[\theta_1^x - \theta_2^x - z\sqrt{\frac{\beta}{\alpha}}(\theta_1^{x-1} - \theta_2^{x-1}) \right] \left[\theta_1^{L-x_0} - \theta_2^{L-x_0} - z\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1}) \right], \\ x_0 \neq 0, \quad x \in \{1, 2, \dots, x_0\}, \\ \left[\theta_1^{x_0} - \theta_2^{x_0} - z\sqrt{\frac{\beta}{\alpha}}(\theta_1^{x_0-1} - \theta_2^{x_0-1}) \right] \left[\theta_1^{L-x} - \theta_2^{L-x} - z\sqrt{\frac{\alpha}{\beta}}(\theta_1^{L-x-1} - \theta_2^{L-x-1}) \right] \left(\frac{\alpha}{\beta} \right)^{x-x_0}, \\ x_0 \neq L, \quad x \in \{x_0, x_0+1, x_0+2, \dots, L-1\}, \end{cases} \end{aligned} \quad (3.8)$$

$$\begin{aligned}
 \sum_{n=0}^{\infty} z^n p_n(0 \mid x_0) &= \\
 &= \begin{cases} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}x_0} \left[z \sqrt{\frac{\alpha}{\beta}} \left(\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1} \right) - \left(\theta_1^{L-x_0} - \theta_2^{L-x_0} \right) \right], & x_0 \neq 0, L \\ 1 + \frac{z \sqrt{\frac{\beta}{\alpha}}}{\Delta_1} \left[z \sqrt{\frac{\alpha}{\beta}} \left(\theta_1^{L-2} - \theta_2^{L-2} \right) - \left(\theta_1^{L-1} - \theta_2^{L-1} \right) \right], & x_0 = 0 \\ \frac{z(\theta_2 - \theta_1)}{\Delta_1} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}(L-1)}, & x_0 = L, \end{cases} \quad (3.9)
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=0}^{\infty} z^n p_n(L \mid x_0) &= \\
 &= \begin{cases} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}(L-x_0)} \left[z \sqrt{\frac{\beta}{\alpha}} \left(\theta_1^{x_0-1} - \theta_2^{x_0-1} \right) - \left(\theta_1^{x_0} - \theta_2^{x_0} \right) \right], & x_0 \neq 0, L \\ \frac{z(\theta_2 - \theta_1)}{\Delta_1} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}(L-1)}, & x_0 = 0, \\ 1 + \frac{z \sqrt{\frac{\alpha}{\beta}}}{\Delta_1} \left[z \sqrt{\frac{\beta}{\alpha}} \left(\theta_1^{L-2} - \theta_2^{L-2} \right) - \left(\theta_1^{L-1} - \theta_2^{L-1} \right) \right], & x_0 = L \end{cases} \quad (3.10)
 \end{aligned}$$

where

$$\Delta_1 = - \left[\theta_1^L - \theta_2^L - z \left(\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} \right) \left(\theta_1^{L-1} - \theta_2^{L-1} \right) + z^2 \left(\theta_1^{L-2} - \theta_2^{L-2} \right) \right], \quad (3.11)$$

and θ_1, θ_2 are given as in formula (2.12). This pgf arises from the R.W. between two fully reflecting barriers at 0 and L , starting at x_0 , $x_0 \in S$.

Case (iv). $\rho = 0$ and $0 < \omega \leq 1$. When $\rho = 0$ and $0 < \omega \leq 1$ the pgf $G(z; x \mid x_0)$, for all $x, x_0 \in S$, becomes

$$\begin{aligned}
 \sum_{n=0}^{\infty} z^n p_n(x \mid x_0) &= \frac{\left(\frac{\beta}{\alpha} \right)^{\frac{x_0-x}{2}}}{z \sqrt{\alpha\beta} (\theta_2 - \theta_1) \Delta_2} \times \\
 &\times \begin{cases} [\theta_1^x - \theta_2^x] \left[\theta_1^{L-x_0} - \theta_2^{L-x_0} - z\omega \sqrt{\frac{\alpha}{\beta}} \left(\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1} \right) \right], & x_0 \neq 0, \quad x \in \{1, 2, \dots, x_0\}, \\ [\theta_1^{x_0} - \theta_2^{x_0}] \left[\theta_1^{L-x} - \theta_2^{L-x} - z\omega \sqrt{\frac{\alpha}{\beta}} \left(\theta_1^{L-x-1} - \theta_2^{L-x-1} \right) \right] \left(\frac{\alpha}{\beta} \right)^{x-x_0}, & x_0 \neq L, \quad x \in \{x_0, x_0+1, x_0+2, \dots, L-1\}, \end{cases} \quad (3.12)
 \end{aligned}$$

$$\sum_{n=0}^{\infty} z^n p_n(0 | x_0) = \begin{cases} \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0}}{\Delta_2} \left[z\omega \sqrt{\frac{\alpha}{\beta}} \left(\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1} \right) - \left(\theta_1^{L-x_0} - \theta_2^{L-x_0} \right) \right], & x_0 \neq 0, L \\ 1, & x_0 = 0 \\ \frac{z\omega(\theta_2 - \theta_1)}{\Delta_2} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}(L-1)}, & x_0 = L, \end{cases} \quad (3.13)$$

$$\sum_{n=0}^{\infty} z^n p_n(L | x_0) = \begin{cases} \frac{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-x_0)}}{\Delta_2} (\theta_2^{x_0} - \theta_1^{x_0}), & x_0 \neq 0, L \\ 0, & x_0 = 0 \\ 1 + \frac{z\omega \sqrt{\frac{\alpha}{\beta}}}{\Delta_2} (\theta_2^{L-1} - \theta_1^{L-1}), & x_0 = L, \end{cases} \quad (3.14)$$

where

$$\Delta_2 = z\omega \sqrt{\frac{\alpha}{\beta}} (\theta_1^{L-1} - \theta_2^{L-1}) - (\theta_1^L - \theta_2^L) \quad (3.15)$$

and θ_1, θ_2 are given as in formulae (2.12). This pgf arises from the R.W. in the presence of perfect absorbing barrier at 0 and imperfect one at L .

Case (v). $\omega = 0$ and $0 < \rho \leq 1$. When $\omega = 0$ and $0 < \rho \leq 1$, the pgf $G(z; x | x_0)$, for all $x, x_0 \in S$, becomes

$$\sum_{n=0}^{\infty} z^n p_n(x | x_0) = \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{x_0-x}{2}}}{z\sqrt{\alpha\beta}(\theta_2 - \theta_1)\Delta_3} \times \begin{cases} \left[\theta_1^{L-x_0} - \theta_2^{L-x_0} \right] \left[\theta_1^x - \theta_2^x - z\rho \sqrt{\frac{\beta}{\alpha}} (\theta_1^{x-1} - \theta_2^{x-1}) \right], & x_0 \neq 0, x \in \{1, 2, \dots, x_0\} \\ \left[\theta_1^{L-x} - \theta_2^{L-x} \right] \left[\theta_1^{x_0} - \theta_2^{x_0} - z\rho \sqrt{\frac{\beta}{\alpha}} (\theta_1^{x_0-1} - \theta_2^{x_0-1}) \right] \left(\frac{\alpha}{\beta} \right)^{x-x_0}, & x_0 \neq L, x \in \{x_0, x_0+1, x_0+2, \dots, L-1\}, \\ 0, & x_0 = L, \end{cases} \quad (3.16)$$

$$\sum_{n=0}^{\infty} z^n p_n(0 | x_0) = \begin{cases} \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0}}{\Delta_3} (\theta_2^{L-x_0} - \theta_1^{L-x_0}), & x_0 \neq 0, L \\ 1 + \frac{z\rho \sqrt{\frac{\beta}{\alpha}}}{\Delta_3} (\theta_2^{L-1} - \theta_1^{L-1}), & x_0 = 0 \\ 0, & x_0 = L, \end{cases} \quad (3.17)$$

$$\sum_{n=0}^{\infty} z^n p_n(L | x_0) = \begin{cases} \frac{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-x_0)}}{\Delta_3} \left[z\rho \sqrt{\frac{\beta}{\alpha}} (\theta_1^{x_0-1} - \theta_2^{x_0-1}) - (\theta_1^{x_0} - \theta_2^{x_0}) \right], & x_0 \neq 0, L \\ \frac{z\rho(\theta_2 - \theta_1)}{\Delta_3} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}(L-1)}, & x_0 = 0 \\ 1, & x_0 = L, \end{cases} \quad (3.18)$$

where

$$\Delta_3 = z\rho\sqrt{\frac{\beta}{\alpha}}(\theta_1^{L-1} - \theta_2^{L-1}) - (\theta_1^L - \theta_2^L), \quad (3.19)$$

and θ_1, θ_2 are given as in formulae (2.12). This pgf arises from the R.W. in the presence of imperfectly absorbing barrier at 0 and perfect absorbing one at L .

4. Closed form expressions for $p_n(x \mid x_0)$, $x = 0, L$

Theorem 2.1 enables us to find explicit expressions for the n -step probabilities $p_n(x \mid x_0)$ for all $x, x_0 \in S$. Only closed-form expressions for $p_n(0 \mid x_0)$ and $p_n(L \mid x_0)$ will be formally established here.

From Theorem 2.1 for $x = 0$ and $x_0 \in S$, $x_0 \neq 0, L$

$$\begin{aligned} \sum_{n=0}^{\infty} z^n p_n(0 \mid x_0) &= \\ &= \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0} \left[z\omega\sqrt{\frac{\alpha}{\beta}}(\theta_2^{x_0+1} - \theta_2^{2L-x_0-1}) - (\theta_2^{x_0} - \theta_2^{2L-x_0}) \right]}{(1 - \rho\omega z^2)(\theta_2^2 - \theta_2^{2L-2}) - \left[1 - \frac{(\beta\rho + \alpha\omega)z^2}{1-z\gamma} \right] (1 + \theta_2^2)(1 - \theta_2^{2L-2})}. \end{aligned} \quad (4.1)$$

Hence from (2.4a)

$$\begin{aligned} p_n(0 \mid x_0) &= \\ &= \frac{1}{2\pi i} \oint_{|z|<1} \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0} \left[z\omega\sqrt{\frac{\alpha}{\beta}}(\theta_2^{x_0+1} - \theta_2^{2L-x_0-1}) - (\theta_2^{x_0} - \theta_2^{2L-x_0}) \right]}{(1 - \rho\omega z^2)(\theta_2^2 - \theta_2^{2L-2}) - \left[1 - \frac{(\beta\rho + \alpha\omega)z^2}{1-z\gamma} \right] (1 + \theta_2^2)(1 - \theta_2^{2L-2})} \frac{dz}{z^{n+1}} \\ &= \frac{\sqrt{\alpha\beta} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0}}{2\pi i} \oint_{|\theta|<1} \frac{\left[\theta^{x_0-n-1} \left(\frac{\omega\sqrt{\frac{\alpha}{\beta}}(\theta^2 - \theta^{2L-2x_0})}{\gamma\theta + \sqrt{\alpha\beta}(\theta^2 + 1)} - (1 - \theta^{2L-2x_0}) \right) \right]}{\left\{ \frac{\theta^{2L} - 1 + \frac{\theta^2}{[\gamma\theta + \sqrt{\alpha\beta}(\theta^2 + 1)]^2}}{(1 - \theta^{2L-2})(\gamma\theta + \sqrt{\alpha\beta}(\theta^2 + 1))} \right\}} d\theta \\ &\quad \left\{ \left[\left(\rho\sqrt{\frac{\beta}{\alpha}} + \omega\sqrt{\frac{\alpha}{\beta}} \right) - \rho\omega(\theta^2 - \theta^{2L-2}) \right] \right\} \end{aligned} \quad (4.2)$$

(using the substitution $\frac{1}{z} = \gamma + \sqrt{\alpha\beta} \left(\theta + \frac{1}{\theta} \right)$), or

$$\begin{aligned}
 p_n(0 | x_0) &= \\
 &= \text{coef. of } \psi^{\frac{1}{2}(n-x_0)} \text{ in } \sqrt{\alpha\beta} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}x_0} \times \\
 &\quad \times \frac{\left\{ \left[\gamma\sqrt{\psi} + \sqrt{\alpha\beta}(1+\psi) \right]^n (1-\psi) \right.}{\left. \left[\omega\sqrt{\frac{\alpha}{\beta}}(\psi + \psi^{L-x_0}) - (\gamma\sqrt{\psi} + \sqrt{\alpha\beta}(1+\psi)) (1-\psi^{L-x_0}) \right] \right\}} \\
 &\quad \times \frac{\left\{ \left[\gamma\sqrt{\psi} + \sqrt{\alpha\beta}(1+\psi) \right]^2 (\psi^L - 1) + \right.}{\left. \psi \left[\left(\rho\sqrt{\frac{\beta}{\alpha}} + \omega\sqrt{\frac{\alpha}{\beta}} \right) (\gamma\sqrt{\psi} + \sqrt{\alpha\beta}(1+\psi)) (1-\psi^{L-1}) - \rho\omega(\psi - \psi^{L-1}) \right] \right\}} \\
 &= \text{coef. of } \psi^{\frac{1}{2}(n-x_0)} \text{ in } (\alpha\beta)^{\frac{n}{2}} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}x_0} \times \\
 &\quad \times \left\{ \frac{\left[1 + \sqrt{\psi} \left(\sqrt{\psi} + \frac{\gamma}{\sqrt{\alpha\beta}} \right) \right]^n (1-\psi)}{1 - \sqrt{\psi} \left(\sqrt{\psi} \left(\frac{\rho-\alpha}{\alpha} \right) - \frac{\gamma}{\sqrt{\alpha\beta}} \right)} \left(1 + \frac{\left[\frac{\omega-\beta}{\beta} - \sqrt{\psi} \left(\sqrt{\psi} + \frac{\gamma}{\sqrt{\alpha\beta}} \right) \right] \psi^{L-x_0}}{\left[1 - \sqrt{\psi} \left(\sqrt{\psi} \left(\frac{\omega-\beta}{\beta} \right) - \frac{\gamma}{\sqrt{\alpha\beta}} \right) \right]} \right) \right\}. \tag{4.3}
 \end{aligned}$$

Hence

$$\begin{aligned}
 p_n(0 | x_0) &= (\alpha\beta)^{\frac{n}{2}} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}x_0} \times \\
 &\quad \times \left\{ \sum_{i=0}^{\infty} \binom{n}{\frac{1}{2}(n-x_0)-i} \left(\frac{\rho-\alpha}{\alpha} \right)^i - \sum_{i=0}^{\infty} \binom{n}{\frac{1}{2}(n-x_0)-i-1} \left(\frac{\rho-\alpha}{\alpha} \right)^i \right. \\
 &\quad \times \left\{ + \frac{\omega-\beta}{\beta} \sum_{i=0}^{\infty} \sum_{k=0}^i \left\{ \binom{n}{\frac{1}{2}(n+x_0)-i-L} - \binom{n}{\frac{1}{2}(n+x_0)-i-L-1} \right\} \left(\frac{\rho-\alpha}{\alpha} \right)^{i-k} \left(\frac{\omega-\beta}{\beta} \right)^k \\
 &\quad - \frac{\gamma}{\sqrt{\alpha\beta}} \sum_{i=0}^{\infty} \sum_{k=0}^i \left\{ \binom{n}{\frac{1}{2}(n+x_0-1)-i-L} - \binom{n}{\frac{1}{2}(n+x_0-1)-i-L-1} \right\} \left(\frac{\rho-\alpha}{\alpha} \right)^{i-k} \left(\frac{\omega-\beta}{\beta} \right)^k \\
 &\quad \left. - \sum_{i=0}^{\infty} \sum_{k=0}^i \left\{ \binom{n}{\frac{1}{2}(n+x_0)-i-L-1} - \binom{n}{\frac{1}{2}(n+x_0)-i-L-2} \right\} \left(\frac{\rho-\alpha}{\alpha} \right)^{i-k} \left(\frac{\omega-\beta}{\beta} \right)^k \right\} \\
 &= (\alpha\beta)^{\frac{n}{2}} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}x_0} \times \\
 &\quad \times \left\{ \sum_{i=0}^{\infty} \binom{n+1}{\frac{1}{2}(n-x_0)-i} \frac{x_0+2i+1}{n+1} \left(\frac{\rho-\alpha}{\alpha} \right)^i + \sum_{i=0}^{\infty} \sum_{k=0}^i \left(\frac{\rho-\alpha}{\alpha} \right)^{i-k} \left(\frac{\omega-\beta}{\beta} \right)^k \right. \\
 &\quad \times \left\{ \left\{ \frac{\omega-\beta}{\beta} \binom{n+1}{\frac{1}{2}(n+x_0)-L-i} \frac{2L-x_0+2i+1}{n+1} - \binom{n+1}{\frac{1}{2}(n+x_0)-L-i-1} \frac{2L-x_0+2i+3}{n+1} \right\} \right. \\
 &\quad \left. \left. - \frac{\gamma}{\sqrt{\alpha\beta}} \binom{n+1}{\frac{1}{2}(n+x_0-1)-L-i} \frac{2L-x_0+2i+2}{n+1} \right\} \right\}. \tag{4.4}
 \end{aligned}$$

$p_n(L | x_0)$ can be obtained from $p_n(0 | x_0)$ by the transformation $\alpha \leftrightarrow \beta$, $\rho \leftrightarrow \omega$ and $x_0 \mapsto L - x_0$. Therefore $p_n(L | x_0)$ has the following form

$$\begin{aligned}
 p_n(L | x_0) &= (\alpha\beta)^{\frac{n}{2}} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}x_0} \times \\
 &\quad \times \left\{ \sum_{i=0}^{\infty} \binom{n+1}{\frac{1}{2}(n-L+x_0)-i} \frac{L-x_0+2i+1}{n+1} \left(\frac{\omega-\beta}{\beta} \right)^i + \sum_{i=0}^{\infty} \sum_{k=0}^i \left(\frac{\rho-\alpha}{\alpha} \right)^k \left(\frac{\omega-\beta}{\beta} \right)^{i-k} \right. \\
 &\quad \times \left\{ \left\{ \frac{\rho-\alpha}{\alpha} \binom{n+1}{\frac{1}{2}(n-L+x_0)-i} \frac{L-x_0+2i+1}{n+1} - \binom{n+1}{\frac{1}{2}(n-L+x_0)-i-1} \frac{L-x_0+2i+3}{n+1} \right\} \right. \\
 &\quad \left. \left. - \frac{\gamma}{\sqrt{\alpha\beta}} \binom{n+1}{\frac{1}{2}(n-L+x_0-1)-i} \frac{L-x_0+2i+2}{n+1} \right\} \right\}. \tag{4.5}
 \end{aligned}$$

Explicit expressions of $p_n(0 \mid x_0)$, $p_n(L \mid x_0)$ for $x_0 \in \{0, L\}$ and $p_n(x \mid x_0)$ for $x, x_0 \in S$, $x \neq 0, L$ can be similarly obtained from Theorem 2.1, using the technique we have used to get (4.4).

5. The mean occupation total and the absorption probabilities

Explicit expressions for the mean number of times the position x , $x \in S$ is occupied, given the starting position x_0 , $x_0 \in S$, denoted by $E[N_{x_0x}]$, can be easily obtained from Theorem 2.1 by putting $z = 1$. Thus

$$E[N_{0x}] = \frac{\rho}{\sum} \begin{cases} 1 - \left(\frac{\beta}{\alpha}\right)^{L-x} - \omega \left[1 - \left(\frac{\beta}{\alpha}\right)^{L-x-1}\right], & \alpha > \beta \\ \left(\frac{\alpha}{\beta}\right)^x - \left(\frac{\alpha}{\beta}\right)^L - \omega \left[\left(\frac{\alpha}{\beta}\right)^{x+1} - \left(\frac{\alpha}{\beta}\right)^L\right], & \alpha < \beta \\ (1 - \omega)(L - x) + \omega, & \alpha = \beta \end{cases} \quad (5.1)$$

for $x \in S$, $x \neq 0, L$;

$$E[N_{Lx}] = \frac{\omega}{\sum} \begin{cases} \left(\frac{\beta}{\alpha}\right)^{L-x} - \left(\frac{\beta}{\alpha}\right)^L - \rho \left[\left(\frac{\beta}{\alpha}\right)^{L-x+1} - \left(\frac{\beta}{\alpha}\right)^L\right], & \alpha > \beta \\ 1 - \left(\frac{\alpha}{\beta}\right)^x - \rho \left[1 - \left(\frac{\alpha}{\beta}\right)^{x-1}\right], & \alpha < \beta \\ (1 - \rho)x + \rho, & \alpha = \beta \end{cases} \quad (5.2)$$

for $x \in S$, $x \neq 0, L$;

$$E[N_{x_0x}] = \frac{1}{\sum} \begin{cases} \frac{\left(\frac{\beta}{\alpha}\right)^{x_0-x}}{\alpha-\beta} \left[1 - \left(\frac{\beta}{\alpha}\right)^x - \frac{\rho\beta}{\alpha} \left(1 - \left(\frac{\beta}{\alpha}\right)^{x-1}\right)\right] \times \\ \times \left[1 - \left(\frac{\beta}{\alpha}\right)^{L-x_0} - \omega \left(1 - \left(\frac{\beta}{\alpha}\right)^{L-x_0-1}\right)\right], & \alpha > \beta \\ \frac{1}{\beta-\alpha} \left[1 - \left(\frac{\alpha}{\beta}\right)^x - \rho \left(1 - \left(\frac{\alpha}{\beta}\right)^{x-1}\right)\right] \times \\ \times \left[1 - \left(\frac{\alpha}{\beta}\right)^{L-x_0} - \frac{\omega\alpha}{\beta} \left(1 - \left(\frac{\alpha}{\beta}\right)^{L-x_0-1}\right)\right], & \alpha < \beta \\ \frac{1}{\alpha} [(1 - \rho)x + \rho] [(1 - \omega)(L - x_0) + \omega], & \alpha = \beta \end{cases} \quad (5.3)$$

for $x_0 \in S$, $x_0 \neq 0, L$, $x \in \{1, 2, \dots, x_0\}$;

$$E[N_{x_0x}] = \frac{1}{\sum} \begin{cases} \frac{1}{\alpha - \beta} \left[1 - \left(\frac{\beta}{\alpha} \right)^{x_0} - \frac{\rho\beta}{\alpha} \left(1 - \left(\frac{\beta}{\alpha} \right)^{x_0-1} \right) \right] \times \\ \times \left[1 - \left(\frac{\beta}{\alpha} \right)^{L-x} - \omega \left(1 - \left(\frac{\beta}{\alpha} \right)^{L-x-1} \right) \right], & \alpha > \beta \\ \frac{\left(\frac{\alpha}{\beta} \right)^{x-x_0}}{\beta - \alpha} \left[1 - \left(\frac{\alpha}{\beta} \right)^{x_0} - \rho \left(1 - \left(\frac{\alpha}{\beta} \right)^{x_0-1} \right) \right] \times \\ \times \left[1 - \left(\frac{\alpha}{\beta} \right)^{L-x} - \frac{\omega\alpha}{\beta} \left(1 - \left(\frac{\alpha}{\beta} \right)^{L-x-1} \right) \right], & \alpha < \beta \\ \frac{1}{\alpha} [(1 - \rho)x_0 + \rho] [(1 - \omega)(L - x) + \omega], & \alpha = \beta \end{cases} \quad (5.4)$$

for $x_0 \in S$, $x_0 \neq 0, L$, $x \in \{x_0, x_0 + 1, \dots, L - 1\}$.

Further, for any x_0 , $x_0 \in S$, we obtain

$$E[N_{x_00}] = \frac{1}{\sum} \begin{cases} \left(\frac{\beta}{\alpha} \right)^{x_0} - \left(\frac{\beta}{\alpha} \right)^L - \omega \left[\left(\frac{\beta}{\alpha} \right)^{x_0} - \left(\frac{\beta}{\alpha} \right)^{L-1} \right], & \alpha > \beta \\ 1 - \left(\frac{\alpha}{\beta} \right)^{L-x_0} - \frac{\omega\alpha}{\beta} \left[1 - \left(\frac{\alpha}{\beta} \right)^{L-x_0-1} \right], & \alpha < \beta \\ (1 - \omega)(L - x_0) + \omega, & \alpha = \beta \end{cases} \quad (5.5)$$

and

$$E[N_{x_0L}] = \frac{1}{\sum} \begin{cases} 1 - \left(\frac{\beta}{\alpha} \right)^{x_0} - \frac{\rho\beta}{\alpha} \left[1 - \left(\frac{\beta}{\alpha} \right)^{x_0-1} \right], & \alpha > \beta \\ \left(\frac{\alpha}{\beta} \right)^{L-x_0} - \left(\frac{\alpha}{\beta} \right)^L - \rho \left[\left(\frac{\alpha}{\beta} \right)^{L-x_0} - \left(\frac{\alpha}{\beta} \right)^{L-1} \right], & \alpha < \beta \\ (1 - \rho)x_0 + \rho, & \alpha = \beta, \end{cases} \quad (5.6)$$

where

$$\sum = \begin{cases} 1 - \left(\frac{\beta}{\alpha} \right)^L + \left(\omega + \frac{\rho\beta}{\alpha} \right) \left(1 - \left(\frac{\beta}{\alpha} \right)^{L-1} \right) + \frac{\rho\omega\beta}{\alpha} \left(1 - \left(\frac{\beta}{\alpha} \right)^{L-2} \right), & \alpha > \beta \\ 1 - \left(\frac{\alpha}{\beta} \right)^L + \left(\rho + \frac{\omega\alpha}{\beta} \right) \left(1 - \left(\frac{\alpha}{\beta} \right)^{L-1} \right) + \frac{\rho\omega\alpha}{\beta} \left(1 - \left(\frac{\alpha}{\beta} \right)^{L-2} \right), & \alpha < \beta \\ 1 - (\rho + \omega)(L - 1) + \rho\omega(L - 2), & \alpha = \beta. \end{cases} \quad (5.7)$$

Formulae (5.1)–(5.7), in the case $\rho = \omega = 0$, agree with the well-known results for a R.W. between fully absorbing barriers at 0 and L and starting at x_0 , $x_0 \in S$, $x_0 \neq 0, L$ (cf. Kemeny and Snell [14, p. 151] and Iosifescu

[11, p. 101] in the case $\gamma = 0$). In this case

$$E[N_{x_0x}] = \begin{cases} \frac{((\frac{\alpha}{\beta})^x - 1)((\frac{\alpha}{\beta})^{L-x_0} - 1)}{(2\alpha - 1)((\frac{\beta}{\alpha})^L - 1)}, & x \in \{1, 2, \dots, x_0\}, \alpha \neq \beta \\ \frac{((\frac{\alpha}{\beta})^{x_0} - 1)((\frac{\alpha}{\beta})^{L-x_0} - (\frac{\alpha}{\beta})^{x-x_0})}{(2\alpha - 1)((\frac{\beta}{\alpha})^L - 1)}, & x \in \{x_0, x_0 + 1, \dots, L - 1\}, \alpha \neq \beta, \end{cases} \quad (5.8)$$

$$E[N_{x_0x}] = \frac{2}{L} \begin{cases} x(L - x_0), & x \in \{1, 2, \dots, x_0\}, \alpha = \beta \\ x_0(L - x), & x \in \{x_0, x_0 + 1, \dots, L - 1\}, \alpha = \beta \end{cases} \quad (5.9)$$

and $E[N_{00}] = E[N_{LL}] = 1$.

Notice that formula (5.9) can be obtained from (5.8) by interchanging α and β ; ρ and ω and replacing x_0 by $L - x_0$. Starting from x_0 , $x_0 \in S$, the probabilities of absorption "with probabilities $1 - \rho$ and $1 - \omega$ " which occur at 0 and L , respectively, will be denoted by $q_{x_0}(j)$, $j = 0, L$. Using (5.5) and (5.6) we get

$$q_{x_0}(0) = \begin{cases} \frac{(\frac{\beta}{\alpha})^{x_0} - (\frac{\beta}{\alpha})^L - \omega[(\frac{\beta}{\alpha})^{x_0} - (\frac{\beta}{\alpha})^{L-1}]}{\frac{1-\omega}{1-\rho}(1 - \frac{\rho\beta}{\alpha}) - (1 - \frac{\omega\alpha}{\beta})(\frac{\beta}{\alpha})^L}, & \alpha > \beta \\ \frac{1 - (\frac{\alpha}{\beta})^{L-x_0} - \frac{\omega\alpha}{\beta}(1 - (\frac{\alpha}{\beta})^{L-x_0-1})}{\frac{1 - \omega\alpha}{1-\rho} - \frac{1-\omega}{1-\rho}(1 - \frac{\rho\beta}{\alpha})(\frac{\alpha}{\beta})^L}, & \alpha < \beta \\ \frac{L - x_0 + \frac{1-\omega}{1-\rho}}{L + \frac{\rho}{1-\rho} + \frac{1-\omega}{1-\rho}}, & \alpha = \beta, \end{cases} \quad (5.10)$$

$$q_{x_0}(L) = \begin{cases} \frac{1 - (\frac{\beta}{\alpha})^{x_0} - \frac{\rho\beta}{\alpha}[1 - (\frac{\beta}{\alpha})^{x_0-1}]}{(1 - \frac{\rho\beta}{\alpha}) - \frac{1-\rho}{1-\omega}(1 - \frac{\omega\alpha}{\beta})(\frac{\beta}{\alpha})^L}, & \alpha > \beta \\ \frac{(\frac{\alpha}{\beta})^{L-x_0} - (\frac{\alpha}{\beta})^L - \rho[(\frac{\alpha}{\beta})^{L-x_0} - (\frac{\alpha}{\beta})^{L-1}]}{\frac{1-\rho}{1-\omega}(1 - \frac{\omega\alpha}{\beta}) - (1 - \frac{\rho\beta}{\alpha})(\frac{\alpha}{\beta})^L}, & \alpha < \beta \\ \frac{x_0 + \frac{\rho}{1-\rho}}{L + \frac{\rho}{1-\rho} + \frac{1-\omega}{1-\rho}}, & \alpha = \beta. \end{cases} \quad (5.11)$$

By an alternative method, formulae (5.10) and (5.11) can be obtained respectively by solving the following difference equations:

$$q_{x_0}(0) = \begin{cases} (1 - \rho) \left[1 + \frac{\rho}{1-\rho} q_1(0) \right], & x_0 = 0 \\ \alpha q_{x_0+1}(0) + \gamma q_{x_0}(0) + \beta q_{x_0-1}(0), & x_0 \in \{1, 2, \dots, L - 1\} \\ \omega q_{L-1}(0), & x_0 = L \end{cases} \quad (5.12)$$

and

$$q_{x_0}(L) = \begin{cases} \rho q_1(L), & x_0 = 0 \\ \alpha q_{x_0+1}(L) + \gamma q_{x_0}(L) + \beta q_{x_0-1}(L), & x_0 \in \{1, 2, \dots, L - 1\} \\ (1 - \omega) \left[1 + \frac{\omega}{1-\omega} q_{L-1}(L) \right], & x_0 = L. \end{cases} \quad (5.13)$$

Obviously $q_{x_0}(0) + q_{x_0}(L) = 1$. We see that formulae (5.10) and (5.11) in the case $\gamma = 0$ agree with that of El-Shehawey (2000) and with that of Percus (1985) in the case $\rho = \omega$ (see also Feller [10, p. 344–349], Kemeney and Snell [14, p. 153] and Iosifescu [11, p. 180] in the case $\rho = \omega = 0$ and $\gamma = 0$).

6. The one-boundary case ($L \rightarrow \infty$)

Let $v_n(x \mid x_0)$ be the n -step probability that the particle is at position x at time n , given that its initial position was x_0 , for the case L infinite. Let $\Phi(z; x \mid x_0)$ denotes its pgf such that

$$\Phi(z; x \mid x_0) = \sum_{n=0}^{\infty} z^n v_n(x \mid x_0), \quad |z| < 1. \quad (6.1)$$

Theorem 2.1 can be easily modified to the one-boundary case by taking the limit as $L \rightarrow \infty$, to obtain on the semi-infinte lattice segment $0, 1, 2, \dots$

$$\Phi(z; x \mid x_0) = \frac{\frac{\rho}{\alpha}}{1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}}} \left(\theta\sqrt{\frac{\alpha}{\beta}} \right)^x, \quad 0 < x < \infty, \quad (6.2)$$

$$\Phi(z; x \mid \infty) = \frac{\frac{\omega}{\beta} \left[1 - \theta^{2x} - z\rho\theta\sqrt{\frac{\beta}{\alpha}} (1 - \theta^{2x-2}) \right]}{\left(1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}} \right) \left(1 - z\omega\theta\sqrt{\frac{\alpha}{\beta}} \right)} \lim_{L \rightarrow \infty} \left(\theta\sqrt{\frac{\beta}{\alpha}} \right)^{L-x}, \quad (6.3)$$

$0 < x < \infty$

$$\Phi(z; 0 \mid x_0) = \begin{cases} \frac{\left(\theta\sqrt{\frac{\alpha}{\beta}} \right)^{x_0}}{1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}}}, & 0 \leq x_0 < \infty \\ \frac{z\omega(1-\theta^2)}{\left(1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}} \right) \left(1 - z\omega\theta\sqrt{\frac{\alpha}{\beta}} \right)} \lim_{L \rightarrow \infty} \left(\theta\sqrt{\frac{\beta}{\alpha}} \right)^{L-1}, & x_0 = \infty, \end{cases} \quad (6.4)$$

$$\Phi(z; \infty \mid x_0) = \begin{cases} \frac{1 - \theta^{2x_0} - z\rho\theta\sqrt{\frac{\beta}{\alpha}} (1 - \theta^{2x_0-2})}{\left(1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}} \right) \left(1 - z\omega\theta\sqrt{\frac{\alpha}{\beta}} \right)} \lim_{L \rightarrow \infty} \left(\theta\sqrt{\frac{\alpha}{\beta}} \right)^{L-x_0}, & 0 \leq x_0 < \infty \\ \frac{1}{1 - z\omega\theta\sqrt{\frac{\alpha}{\beta}}}, & x_0 = \infty \end{cases} \quad (6.5)$$

and

$$\begin{aligned} \Phi(z; x \mid x_0) &= \frac{\theta}{z\sqrt{\alpha\beta}(1-\theta^2)} \times \\ &\times \begin{cases} \left[1 - \frac{(\theta - z\rho\sqrt{\frac{\beta}{\alpha}})^{\theta^{2x-1}}}{1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}}} \right] \left(\theta\sqrt{\frac{\beta}{\alpha}} \right)^{x_0-x}, & x \leq x_0, \quad 0 < x_0 < \infty \\ \left[1 - \frac{(\theta - z\rho\sqrt{\frac{\beta}{\alpha}})^{\theta^{2x_0-1}}}{1 - z\rho\theta\sqrt{\frac{\beta}{\alpha}}} \right] \left(\theta\sqrt{\frac{\alpha}{\beta}} \right)^{x-x_0}, & x \geq x_0, \quad 0 < x_0 < \infty. \end{cases} \end{aligned} \quad (6.6)$$

For $x, x_0 \geq 0$ define T_{x_0x} as the number of times the position x is occupied given the starting position x_0 . Therefore explicit expressions for the mean, $E[T_{x_0x}]$, can be easily obtained from (6.2)–(6.6) by setting $z = 1$ and

$$\theta = \begin{cases} \sqrt{\frac{\beta}{\alpha}}, & \alpha \geq \beta \\ \sqrt{\frac{\alpha}{\beta}}, & \alpha \leq \beta. \end{cases}$$

Thus

$$E[T_{0x}] = \frac{\rho}{\alpha} \begin{cases} \frac{1}{1-\rho(\frac{\beta}{\alpha})}, & \alpha \geq \beta \\ \frac{1}{1-\rho} \left(\frac{\alpha}{\beta} \right)^x, & \alpha \leq \beta \end{cases} \quad (6.7)$$

and

$$E[T_{\infty x}] = \frac{\omega}{\beta} \begin{cases} \frac{1}{1-\omega(\frac{\alpha}{\beta})} \left[1 - \frac{1-\rho(\frac{\beta}{\alpha})}{1-\rho} \left(\frac{\alpha}{\beta} \right)^x \right], & \alpha < \beta \\ 0, & \alpha \geq \beta \end{cases} \quad (6.8)$$

for $0 < x < \infty$;

$$E[T_{x_0 0}] = \begin{cases} \frac{1}{1-\rho(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^{x_0}, & \alpha > \beta \\ \frac{1}{1-\rho}, & \alpha \leq \beta \end{cases} \quad (6.9)$$

and

$$E[T_{x_0 \infty}] = \begin{cases} \frac{1}{1-\omega} \left[1 - \frac{1-\rho}{1-\rho(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^{x_0} \right], & \alpha > \beta \\ 0, & \alpha \leq \beta \end{cases} \quad (6.10)$$

for $0 \leq x_0 < \infty$;

$$E[T_{x_0 x}] = \begin{cases} \frac{1}{\alpha-\beta} \left[1 - \frac{1-\rho}{1-\rho(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^x \right] \left(\frac{\beta}{\alpha} \right)^{x_0-x}, & \alpha > \beta \\ \frac{1}{\beta-\alpha} \left[1 - \frac{1-\rho(\frac{\beta}{\alpha})}{1-\rho} \left(\frac{\alpha}{\beta} \right)^x \right], & \alpha < \beta \\ \frac{1}{\alpha} \left[x + \frac{\rho}{1-\rho} \right], & \alpha = \beta \end{cases} \quad (6.11)$$

for $0 < x \leq x_0$, $0 < x_0 < \infty$;

$$E[T_{x_0x}] = \begin{cases} \frac{1}{\alpha-\beta} \left[1 - \frac{1-\rho}{1-\rho(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^{x_0} \right], & \alpha > \beta \\ \frac{1}{\beta-\alpha} \left[1 - \frac{1-\rho(\frac{\beta}{\alpha})}{1-\rho} \left(\frac{\alpha}{\beta} \right)^{x_0} \right] \left(\frac{\alpha}{\beta} \right)^{x-x_0}, & \alpha < \beta \\ \frac{1}{\beta} \left[x_0 + \frac{\rho}{1-\rho} \right], & \alpha = \beta \end{cases} \quad (6.12)$$

for $x_0 \leq x$, $0 < x_0 < \infty$;

$$E[T_{\infty 0}] = \begin{cases} \frac{\omega}{1-\rho} \left[\frac{1-\frac{\alpha}{\beta}}{1-\omega(\frac{\alpha}{\beta})} \right], & \alpha < \beta \\ 0, & \alpha \geq \beta \end{cases} \quad (6.13)$$

and

$$E[T_{\infty \infty}] = \begin{cases} \frac{1}{1-\omega}, & \alpha \geq \beta \\ \frac{1}{1-\omega(\frac{\alpha}{\beta})}, & \alpha < \beta. \end{cases} \quad (6.14)$$

Notice that $(1-\rho)E[T_{x_0 0}]$ and $(1-\omega)E[T_{x_0 \infty}]$ are, respectively, the probability of absorption at 0 and the survival probability "the probability never to have visited the absorbing barrier", they, respectively, denoted by $B_{x_0}(0)$ and $B_{x_0}(\infty)$. Therefore

$$B_{x_0}(0) = \begin{cases} \frac{1-\rho}{1-\rho(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^{x_0}, & \alpha > \beta, \quad 0 \leq x_0 < \infty \\ 1, & \alpha \leq \beta, \quad 0 \leq x_0 < \infty \\ \frac{\omega(1-\frac{\alpha}{\beta})}{1-\omega(\frac{\alpha}{\beta})}, & \alpha < \beta, \quad x_0 = \infty \\ 0, & \alpha \geq \beta, \quad x_0 = \infty, \end{cases} \quad (6.15)$$

and

$$B_{x_0}(\infty) = \begin{cases} 1 - \frac{1-\rho}{1-\rho(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^{x_0}, & \alpha > \beta, \quad 0 \leq x_0 < \infty \\ 0, & \alpha \leq \beta, \quad 0 \leq x_0 < \infty \\ \frac{1-\omega}{1-\omega(\frac{\alpha}{\beta})}, & \alpha < \beta, \quad x_0 = \infty \\ 1, & \alpha \geq \beta, \quad x_0 = \infty. \end{cases} \quad (6.16)$$

Obviously $B_{x_0}(0) + B_{x_0}(\infty) = 1$.

Starting from x_0 , the mean number of steps the particle is at x after time n has elapsed; $x, x_0 \in \{0, 1, \dots\}$ may be obtained from

$$m_{x_0x} = \lim_{L \rightarrow \infty} \sum_{n=0}^{\infty} np_n(x | x_0) = \left[\frac{d}{dz} \Phi(z; x | x_0) \right]_{z=1} \quad (6.17)$$

and are found to be

$$m_{0x} = \frac{\rho}{|\alpha - \beta|} \begin{cases} \frac{1-\rho}{1-\rho(\frac{\beta}{\alpha})} \left[x + \frac{\rho(\frac{\beta}{\alpha})(\gamma+2\alpha)}{1-\rho(\frac{\beta}{\alpha})} \right], & \alpha > \beta \\ \frac{1}{1-\rho} \left[x + \frac{\rho(\gamma+2\beta)}{1-\rho} \right] \left(\frac{\alpha}{\beta} \right)^x, & \alpha < \beta \\ \infty, & \alpha = \beta \end{cases} \quad (6.18)$$

for $0 < x < \infty$; and

$$m_{x_0 0} = \frac{1}{|\alpha - \beta|} \begin{cases} \frac{1}{1-\rho(\frac{\beta}{\alpha})} \left[x_0 + \frac{\rho(\frac{\beta}{\alpha})(\gamma+2\alpha)}{1-\rho(\frac{\beta}{\alpha})} \right] \left(\frac{\alpha}{\beta} \right)^{x_0}, & \alpha > \beta \\ \frac{1}{1-\rho} \left[x_0 + \frac{\rho(\gamma+2\beta)}{1-\rho} \right], & \alpha < \beta \\ \infty, & \alpha = \beta \end{cases} \quad (6.19)$$

for $0 < x_0 < \infty$;

$$m_{00} = \frac{\rho}{|\alpha - \beta|} \begin{cases} \frac{\beta(\gamma+2\alpha)}{\left(1-\rho(\frac{\beta}{\alpha})\right)^2}, & \alpha > \beta \\ \frac{\alpha(\gamma+2\beta)}{(1-\rho)^2}, & \alpha < \beta \\ \infty, & \alpha = \beta, \end{cases} \quad (6.20)$$

$$m_{x_0 x} = \frac{1}{(|\alpha - \beta|)^2} \begin{cases} \left(\frac{\beta}{\alpha} \right)^{x_0-x} U_{x_0, x}(\beta, \alpha) - \left(\frac{\beta}{\alpha} \right)^{x_0} V_{x, x_0} \left(\beta, \alpha; \frac{\rho\beta}{\alpha}, \rho \right), & \alpha > \beta \\ U_{x_0, x}(\alpha, \beta) - \left(\frac{\alpha}{\beta} \right)^x V_{x, x_0} \left(\alpha, \beta; \rho, \frac{\rho\beta}{\alpha} \right), & \alpha < \beta \end{cases} \quad (6.21)$$

for $0 < x_0 < \infty$, $x \in \{1, 2, \dots, x_0\}$; and

$$m_{x_0 x} = \frac{1}{(|\alpha - \beta|)^2} \begin{cases} U_{x, x_0}(\beta, \alpha) - \left(\frac{\beta}{\alpha} \right)^{x_0} V_{x, x_0} \left(\beta, \alpha; \frac{\rho\beta}{\alpha}, \rho \right), & \alpha > \beta \\ \left(\frac{\beta}{\alpha} \right)^{x-x_0} U_{x, x_0}(\alpha, \beta) - \left(\frac{\alpha}{\beta} \right)^x V_{x, x_0} \left(\alpha, \beta; \rho, \frac{\rho\beta}{\alpha} \right), & \alpha < \beta \end{cases} \quad (6.22)$$

for $0 < x_0 < \infty$, $x \in \{x_0, x_0 + 1, \dots, L - 1\}$; where

$$U_{x, x_0}(\alpha, \beta) = x - x_0 + \gamma + \frac{2\alpha(\gamma+2\beta)}{\beta-\alpha},$$

$$V_{x, x_0} \left(\alpha, \beta; \rho, \frac{\rho\beta}{\alpha} \right) = \frac{1-\frac{\rho\beta}{\alpha}}{1-\rho} \left[x + x_0 - \beta + \alpha + \frac{2\alpha}{\beta-\alpha} + \frac{1-\frac{\rho\beta}{\alpha} \left[\beta \left(1 - \frac{\alpha}{\beta} \right)^2 + \rho \right]}{(1-\rho) \left(1 - \frac{\rho\beta}{\alpha} \right)} \right]. \quad (6.23)$$

If we denote by $E_{x_0 0}$ the mean number of steps taken until being annihilated given that the particle is annihilated with probability $1 - \rho$ at the origin. Then for $0 < x_0 < \infty$,

$$E_{x_0 0} = \frac{(1-\rho) m_{x_0 0}}{B_{x_0}(0)} = \frac{1}{|\alpha - \beta|} \begin{cases} x_0 + \frac{\rho(\frac{\beta}{\alpha})(\gamma+2\alpha)}{1-\rho(\frac{\beta}{\alpha})}, & \alpha > \beta \\ x_0 + \frac{\rho(\gamma+2\beta)}{1-\rho}, & \alpha < \beta \\ \infty, & \alpha = \beta \end{cases} \quad (6.24)$$

Using fomulae (6.2)–(6.6) and $u = \frac{1}{2z\alpha} \left[1 - z\gamma - \sqrt{(1 - z\gamma)^2 - 4\alpha\beta z^2} \right]$ we can now obtain explicit expressions for $v_n(x \mid x_0)$, $x \geq 1$, $x_0 \geq 0$. Notice first that from (6.4) and (6.6) we get

$$\begin{aligned} \Phi(z; x \mid x_0) &= \frac{\Phi(z; 0 \mid x_0)}{1 - z\gamma} \left[\frac{\alpha - \rho(1 - z\gamma)}{\beta^{x-1}} \left(\frac{(u\alpha)^{x-1} - \left(\frac{\beta}{u}\right)^{x-1}}{u\alpha - \frac{\beta}{u}} \right) + \frac{1}{\beta^x} \left(\frac{(u\alpha)^{x+1} - \left(\frac{\beta}{u}\right)^{x+1}}{u\alpha - \frac{\beta}{u}} \right) \right] \\ &\quad - \frac{1}{z\beta^{x-x_0}} \left(\frac{(u\alpha)^{x-x_0} - \left(\frac{\beta}{u}\right)^{x-x_0}}{u\alpha - \frac{\beta}{u}} \right) \begin{cases} 0, & x = 1, 2, \dots, x_0 - 1 \\ 1, & x = x_0, x_0 + 1, \dots \end{cases} \end{aligned} \quad (6.25)$$

Hence using the same technique as in Section 4 we get for $x \geq 1$, $x_0 \geq 0$

$$\begin{aligned} v_n(x \mid x_0) &= \frac{1}{2\pi i} \oint_{|u| < 1} \frac{(\alpha u^2 + u\gamma + \beta)^n}{u^{n+x-x_0+1}} \times \\ &\quad \times \left\{ \frac{(\alpha u^2 + u\gamma + \beta) \left[\beta + \alpha u^2 \left[1 - \left(\frac{\alpha}{\beta} u^2\right)^{x-1} \right] - \beta \left(\frac{\alpha}{\beta} u^2\right)^{x+1} \right]}{(\beta + \alpha u^2) [(\alpha - \rho) u^2 + \beta + u\gamma]} \right. \\ &\quad \left. - \frac{\rho u^2 \left[1 - \left(\frac{\alpha}{\beta} u^2\right)^{x-1} \right]}{[(\alpha - \rho) u^2 + \beta + u\gamma]} - \delta \left[1 - \left(\frac{\alpha}{\beta} u^2\right)^{x-x_0} \right] \right\} du, \\ &= \text{coef. of } \psi^{\frac{1}{2}(n+x-x_0)} \\ &\text{in } \left(\beta + \gamma\sqrt{\psi} + \alpha\psi \right)^n \left\{ \frac{(\beta + \gamma\sqrt{\psi} + \alpha\psi) \left[(\beta + \alpha\psi) - \beta \left(\frac{\psi\alpha}{\beta}\right)^x - \alpha\psi \left(\frac{\psi\alpha}{\beta}\right)^x \right]}{(\beta + \alpha\psi) [(\alpha - \rho)\psi + \gamma\sqrt{\psi} + \beta]} \right. \\ &\quad \left. - \frac{\rho\psi \left[1 - \left(\frac{\psi\alpha}{\beta}\right)^{x-1} \right]}{[(\alpha - \rho)\psi + \gamma\sqrt{\psi} + \beta]} - \delta \left[1 - \left(\frac{\psi\alpha}{\beta}\right)^{x-x_0} \right] \right\} \\ &= \text{coef. of } \psi^{\frac{1}{2}(n+x-x_0)} \\ &\text{in } \beta^n \left(1 + \sqrt{\psi} \left(\frac{\gamma + \alpha\sqrt{\psi}}{\beta} \right) \right)^n \left\{ 1 + \frac{\left[\frac{\rho - \alpha}{\alpha} - \sqrt{\psi} \left(\frac{\gamma + \alpha\sqrt{\psi}}{\beta} \right) \left(\frac{\psi\alpha}{\beta} \right)^x \right]}{1 - \sqrt{\psi} \left(\frac{\rho - \alpha}{\beta} \sqrt{\psi} - \gamma \right)} \right. \\ &\quad \left. - \delta \left[1 - \left(\frac{\psi\alpha}{\beta}\right)^{x-x_0} \right] \right\} \end{aligned} \quad (6.26)$$

where

$$\delta = \begin{cases} 0, & x = 1, 2, \dots, x_0 - 1 \\ 1, & x = x_0, x_0 + 1, \dots \end{cases}$$

Therefore

$$v_n(x | x_0) = (\alpha\beta)^{\frac{n}{2}} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(x-x_0)} \left\{ \begin{aligned} & \left(\binom{n}{\frac{1}{2}(n+x-x_0)} - \binom{n}{\frac{1}{2}(n+x+x_0)+1} + \sum_{i=1}^{\infty} \left(\frac{\rho-\alpha}{\alpha}\right)^i \times \right. \\ & \times \left[\binom{n}{\frac{1}{2}(n+x+x_0)+i-1} - \binom{n}{\frac{1}{2}(n+x+x_0)+i+1} \right] - \\ & \left. \frac{\gamma}{\sqrt{\alpha\beta}} \sum_{i=0}^{\infty} \binom{n}{\frac{1}{2}(n+x+x_0+1)+i} \left(\frac{\rho-\alpha}{\alpha}\right)^i \right\}. \end{aligned} \right. \quad (6.27)$$

Notice that

$$\begin{aligned} & \sum_{i=1}^{\infty} \left(\frac{\rho-\alpha}{\alpha}\right)^i \left[\binom{n}{\frac{1}{2}(n+x+x_0)+i-1} - \binom{n}{\frac{1}{2}(n+x+x_0)+i+1} \right] \\ & \quad - \binom{n}{\frac{1}{2}(n+x+x_0)+1} \\ & = \frac{\rho-\alpha}{\alpha} \binom{n}{\frac{1}{2}(n+x+x_0)} + \frac{\rho}{\alpha} \left(\frac{\rho-2\alpha}{\rho-\alpha}\right) \sum_{i=1}^{\infty} \binom{n}{\frac{1}{2}(n+x+x_0)+i} \left(\frac{\rho-\alpha}{\alpha}\right)^i. \end{aligned} \quad (6.28)$$

Thus

$$v_n(x | x_0) = (\alpha\beta)^{\frac{n}{2}} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(x-x_0)} \left\{ \begin{aligned} & \left(\binom{n}{\frac{1}{2}(n+x-x_0)} - \frac{\rho-\alpha}{\alpha} \binom{n}{\frac{1}{2}(n+x+x_0)} + \right. \\ & \frac{\rho}{\alpha} \left(\frac{\rho-2\alpha}{\rho-\alpha}\right) \sum_{i=1}^{\infty} \binom{n}{\frac{1}{2}(n+x+x_0)+i} \left(\frac{\rho-\alpha}{\alpha}\right)^i \\ & \left. - \frac{\gamma}{\sqrt{\alpha\beta}} \sum_{i=0}^{\infty} \binom{n}{\frac{1}{2}(n+x+x_0+1)+i} \left(\frac{\rho-\alpha}{\alpha}\right)^i \right\} \end{aligned} \right. \quad (6.29)$$

whereas $v_n(0 | x_0)$ is obtained immediately from the relation $v_n(0 | x_0) = \beta v_{n-1}(1 | x_0)$, and formula (6.29) with $x = 1$:

$$v_n(0 | x_0) = (\alpha\beta)^{\frac{n}{2}} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}x_0} \left\{ \begin{aligned} & \sum_{i=0}^{\infty} \left(\frac{\rho-\alpha}{\alpha}\right)^i \binom{n+1}{\frac{1}{2}(n+x_0)+i+1} \frac{x_0+2i+1}{n+1} - \\ & \frac{\gamma}{\sqrt{\alpha\beta}} \sum_{i=0}^{\infty} \binom{n-1}{\frac{1}{2}(n+x_0+1)+i} \left(\frac{\rho-\alpha}{\alpha}\right)^i \end{aligned} \right\}. \quad (6.30)$$

Appendix

We present here details of the calculations leading to the proof of Theorem 2.1. Solving the recursion (2.5) with the boundary conditions (2.6) systematically, starting with the following formula

$$G(z; 2 | x_0) = \frac{1}{\mu(z)} \left[(1 - \varphi_0(z)) G(z; 1 | x_0) - \delta_{1,x_0}^\circ(z) - \rho^\circ(z) \delta_{0,x_0} \right], \quad (1)$$

where

$$\varphi_0(z) = \mu_0(z)\rho^\circ(z).$$

Equation (2.5) can be reduced from the second order to the first order by successively eliminating one of the $G(z; x \mid x_0)$ in each equation of the system, to obtain finally

$$\begin{aligned} G(z; x \mid x_0) &= \\ &= \frac{1}{\mu(z)(1 - \varphi_{x-3}(z))} \left[(1 - \varphi_{x-2}(z)) G(z; x-1 \mid x_0) \right. \\ &\quad \left. - \sum_{k=0}^{x-2} \delta_{k+1, x_0}^\circ(z) (1 - \varphi_{k-1}(z)) \lambda^{x-k-2}(z) - \rho^\circ(z) \lambda^{x-2}(z) \delta_{0, x_0} \right], \end{aligned} \quad (2)$$

for $x \in \{2, 3, \dots, L-2\}$, where $\varphi_j(z)$ satisfies the following difference equations

$$\varphi_j(z) = \begin{cases} \rho^\circ(z)\mu_0(z), & j = 0 \\ 0, & j = -1 \\ \varphi_{j-1}(z) + \lambda(z)\mu(z)(1 - \varphi_{j-2}(z)), & j = 1, 2, \dots, L-2 \\ \varphi_{j-2}(z) + \lambda_0(z)\omega^\circ(z)(1 - \varphi_{L-3}(z)), & j = L-1. \end{cases} \quad (3)$$

On account of the boundary conditions (2.6), we obtain

$$G(z; L-1 \mid x_0) = \frac{1}{1 - \varphi_L^*(z)} [\lambda(z)G(z; L-2 \mid x_0) - \delta_{L-1, x_0}^\circ(z) + \omega^\circ(z)\delta_{L, x_0}], \quad (4)$$

where

$$\varphi_L^*(z) = \lambda_0(z)\omega^\circ(z).$$

Now, $G(z; L-1 \mid x_0)$ can easily be expressed in terms of $\varphi_j(z)$, as

$$\begin{aligned} G(z; L-1 \mid x_0) &= \\ &= \frac{1}{1 - \Phi(z)} \left[\sum_{k=0}^{L-2} \delta_{k+1, x_0}^\circ(z) (1 - \varphi_{k-1}(z)) \lambda^{L-k-2}(z) \right. \\ &\quad \left. + \rho^\circ(z) \lambda^{L-2}(z) \delta_{0, x_0} + (1 - \varphi_{L-3}(z)) \omega^\circ(z) \delta_{L, x_0} \right], \end{aligned} \quad (5)$$

where

$$\Phi(z) = \varphi_{L-2}(z) + \varphi_L^*(z)(1 - \varphi_{L-3}(z)). \quad (6)$$

Reversing the recursive process which leads to (2), we obtain general expressions for all $G(z; x \mid x_0)$, $x, x_0 \in S$ in the form

$$G(z; x \mid x_0) = \frac{1}{1 - \Phi(z)} \left\{ \begin{aligned} & (1 - \varphi_{x-2}(z)) (1 - \varphi_{x+2}^*(z)) \delta_{x,x_0}^\circ(z) \\ & + (1 - \varphi_{x+2}^*(z)) \left[\rho^\circ(z) \lambda^{x-1}(z) \delta_{0,x_0} \right. \\ & \quad \left. + \sum_{k=0}^{x-2} \delta_{k+1,x_0}^\circ(z) (1 - \varphi_{k-1}(z)) \lambda^{x-k-1}(z) \right] \\ & + (1 - \varphi_{x-2}(z)) \left[\omega^\circ(z) \delta_{L,x_0} \mu^{L-x-1}(z) \right. \\ & \quad \left. + \sum_{k=0}^{L-x-2} \delta_{L-k-1,x_0}^\circ(z) (1 - \varphi_{L-k+1}^*(z)) \mu^{L-x-k-1}(z) \right] \end{aligned} \right\} \quad (7)$$

for $x \in \{2, 3, \dots, L-2\}$,

$$\begin{aligned} G(z; 1 \mid x_0) &= \frac{1}{1 - \Phi(z)} \left[(1 - \varphi_3^*(z)) \rho^\circ(z) \delta_{0,x_0} \right. \\ & \quad \left. + \sum_{k=0}^{L-2} \delta_{L-k-1,x_0}^\circ(z) (1 - \varphi_{L-k+1}^*(z)) \mu^{L-k-2}(z) \right. \\ & \quad \left. + \omega^\circ(z) \delta_{L,x_0} \mu^{L-2}(z) \right], \end{aligned} \quad (8)$$

$$\begin{aligned} G(z; L-1 \mid x_0) &= \frac{1}{1 - \Phi(z)} \left[\rho^\circ(z) \delta_{0,x_0} \lambda^{L-2}(z) + \sum_{k=0}^{L-2} \delta_{k+1,x_0}^\circ(z) (1 - \varphi_{k-1}(z)) \lambda^{L-k-2}(z) \right. \\ & \quad \left. + \omega^\circ(z) \delta_{L,x_0} (1 - \varphi_{L-3}(z)) \right], \end{aligned} \quad (9)$$

$G(z; 0 \mid x_0)$ and $G(z; L \mid x_0)$ can be easily obtained from formulae (2.6), (8) and (9), where $\varphi_j^*(z)$ satisfies the following recurrence relations

$$\varphi_{L-j}^*(z) = \begin{cases} \lambda_0(z) \omega^\circ(z), & j = 0 \\ \varphi_{L-j+1}^*(z) + \lambda(z) \mu(z) (1 - \varphi_{L-j+2}^*(z)), & j = 1, 2, \dots, L-2 \\ \varphi_2^*(z) + \mu_0(z) \rho^\circ(z) (1 - \varphi_3^*(z)), & j = L-1. \end{cases} \quad (10)$$

Explicit expressions for $\varphi_j(z)$, $\varphi_j^*(z)$ are immediately obtained by the standard method of solution of the second order linear recurrence relation (cf. MICKEN (1990) and ELAYDI (1999)):

$$1 - \varphi_j(z) = \frac{1}{\theta_1 - \theta_2} \left[\left(\theta_1^{j+2} - \theta_2^{j+2} \right) - z \rho \sqrt{\frac{\beta}{\alpha}} \left(\theta_1^{j+1} - \theta_2^{j+1} \right) \right] \left(\frac{z \sqrt{\alpha \beta}}{1 - z \gamma} \right)^{j+1}, \quad (11)$$

$$1 - \varphi_{L-j}^*(z) = \frac{1}{\theta_1 - \theta_2} \left[\left(\theta_1^{j+2} - \theta_2^{j+2} \right) - z\omega \sqrt{\frac{\alpha}{\beta}} \left(\theta_1^{j+1} - \theta_2^{j+1} \right) \right] \left(\frac{z\sqrt{\alpha\beta}}{1 - z\gamma} \right)^{j+1}, \quad (12)$$

where θ_1, θ_2 are given as (2.12). We assume that z is real and positive and that the function inside the square root (2.12) is positive, i.e.

$$(1 - z\gamma)^2 > 4z^2\alpha\beta \quad \text{or} \quad 0 < z < \frac{1}{1 - (\sqrt{\alpha} - \sqrt{\beta})^2}.$$

Inserting formulae (11) and (12) into (6), we obtain

$$1 - \Phi(z) = \frac{1}{\theta_1 - \theta_2} \Delta_0(z; \alpha, \beta, \rho, \omega) \left(\frac{z\sqrt{\alpha\beta}}{1 - z\gamma} \right)^{L-1}, \quad (13)$$

where

$$\begin{aligned} \Delta_0(z; \alpha, \beta, \rho, \omega) &= \theta_1^L - \theta_2^L - \frac{z}{\sqrt{\alpha\beta}} (\beta\rho + \alpha\omega) (\theta_1^{L-1} - \theta_2^{L-1}) \\ &\quad + \rho\omega z^2 (\theta_1^{L-2} - \theta_2^{L-2}) \\ &= \Delta_0(z; \beta, \alpha, \omega, \rho) \end{aligned} \quad (14)$$

which will be abbreviated to Δ_0 .

Formulae (2.6), (7)–(9), and (11)–(14) enable us to find explicit expressions for all $G(z; x \mid x_0)$, $x, x_0 \in S$ as

$$\begin{aligned} G(z; 0 \mid x_0) &= \\ &= \frac{1}{\Delta_0} \begin{cases} \Delta_0 + z\rho\sqrt{\frac{\beta}{\alpha}} \left[\theta_1^{L-1} - \theta_2^{L-1} - z\omega\sqrt{\frac{\alpha}{\beta}} (\theta_1^{L-2} - \theta_2^{L-2}) \right], & x_0 = 0 \\ \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}x_0} \left[\theta_1^{L-x_0} - \theta_2^{L-x_0} - z\omega\sqrt{\frac{\alpha}{\beta}} (\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1}) \right], & x_0 \neq 0, L \\ z\omega (\theta_1 - \theta_2) \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}(L-1)}, & x_0 = L, \end{cases} \end{aligned} \quad (15)$$

$$\begin{aligned} G(z; x \mid x_0) &= \frac{\left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}(x_0-x)}}{z\sqrt{\alpha\beta} (\theta_1 - \theta_2) \Delta_0} \times \\ &\times \begin{cases} \left[\theta_1^{x-x_0} - \theta_2^{x-x_0} - z\rho\sqrt{\frac{\beta}{\alpha}} (\theta_1^{x-1-x_0} - \theta_2^{x-1-x_0}) \right] \left[\theta_1^{L-x_0} - \theta_2^{L-x_0} - z\omega\sqrt{\frac{\alpha}{\beta}} (\theta_1^{L-x_0-1} - \theta_2^{L-x_0-1}) \right], \\ x_0 \in S, \quad x_0 \neq 0, \quad x \in \{1, 2, \dots, x_0\} \\ \left[\theta_1^{x_0-x} - \theta_2^{x_0-x} - z\rho\sqrt{\frac{\beta}{\alpha}} (\theta_1^{x_0-1-x} - \theta_2^{x_0-1-x}) \right] \left[\theta_1^{L-x} - \theta_2^{L-x} - z\omega\sqrt{\frac{\alpha}{\beta}} (\theta_1^{L-x-1} - \theta_2^{L-x-1}) \right] \left(\frac{\alpha}{\beta} \right)^{x-x_0}, \\ x_0 \in S, \quad x_0 \neq L, \quad x \in \{x_0, x_0 + 1, x_0 + 2, \dots, L-1\} \end{cases} \end{aligned} \quad (16)$$

and

$$\begin{aligned}
 G(z; L \mid x_0) &= \\
 &= \frac{1}{\Delta_0} \begin{cases} z\rho(\theta_1 - \theta_2) \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-1)}, & x_0 = 0 \\ \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}(L-x_0)} \left[\theta_1^{x_0} - \theta_2^{x_0} - z\rho\sqrt{\frac{\beta}{\alpha}}(\theta_1^{x_0-1} - \theta_2^{x_0-1}) \right], & x_0 \neq 0, L \\ \Delta_0 + z\omega\sqrt{\frac{\alpha}{\beta}} \left[\theta_1^{L-1} - \theta_2^{L-1} - z\rho\sqrt{\frac{\beta}{\alpha}}(\theta_1^{L-2} - \theta_2^{L-2}) \right], & x_0 = L \end{cases} \quad (17)
 \end{aligned}$$

where Δ_0 , θ_1 and θ_2 are respectively given as in formulae (14) and (2.12). Obviously, formula (17) can easily be obtained from formula (15) on replacing $\alpha, \beta, \rho, \omega$ and x_0 by $\beta, \alpha, \omega, \rho$ and $L - x_0$, respectively.

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Department of Mathematics
Damietta Faculty of Science
P.O. Box 6
New Damietta
EGYPT
E-mail: el_shehawy@mans.edu.eg