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TWISTED SIGNATURES OF FLAG MANIFOLDS

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ABSTRACT. A product formula for some twisted signatures of flag manifolds is proved. The result is used to compute twisted signatures of some flag manifolds from those of Grassmannians, and by that to deduce some upper bounds of the stable span.

1. Introduction

Two real vector bundles ξ and η over a space X are called stably equivalent, if there exist trivial vector bundles ϵ^r and ϵ^s over X, such that $\xi \oplus \epsilon^r$ and $\eta \oplus \epsilon^s$ are isomorphic. The geometric dimension of a real vector bundle ξ is the minimal fibre dimension of a vector bundle η, such that η is stably equivalent to ξ. The difference of the fibre dimension of ξ and its geometric dimension is called the stable span of ξ. Now let M be a closed smooth connected manifold. The stable span of M, denoted by $\operatorname{stable\,span}(M)$, is defined as the stable span of the tangent bundle of M. The span of M, denoted by $\operatorname{span}(M)$, is by definition the maximal number of vector fields on M which are linearly independent in all points of M. Clearly $\operatorname{span}(M) \leq \operatorname{stable\,span}(M)$. Methods and results concerning *span* and stable span are described in the survey article [8] by J. Korbaš and P. Zvengrowski. When M is orientable and has dimension divisible by 4, then the signature of M may be used to determine upper bounds for $\operatorname{stable\,span}(M)$ ([1], [6], [10]). Actually it is possible to use twisted signatures

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for this purpose. Sometimes twisted signatures allow to get results for a manifold, where the signature fails. We state the theorem for twisted signatures in the form, in which it will be used in the sequel.

In Section 1 we assume, that M is a closed oriented smooth manifold of dimension $2n$. The groups $K(M)$, $KO(M)$ and $KSp(M)$ are the K-groups of complex, real, and quaternionic vector bundles over M. The inclusions $i\colon O(l) \to U(l)$ and $j\colon Sp(l) \to U(2l)$ define homomorphisms $i_*\colon KO(M) \to K(M)$ and $j_*\colon KSp(M) \to K(M)$. $ch\colon K(M) \to H^*(M;\mathbb{Q})$ is the Chern character homomorphism. We write $ch(M)$ instead of $ch(K(M))$, $ch\,O(M)$ for $ch(i_*KO(M))$, and $ch\,Sp(M)$ for $ch(j_*KSp(M))$. All $z \in ch(M)$ have the form $z = \sum_{i=0}^{n} a_i$ with $a_i \in H^{2i}(M;\mathbb{Q})$. As usual we define $z^{(2)} = \sum_{i=0}^{n} 2^i a_i$. Let $\mathscr{L}(M)$ denote the Hirzebruch $\mathscr{L}$-class of M associated to the power series $x/tanh(x)$. The signature of M twisted by z is defined as the rational number

$$L(M,z) = \left\langle z^{(2)}\mathscr{L}(M), [M]\right\rangle$$

which in fact is an integer.

It is well known that $\operatorname{stable\,span}(M) = r > 0$ implies divisibility conditions for the signature and also for the twisted signatures. These divisibility conditions may be deduced from a general integrality theorem, which was proved in [10] by using elliptic differential operators. For the signature these conditions are stated and proved by other methods in [1] and [6]. For the twisted signatures with $z \in ch(M)$ and $z \in chO(M)$ a proof by the same methods as in [1] was given in [4]. A similar theorem exists with $z \in ch\,Sp(M)$. Once the signature or a twisted signature is known and different from zero, the power of 2 in the integer factorization gives rise to upper bounds for the stable span. To state the theorem three series of integers are introduced.

$$\begin{array}{lllll} a_1 = 1, & a_2 = 2, & a_3 = 4, & a_4 = 8, & a_{i+4} = a_i + 8 \\ a'_1 = 1, & a'_2 = 2, & a'_3 = 3, & a'_4 = 7, & a'_{i+4} = a'_i + 8 \\ b_1 = 0, & b_2 = 4, & b_3 = 5, & b_4 = 6, & b_{i+4} = b_i + 8. \end{array}$$

The series a'_i differs from the series a_i for $i \equiv 0, 3 \mod 4$. This difference comes from the fact that for $\operatorname{stable\,span}(M)$ divisible by 4 an additional factor 2 occurs in the signature, which is not necessarily contained in the twisted signature. Using the divisibility conditions mentioned above, one proves the following theorem.

Proposition 1.1. *Let M be a closed oriented smooth manifold of dimension $2n, z \in ch(M)$, and assume that $L(M,z) = 2^i u$, where u is an odd integer and $i \geq 0$. Then* $\operatorname{stable\,span}(M) \leq 2i$.

For n even, and $i > 0$, and additional assumptions the following stronger inequalities hold.

(i) *When* $z \in ch\, O(M)$, *then* $\operatorname{stable\,span}(M) \leq a_i$.

(ii) *When* $z = 1$, *then* $\operatorname{stable\,span}(M) \leq a'_i$.

(iii) *When* $z \in ch\, Sp(M)$, *then* $\operatorname{stable\,span}(M) \leq b_i$.

The proof is a consequence of the divisibility conditions for twisted signatures resp. signatures.

Since the signature is a multiplicative genus, it is computable for the flag manifolds once it is known for the Grassmannians. The signatures of the Grassmannians were computed by Mong [14] and Shanahan [15], using the Atiyah-Bott-Singer fixed point theorem. We use the same technique to compute some twisted signatures of Grassmannians, and to show that certain twisted signatures of flag manifolds occur as the product of an appropriate twisted signature of a Grassmannian with signatures of Grassmannians.

In Section 2 we establish a product theorem for certain twisted signatures of flag manifolds. In Section 3 we compute some twisted signatures of the Grassmannians. In Section 4 upper bounds for the stable span of some flag manifolds are established.

The way of computation is described in [11], and for details of the method we refer to that paper. We use the opportunity to correct an error in [11]. In equation (2.4) on the right hand side a sign $(-1)^n$ is missing. This change effects a change of signs in the case of complex Grassmannians, but does not effect the results of the paper.

2. A product formula for some twisted signatures of flag manifolds

First the fixed point formula is stated, which will be used in the sequel. Let M be an oriented closed smooth manifold of dimension $2n$, and assume that M admits a smooth action of the circle group S^1 with only a finite number of fixed points $p_1, \ldots, p_s$. The restriction of the S^1-action on the tangent bundle TM to the fibre M_{p_i} is completely determined by a set of, not necessarily distinct, positive integers $m_1, \ldots, m_n$. It is possible to choose the orientation of M_{p_i} so that M_{p_i} can be expressed as a direct sum

$$M_{p_i} = \bigoplus_{j=1}^{n} V(m_j), \tag{2.1}$$

where the $V(m_j)$ are 1-dimensional complex vector spaces, and S^1 operates on $V(m_j)$ by $g \cdot v = g^{m_j} v$ for all $g \in S^1$ and all $v \in V(m_j)$. Let η be an element of the equivariant K-group of M. The restriction of η to p_i gives an element of the complex representation ring $R(S^1)$ with character χ_i. The local expression, which occurs in the fixed point formula leading to the twisted signature is

$$\gamma_i(g) = \epsilon_i \chi_i(g) \prod_{j=1}^{n} \frac{g^{-m_j/2} + g^{m_j/2}}{g^{-m_j/2} - g^{m_j/2}} \tag{2.2}$$

where $\epsilon_i = 1$, when the natural orientation of M_{p_i} is the same as the orientation defined by the complex structure introduced in (2.1), and $\epsilon_i = -1$, when M_{p_i} has the opposite orientation. The following formula is proved by using the Atiyah–Singer fixed point formula [3] and ideas used in [2]. For all details we refer to [11] and [12].

Theorem 2.1. *With notations introduced above, $z = ch(\eta)$, and*

$$\Gamma(g) = \sum_{i=1}^{s} \gamma_i(g), \tag{2.3}$$

the following equality holds

$$L(M, z) = (-1)^n \lim_{g \to 1} \Gamma(g). \tag{2.4}$$

The theorem will be applied to the computation of twisted signatures of flag manifolds. Let $\mathbb{F}$ be either the field of real numbers, the field of complex numbers or the skew field of quaternions, and let $d(\mathbb{F}) = \dim_{\mathbb{R}}(\mathbb{F})$. The vector space $\mathbb{F}^n$ is equipped with the usual scalar product. When $\mathbb{F} = \mathbb{H}$, all vector spaces over $\mathbb{H}$ are assumed to be right vector spaces. For all s-tuples $(n_1, \dots, n_s)$ of positive integers and $n = n_1 + \dots + n_s$ the flag-manifold $G_{\mathbb{F}}(n_1, \dots, n_s)$ is defined as the space of mutually orthogonal subspaces $(X_1, \dots, X_s)$ of $\mathbb{F}^n$, and $\dim_{\mathbb{F}} X_i = n_i$. The manifolds $G_{\mathbb{F}}(n_1, \dots, n_s)$ are in a natural way closed smooth manifolds of dimension $d(\mathbb{F}) = \sum_{i<j} n_i n_j$ (see [9]). There are natural vector bundles $\xi_1, \dots, \xi_s$ over $G_{\mathbb{F}}(n_1, \dots, n_s)$. The bundle ξ_i is defined as the $\mathbb{F}$-vector bundle whose fibre at the point $(X_1, \dots, X_s)$ is the vector space X_i. According to Lam [9] the tangent bundle of $G_{\mathbb{F}}(n_1, \dots, n_s)$ is isomorphic to $\bigoplus_{1 \le i < j \le s} \mathrm{Hom}_{\mathbb{F}}(\xi_i, \xi_j)$ as $Z(\mathbb{F})$-vector bundles, where $Z(\mathbb{F})$ denotes the center of $\mathbb{F}$.

Since in the real case we consider only flag manifolds with n_i even, it will be sometimes convenient to use the unified notation $G(n_1, \dots, n_s)$ for the flag manifolds $G_{\mathbb{F}}(n_1, \dots, n_s)$, when $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, and for $G_{\mathbb{R}}(2n_1, \dots, 2n_s)$. The symbol $U_{\mathbb{F}}(l)$ will be used for one of the groups $U(l)$, $Sp(l)$, or $O(2l)$ according as $\mathbb{F}$ is equal to $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{R}$. Every finite dimensional complex representation α

of the group $U_{\mathbb{F}}(n_1) \times \cdots \times U_{\mathbb{F}}(n_s)$ defines in a natural way a complex vector bundle $\alpha(\xi_1 \oplus \cdots \oplus \xi_s)$ over $G(n_1, \ldots, n_s)$.

Now assume that $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ and let $d_1, \ldots, d_n$ be positive integers and $d_1 < d_2 < \cdots < d_n$. An action of S^1 on $\mathbb{F}^n$ is defined by $g \cdot (z_1, \ldots, z_n) = (g^{d_1} z_1, \ldots, g^{d_n} z_n)$. This action defines a smooth S^1-action on the flag manifold $G_{\mathbb{F}}(n_1, \ldots, n_s)$. Let $N_0 = 0$ and $N_i = n_1 + n_2 + \cdots + n_i$ for $1 \leq i \leq s$. The fixed points of this action are the points $P(\sigma) = \big([e_{\sigma(1)}, \ldots, e_{\sigma(N_1)}], \ldots, [e_{\sigma(N_{s-1}+1)}, \ldots, e_{\sigma(N_s)}]\big)$ for all permutions $\sigma \in S_n$ with $\sigma(N_{i-1}+1) < \sigma(N_{i-1}+2) < \ldots < \sigma(N_i)$ for all $i \in \{1, \ldots, s\}$, and $(e_1, \ldots, e_n)$ the standard base of $\mathbb{F}^n$. By this action the vector bundles $\xi_1, \ldots, \xi_s$ and $\mathrm{Hom}_{\mathbb{F}}(\xi_i, \xi_j)$ become equivariant vector bundles and the isomorphism between the tangent bundle of $G_{\mathbb{F}}(n_1, \ldots, n_s)$ and $\bigoplus_{i<j} \mathrm{Hom}_{\mathbb{F}}(\xi_i, \xi_j)$ is an equivariant isomorphism. The bundle $\alpha(\xi_1 \oplus \cdots \oplus \xi_s)$, which is defined by the complex representation α is in a natural way an equivariant vector bundle.

When $\mathbb{F} = \mathbb{C}$, then $\mathrm{Hom}_{\mathbb{C}}(\xi_i, \xi_j)$ is a complex vector bundle and the orientation of $G_{\mathbb{C}}(n_1, \ldots, n_s)$ is determined by the complex structure. A complex base of

$$\mathrm{Hom}_{\mathbb{C}}([e_{\sigma(N_{i-1}+1)}, \ldots, e_{\sigma(N_i)}], [e_{\sigma(N_{j-1}+1)}, \ldots, e_{\sigma(N_j)}])$$

is given by $f_{\kappa\lambda}$, $1 \leq \kappa \leq n_i$, $1 \leq \lambda \leq n_j$, where $f_{\kappa\lambda}$ is the uniquely defined linear map with $f_{\kappa\lambda}(e_{\sigma(N_{i-1}+\mu)}) = \delta_{\mu\kappa} e_{\sigma(N_{j-1}+\lambda)}$. For any $g \in S^1$ we find that $(g \cdot f_{\kappa\lambda})(x) = g^{d_{\sigma(N_{j-1}+\lambda)} - d_{\sigma(N_{i-1}+\kappa)}} f_{\kappa\lambda}(x)$. The restriction of the S^1-action on ξ_i to the fibre over $P(\sigma)$ gives an element of $R(S^1)$ with character $p_{i\sigma}(g) = g^{d_{\sigma(N_{i-1}+1)}} + \cdots + g^{d_{\sigma(N_i)}}$. The restriction of the S^1-action on $\Lambda^{n_i} \xi_i$ to the fibre over $P(\sigma)$ gives an element of $R(S^1)$ with character $q_{i\sigma}(g) = g^{d_{\sigma(N_{i-1}+1)} + \cdots + d_{\sigma(N_i)}}$. For the general computations, let α be a complex representation of $U(n_1) \times \cdots \times U(n_s)$, and denote by $\chi_\sigma(g)$ the character of the complex representation of S^1, which is defined by the restriction of $\alpha(\xi_1 \oplus \cdots \oplus \xi_s)$ to the fibre over the fixed point $P(\sigma)$. Using these notations the function $\Gamma(g)$ of the theorem has the form

$$\begin{aligned}
&\Gamma(n_1, \ldots, n_s, d_1, \ldots, d_n; g) \\
&= \sum_{\sigma} \chi_\sigma(g) \prod_{1 \leq i < j \leq s} \prod_{\substack{N_{i-1} < \kappa \leq N_i \\ N_{j-1} < \lambda \leq N_j}} \frac{g^{-(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2} + g^{(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2}}{g^{-(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2} - g^{(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2}} \\
&= \sum_{\sigma} \chi_\sigma(g) \prod_{1 \leq i < j \leq s} \prod_{\substack{N_{i-1} < \kappa \leq N_i \\ N_{j-1} < \lambda \leq N_j}} \frac{g^{d_{\sigma(\lambda)}} + g^{d_{\sigma(\kappa)}}}{g^{d_{\sigma(\lambda)}} - g^{d_{\sigma(\kappa)}}}.
\end{aligned}$$

When $\mathbb{F} = \mathbb{H}$, we fix an orientation of $G_{\mathbb{H}}(n_1, \dots, n_s)$ by fixing an orientation of $\mathrm{Hom}_{\mathbb{H}}(\xi_i, \xi_j)$. Take $X = (X_1, \dots, X_s) \in G_{\mathbb{H}}(n_1, \dots, n_s)$ and let $a_1, \dots, a_{n_i}$ be a $\mathbb{H}$-base of $X_i = [a_1, \dots, a_{n_i}]$ and let $b_1, \dots, b_{n_j}$ be an $\mathbb{H}$-base of $X_j = [b_1, \dots, b_{n_j}]$. Let $f^{\rho}_{\mu\nu}$ be the uniquely defined $\mathbb{H}$-homomorphism with $f^{\rho}_{\mu\nu}(a_\kappa) = \delta_{\kappa\mu} b_\nu \rho$ for all $\kappa, \mu \in \{1, \dots, n_i\}$, $\nu \in \{1, \dots, n_j\}$ and $\rho \in \{1, i, j, k\}$. The $f^{\rho}_{\mu\nu}$ are a real base of $\mathrm{Hom}_{\mathbb{H}}(X_i, X_j)$. We use the orientation defined by

$$(f^1_{11}, f^i_{11}, f^j_{11}, f^k_{11}, f^1_{12}, f^i_{12}, \dots, f^k_{n_i, n_j}).$$

The reader should not be confused by the fact that the upper indices i, j denote the canonical base element of $\mathbb{H}$ and the lower indices are positive integers. In the fixed points $P(\sigma)$ we use this orientation, and take $a_\kappa = e_{\sigma(N_{i-1}+\kappa)}$, $b_\lambda = e_{\sigma(N_{j-1}+\lambda)}$. Then $(f^1_{11}, f^j_{11}, \dots, f^1_{n_i n_j}, f^j_{n_i n_j})$ is a complex base with $g \cdot f^1_{\kappa\lambda} = g^{d_{\sigma(N_{j-1}+\lambda)} - d_{\sigma(N_{i-1}+\kappa)}} f^1_{\kappa\lambda}$ and $g \cdot f^j_{\kappa\lambda} = g^{d_{\sigma(N_{j-1}+\lambda)} + d_{\sigma(N_{i-1}+\kappa)}} f^j_{\kappa\lambda}$. The restriction of the S^1-action on ξ_i to the fibre over $P(\sigma)$ gives an element of $R(S^1)$ with character

$$p_{i\sigma}(g) = \sum_{N_{i-1} < \kappa \le N_i} (g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}}).$$

For later computations we remark that there exists an element $\alpha_i \in R(Sp(n_i))$, such that the restriction of $\alpha_i(\xi_i)$ to $P(\sigma)$ has character $q_{i\sigma}$ with

$$q_{i\sigma}(g) = \prod_{N_{i-1} < \kappa \le N_i} (g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}})$$

(see e.g. Husemoller [7]).

Now let α be a complex representation of $Sp(n_1) \times \dots \times Sp(n_s)$, and let $\chi_\sigma(g)$ be the character of the representation of S^1, which is defined by the restriction of $\alpha(\xi_1 \oplus \dots \oplus \xi_s)$ to the fixed point $P(\sigma)$. Using these notations, the function $\Gamma(g)$ of the theorem has the form

$$\begin{aligned}
&\Gamma(n_1, \dots, n_s, d_1, \dots, d_n; g) \\
&= \sum_\sigma \chi_\sigma(g) \prod_{1 \le i < j \le s} \prod_{\substack{N_{i-1} < \kappa \le N_i \\ N_{j-1} < \lambda \le N_j}} \frac{g^{-(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2} + g^{(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2}}{g^{-(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2} - g^{(d_{\sigma(\lambda)} - d_{\sigma(\kappa)})/2}} \times \\
&\qquad \times \frac{g^{-(d_{\sigma(\lambda)} + d_{\sigma(\kappa)})/2} + g^{(d_{\sigma(\lambda)} + d_{\sigma(\kappa)})/2}}{g^{-(d_{\sigma(\lambda)} + d_{\sigma(\kappa)})/2} - g^{(d_{\sigma(\lambda)} + d_{\sigma(\kappa)})/2}} \\
&= \sum_\sigma \chi_\sigma(g) \prod_{1 \le i < j \le s} \prod_{\substack{N_{i-1} < \kappa \le N_i \\ N_{j-1} < \lambda \le N_j}} \frac{g^{d_{\sigma(\lambda)}} + g^{-d_{\sigma(\lambda)}} + g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}}}{(g^{d_{\sigma(\lambda)}} + g^{-d_{\sigma(\lambda)}}) - (g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}})}.
\end{aligned}$$

When $\mathbb{F} = \mathbb{R}$, we consider only the flag manifolds $G_{\mathbb{R}}(2n_1, \dots, 2n_s)$, with $n_1 + \dots + n_s = n$. Let $e_1, e_2, \dots, e_{2n}$ be the standard base of $\mathbb{R}^{2n}$. For any $e^{i\varphi} \in S^1$

the action on $\mathbb{R}^{2n}$ is defined by $e^{i\varphi} \cdot e_{2j-1} = \cos(d_j\varphi)e_{2j-1} + \sin(d_j\varphi)e_{2j}$, and $e^{i\varphi} \cdot e_{2j} = -\sin(d_j\varphi)e_{2j-1} + \cos(d_j\varphi)e_{2j}$, $j = 1, \dots, n$, and positive integers d_j as before. This action on $\mathbb{R}^{2n}$ defines a smooth action of S^1 on $G_{\mathbb{R}}(2n_1, \dots, 2n_s)$ with fixed points $P(\sigma) = (X_1, \dots, X_s)$ with

$$X_i = [e_{2\sigma(N_{i-1}+1)-1}, e_{2\sigma(N_{i-1}+1)}, \dots, e_{2\sigma(N_i)-1}, e_{2\sigma(N_i)}],$$

and $\sigma \in S_n$ such that $\sigma(N_{i-1}+1) < \cdots < \sigma(N_i)$ for all $i \in \{1, \dots, s\}$. We define an orientation in the tangent bundle of $G_{\mathbb{R}}(2n_1, \dots, 2n_s)$ by fixing an orientation of the real vector bundles $\mathrm{Hom}(\xi_i, \xi_j)$. For $X = (X_1, \dots, X_s) \in G_{\mathbb{R}}(2n_1, \dots, 2n_s)$ let $a_1, \dots, a_{2n_i}$ be an orthonormal base of X_i and let $b_1, \dots, b_{2n_j}$ be an orthonormal base for X_j. Then let $f_{\kappa\lambda}\colon X_i \to X_j$ be the uniquely defined linear map with $f_{\kappa\lambda}(a_\mu) = \delta_{\kappa\mu} b_\lambda$, and take the orientation of $\mathrm{Hom}(X_i, X_j)$ given by the base

$$(f_{11}, f_{12}, \dots, f_{1n_j}, f_{21}, \dots, f_{2n_i 2n_j}).$$

A short computation shows that the rotation numbers of $\mathrm{Hom}_{\mathbb{R}}(\xi_i, \xi_j)$ are $d_{\sigma(\kappa)} + d_{\sigma(\lambda)}$ and $|d_{\sigma(\lambda)} - d_{\sigma(\kappa)}|$, for $N_{i-1} < \kappa \leq N_i$ and $N_{j-1} < \lambda \leq N_j$. The orientation determined by the complex structure according to these positive numbers differs from the orientation defined above by $(-1)^{W(i,j,\sigma)}$, where $W(i,j,\sigma)$ denotes the number of pairs (κ, λ), $N_{i-1} < \kappa \leq N_i$, $N_{j-1} < \lambda \leq N_j$, such that $\kappa < \lambda$ and $\sigma(\kappa) > \sigma(\lambda)$. The restriction of $\xi_i \otimes \mathbb{C}$ to the fibre over the fixed point $P(\sigma)$ gives an element $R(S^1)$ with character

$$p_{i\sigma}(g) = \sum_{N_{i-1} < \kappa \leq N_i} (g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}}).$$

Just as in the symplectic case there exists an element $\alpha_i \in R(O(2n_i))$, such that the restriction of the S^1-action on the virtual bundle $\alpha_i(\xi_i)$ to $P(\sigma)$ gives rise to an element of $R(S^1)$ with character

$$q_{i\sigma}(g) = \prod_{N_{i-1} < \kappa \leq N_i} (g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}})$$

(see [7]).

Now let α be a complex representation of $O(2n_1) \times \cdots \times O(2n_s)$ and let χ_σ be the character of the restriction of $\alpha(\xi_1 \oplus \cdots \oplus \xi_s)$ to the fibre over the fixed point $P(\sigma)$. With these notations the function $\Gamma(g)$ of the theorem is the same as that for the quaternionic flag manifold $G_{\mathbb{H}}(n_1, \dots, n_s)$.

In the sequel the unified notation $G(n_1, \dots, n_s)$ will be used for all three types of flag manifolds. Let $b_i = g^{d_i}$ in the case of complex flag manifolds and $b_i = g^{d_i} + g^{-d_i}$ in the case of quaternionic or real flag manifolds. We take $t \in \{1, \dots, s-1\}$, $k = n_1 + \cdots + n_t$, and choose a complex representation α of $U_{\mathbb{F}}(n_1 + \cdots + n_t)$. The restriction of the equivariant complex vector bundle

$\alpha(\xi_1 \oplus \cdots \oplus \xi_t)$, to the fixed point $P(\sigma)$ give rise to a complex representation of S^1 with characters $\chi_\sigma(g)$. When the element η of the equivariant complex K-group of $G(n_1, \ldots, n_s)$ is chosen as $\alpha(\xi_1 \oplus \cdots \oplus \xi_t)$, then the expression $\Gamma(g)$ in (2.3) may be written for all three types of flag manifolds in the unified form

$$\Gamma(n_1, \ldots, n_s, d_1, \ldots, d_n; g) = \sum_\sigma \chi_\sigma(g) \prod_{1 \leq i < j \leq s} \prod_{\substack{N_{i-1} < \kappa \leq N_i \\ N_{j-1} < \lambda \leq N_j}} \frac{b_{\sigma(\lambda)} + b_{\sigma(\kappa)}}{b_{\sigma(\lambda)} - b_{\sigma(\kappa)}}.$$

To deduce the product formula, we use the fact, that the sum over all $\sigma \in S_n$ such that $\sigma(N_{i-1}+1) < \cdots < \sigma(N_i)$ for all $i \in \{1, \ldots, s\}$ gives the same result as when one takes the sum over all $\tau \in S_n$ such that $\tau(1) < \cdots < \tau(N_t)$ and $\tau(N_t + 1) < \cdots < \tau(N_s)$ followed by the sum over all permutations ρ of the numbers $\tau(1), \ldots, \tau(N_t)$ and ζ of $\tau(N_t+1), \ldots, \tau(N_s)$ such that $\rho(\tau(1)) < \cdots < \rho(\tau(N_1)), \ldots, \rho(\tau(N_{t-1}+1) < \cdots < \rho(\tau(N_t))$ and $\zeta(\tau(N_t+1)) < \ldots < \zeta(\tau(N_{t+1})), \ldots, \zeta(\tau(N_{s-1}+1)) < \cdots < \zeta(\tau(N_s))$. Using this splitting, we find

$$\begin{aligned}
&\Gamma(n_1, \ldots, n_s, d_1, \ldots, d_n; g) \\
&= \sum_\sigma \left(\chi_\sigma(g) \prod_{\substack{1 \leq \kappa \leq k \\ k+1 \leq \lambda \leq n}} \frac{b_{\sigma(\lambda)} + b_{\sigma(\kappa)}}{b_{\sigma(\lambda)} - b_{\sigma(\kappa)}} \right) \left(\prod_{1 \leq i < j \leq t} \prod_{\substack{N_{i-1} < \kappa \leq N_i \\ N_{j-1} < \lambda \leq N_j}} \frac{b_{\sigma(\lambda)} + b_{\sigma(\kappa)}}{b_{\sigma(\lambda)} - b_{\sigma(\kappa)}} \right) \times \\
&\qquad \times \left(\prod_{t+1 \leq i < j \leq s} \prod_{\substack{N_{i-1} < \kappa \leq N_i \\ N_{j-1} < \lambda \leq N_j}} \frac{b_{\sigma(\lambda)} + b_{\sigma(\kappa)}}{b_{\sigma(\lambda)} - b_{\sigma(\kappa)}} \right) \\
&= \sum_\tau \left(\chi_\tau(g) \prod_{\substack{1 \leq \kappa \leq k \\ k+1 \leq \lambda \leq n}} \frac{b_{\tau(\lambda)} + b_{\tau(\kappa)}}{b_{\tau(\lambda)} - b_{\tau(\kappa)}} \right) \Gamma(n_1, \ldots, n_t, d_{\tau(1)}, \ldots, d_{\tau(k)}; g) \times \\
&\qquad \times \Gamma(n_{t+1}, \ldots, n_s, d_{\tau(k+1)}, \ldots, d_{\tau(n)}; g).
\end{aligned}$$

According to [2] and [12] the last two factors are constant, and except for a sign equal to the signature of $G(n_1, \ldots, n_t)$ resp. $G(n_{t+1}, \ldots, n_s)$. The first factor has for g going to 1 the limit

$$(-1)^{\frac{1}{2} \dim G(k, n-k)} L\big(G(k, n-k), ch(\alpha(\gamma_k))\big),$$

where γ_k is the canonical k-dimensional $\mathbb{F}$-vector bundle over $G(k, n-k)$. Using the notations fixed before, we can state the result.

Proposition 2.1. *Let $n_1, \dots, n_s$ be positive integers, $n = n_1 + \cdots + n_s$, $t \in \{1, \dots, s-1\}$ and $k = n_1 + \cdots + n_t$, and let α be a complex representation of the group $U_{\mathbb{F}}(n_1 + \cdots + n_t)$. Then the following product formula holds*

$$\begin{aligned} &L\big(G(n_1, \dots, n_s), ch(\alpha(\xi_1 \oplus \cdots \oplus \xi_t))\big) \\ &= L\big(G(k, n-k), ch(\alpha(\gamma_k))\big) \times L\big(G(n_1, \dots, n_t)\big) \times L\big(G(n_{t+1}, \dots, n_s)\big). \end{aligned}$$

3. Twisted signatures of Grassmannians

Before starting the computations a lemma is stated, which will be used later, and which may be proved by induction.

Lemma 3.1. *Let S_n denote the symmetric group of n elements, and let $k \in \{1, \dots, n\}$. Then the sum $\sum(-1)^{\sigma(1)+\cdots+\sigma(k)}$ over all elements $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$ is equal to*

$$(-1)^{k(k+1)/2} \begin{pmatrix} [\frac{k}{2}] & + & [\frac{n-k}{2}] \\ & [\frac{k}{2}] & \end{pmatrix},$$

when $k(n-k) \equiv 0 \mod 2$, and 0 otherwise.

In the following computations we use the unified notation as in Section 2. We write k for n_1, and $n-k$ for n_2, and γ_k instead of ξ_1. For $\mathbb{F} = \mathbb{C}$ let $z \in \{\gamma_k, \Lambda^k \gamma_k\}$, and for $\mathbb{F} \in \{\mathbb{H}, \mathbb{R}\}$ let $z \in \{\gamma_k, \alpha_1(\gamma_k)\}$, where α_1 is the element of $R(Sp(k))$ resp. $R(O(2k))$, such that the restriction of $\alpha_1(\gamma_k)$ to the fibre over the fixed point $P(\sigma)$ is an element of $R(S^1)$ with character $\prod_{1 \le \kappa \le k} (g^{d_{\sigma(\kappa)}} + g^{-d_{\sigma(\kappa)}})$. Using the notation $b_i = g^{d_i}$, when $\mathbb{F} = \mathbb{C}$, and $b_i = g^{d_i} + g^{-d_i}$, when $\mathbb{F} \in \{\mathbb{H}, \mathbb{R}\}$, the restriction of z to the fibre over $P(\sigma)$ is $\chi_\sigma(g) = b_{\sigma(1)} + \cdots + b_{\sigma(k)}$ resp. $\chi_\sigma(g) = b_{\sigma(1)} \dots b_{\sigma(k)}$. The character $\Gamma(g)$ for the computation of the twisted signature $L(G(n, n-k), ch(z^m))$, with $m \in \{0, 1\}$, has the form

$$\begin{aligned} \Gamma(g) &= \sum_\sigma \chi_\sigma(g)^m \prod_{\substack{1 \le \kappa \le k \\ k+1 \le \lambda \le n}} \frac{b_{\sigma(\lambda)} + b_{\sigma(\kappa)}}{b_{\sigma(\lambda)} - b_{\sigma(\kappa)}} \\ &= \frac{1}{V} \sum_\sigma (-1)^{W(\sigma)} \chi_\sigma(g)^m \prod_{\substack{1 \le \kappa \le k \\ k+1 \le \lambda \le n}} (b_{\sigma(\lambda)} + b_{\sigma(\kappa)}) \times \\ &\quad \times \prod_{1 \le \kappa < \lambda \le k} (b_{\sigma(\lambda)} - b_{\sigma(\kappa)}) \prod_{k+1 \le \kappa < \lambda \le n} (b_{\sigma(\lambda)} - b_{\sigma(\kappa)}), \end{aligned}$$

where $V = |b_i^0, b_i^1, \dots, b_i^{n-1}|_{i=1,\dots,n}$ is the Vandermonde determinant with lines $b_i^0, \dots, b_i^{n-1}$, and $W(\sigma) = \#\{(i,j) : 1 \le i \le k, \; k+1 \le j \le n, \; \sigma(i) > \sigma(j)\}$.

As before the sum is over the elements $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. The expression $\Gamma(g)$ may be written as

$$\frac{(-1)^{k(k-1)/2}}{V} \sum_{\sigma} (-1)^{W(\sigma)} \chi_\sigma(g)^m \prod_{\substack{1\le\kappa\le k \\ k+1\le\lambda\le n}} (b_{\sigma(\lambda)} - (-b_{\sigma(\kappa)})) \quad \times$$

$$\times \prod_{1\le\kappa<\lambda\le k} (-b_{\sigma(\lambda)} - (-b_{\sigma(\kappa)})) \prod_{k+1\le\kappa<\lambda\le n} (b_{\sigma(\lambda)} - b_{\sigma(\kappa)})$$

$$= \frac{(-1)^{k(k-1)/2}}{V\, k!(n-k)!} \sum_{\sigma,\rho\in S_n} \chi_\sigma(g)^m \operatorname{sign}\sigma \cdot \operatorname{sign}\rho \times$$

$$\times (-b_{\sigma(1)})^{\rho(1)-1} \ldots (-b_{\sigma(k)})^{\rho(k)-1} b_{\sigma(k+1)}^{\rho(k+1)-1} \ldots b_{\sigma(n)}^{\rho(n)-1}$$

$$= \frac{(-1)^{k(k+1)/2}}{V\, k!(n-k)!} \sum_{\sigma,\rho\in S_n} (-1)^{\rho(1)+\cdots+\rho(k)} \operatorname{sign}\sigma \cdot \operatorname{sign}\rho \cdot \chi_\sigma(g)^m b_{\sigma(1)}^{\rho(1)-1} \ldots b_{\sigma(n)}^{\rho(n)-1}.$$

For $m = 0$ this expression becomes.

$$\Gamma(g) = \frac{(-1)^{k(k+1)/2}}{k!(n-k)!} \sum_{\rho\in S_n} (-1)^{\rho(1)+\cdots+\rho(k)}.$$

For $m = 1$ and $\chi_\sigma(g) = b_{\sigma(1)} \ldots b_{\sigma(k)}$, the expression becomes

$$\Gamma(g) = \frac{(-1)^{k(k+1)/2}}{V\, k!(n-k)!} \sum_{\rho\in S_n} (-1)^{\rho(1)+\cdots+\rho(k)} \times$$

$$\times \operatorname{sign}\rho \left| b_i^{\rho(1)}, \ldots, b_i^{\rho(k)}, b_i^{\rho(k+1)-1}, \ldots, b_i^{\rho(n)-1} \right|_{i=1,\ldots,n}.$$

The determinant with lines $b_i^{\rho(1)}, \ldots, b_i^{\rho(k)}, b_i^{\rho(k+1)-1}, \ldots, b_i^{\rho(n)-1}$ does not vanish only if $\rho(k+1), \ldots, \rho(n) \in \{1, \ldots, n-k\}$ and $\rho(1), \ldots, \rho(k) \in \{n-k+1, \ldots, n\}$, and therefore

$$\Gamma(g) = (-1)^{k(n-k)} \frac{|b_i^0, \ldots, b_i^{n-k-1}, b_i^{n-k+1}, \ldots, b_i^n|_{i=1,\ldots,n}}{|b_i^0, \ldots, b_i^{n-1}|_{i=1,\ldots,n}}.$$

For $\chi_\sigma(g) = b_{\sigma(1)} + \cdots + b_{\sigma(k)}$ and $m = 1$ we have

$$\Gamma(g) = \frac{(-1)^{k(k+1)/2}}{V\, k!(n-k)!} \sum_{j=1}^{k} \sum_{\rho\in S_n} (-1)^{\rho(1)+\cdots+\rho(k)} \operatorname{sign}\rho \times$$

$$\times \left| b_i^{\rho(1)-1}, \ldots, b_i^{\rho(j-1)-1}, b_i^{\rho(j)}, b_i^{\rho(j+1)-1}, \ldots, b_i^{\rho(n)-1} \right|_{i=1,\ldots,n}.$$

The last determinant does not vanish only if $\rho(j) = n$. This means that in this case

$$\begin{aligned}\Gamma(g) &= \frac{(-1)^{k(k+1)/2}}{V\,k!(n-k)!}\sum_{j=1}^{k}\sum_{\substack{\rho\in S_n\\ \rho(j)=n}}(-1)^{\rho(1)+\cdots+\rho(k)}\left|b_i^0, b_i^1, \ldots, b_i^{n-2} b_i^n\right|\\ &= \frac{(-1)^{n+k(k+1)/2}}{(k-1)!(n-k)!}\sum_{\rho\in S_{n-1}}(-1)^{\rho(1)+\cdots+\rho(k-1)}\frac{|b_i^0,\ldots,b_i^{n-2},b_i^n|}{|b_i^0,\ldots,b_i^{n-1}|}.\end{aligned}$$

In all expressions the determinant in the numerator is a generalized Vandermonde determinant and the quotient is a polynomial in the b_i. The value in $b_i = 1$, $i = 1,\ldots,n$, is known by a theorem of Mitchell [13], see also [5]. When $s_1 < \cdots < s_n$ are non-negative integers, then

$$\lim_{b_i,\ldots,b_n\to 1}\frac{|b_i^{s_1},\ldots,b^{s_n}|}{|b_i^0,\ldots,b_i^{n-1}|} = \prod_{1\le\kappa<\lambda\le n}\frac{s_\lambda - s_\kappa}{\lambda-\kappa}.$$

Computing the limits $g \to 1$, this means $b_i \to 1$ for all i resp. $b_i \to 2$ for all i, we get the following result for the Grassmannians.

Proposition 3.1.

(i) *For all three types of Grassmannians $G(k, n-k)$ the signature is given by the formula*

$$\operatorname{signature} G(k,n-k) = \binom{[\frac{k}{2}] + [\frac{n-k}{2}]}{[\frac{k}{2}]}$$

when $k(n-k) \equiv 0 \mod 2$, *and* $\operatorname{signature} G(k,n-k) = 0$ *otherwise.*

(ii) *For the complex Grassmannians the following equalities hold*

$$L(G_{\mathbb{C}}(k,n-k), ch(\Lambda^k\gamma_k)) = \binom{n}{k}$$

for all k and n, and

$$L(G_{\mathbb{C}}(k,n-k), ch(\gamma_k)) = n\binom{[\frac{k-1}{2}] + [\frac{n-k}{2}]}{[\frac{k-1}{2}]},$$

when $(k-1)(n-k) \equiv 0 \mod 2$, *and* $L(G(k,n-k), ch(\gamma_k)) = 0$ *otherwise.*

(iii) *For $G(k,n-k) \in \{G_{\mathbb{H}}(k,n-k), G_{\mathbb{R}}(2k,2n-2k)\}$ one has the equalities*

$$L(G(k,n-k), ch(\alpha_1(\gamma_k)) = (-1)^{k(n-k)}2^k\binom{n}{k}$$

for all k and n, where α_1 denotes the element of $R(S^1)$ introduced in Section 2*, and*

$$L(G(k, n-k), ch(\gamma_k)) = (-1)^{n+k} 2n \binom{[\frac{k-1}{2}] + [\frac{n-k}{2}]}{[\frac{k-1}{2}]},$$

when $(k-1)(n-k) \equiv 0 \mod 2$*, and* $L(G(k, n-k), ch(\gamma_k)) = 0$ *otherwise. In the case of real Grassmann manifolds, $ch(\gamma_k)$ in the formula means Chern character of the complexified bundle $\gamma_k \otimes \mathbb{C}$.*

4. Stable span of flag manifolds

With regard to the application of twisted signatures to find upper bounds for the stable span, one would like to find those twisted signatures, which have a minimal power of 2 in their decomposition in prime numbers. Having this intention in mind, we compute some twisted signatures of flag manifolds.

Proposition 4.1.

(i) *When at most one of the integers $n_1, \dots, n_s$ is odd, then*

$$\operatorname{signature} G(n_1, \dots, n_s) = \frac{[\frac{n}{2}]!}{[\frac{n_1}{2}]! \dots [\frac{n_s}{2}]!},$$

where $n = n_1 + \cdots + n_s$, otherwise is $\operatorname{signature} G(n_1, \dots, n_s) = 0$.

(ii) *When the number of odd integers of $n_1, \dots, n_s$ is equal to 2, then assume that n_1 and n_2 are odd, and let $\alpha(\mathbb{F}) = 1$ for $\mathbb{F} = \mathbb{C}$, and $\alpha(\mathbb{F}) = 2$, for $\mathbb{F} \in \{\mathbb{H}, \mathbb{R}\}$. Then*

$$L(G(n_1, \dots, n_s), ch(\xi_1)) = \pm 2^{\alpha(\mathbb{F})} \frac{[\frac{n}{2}]!}{[\frac{n_1}{2}]! \dots [\frac{n_s}{2}]!}.$$

(iii) *Let $n_1, \dots, n_s$ be as in* (ii)*. Using the notation $z_1 = \Lambda^{n_1} \xi_1$ in the complex case, and $z_1 = \alpha_1(\xi_1)$ as defined in* Section 2 *for the real and quaternionic flag manifolds, the following equation holds*

$$L(G(n_1, \dots, n_s), ch(z_1)) = \pm 2^{\beta(\mathbb{F})} \binom{n}{n_1} \frac{[\frac{n-n_1}{2}]!}{[\frac{n_2}{2}]! \dots [\frac{n_s}{2}]!},$$

where $n = n_1 + \cdots + n_s$, and $\beta(\mathbb{F}) = 0$, when $\mathbb{F} = \mathbb{C}$, and $\beta(\mathbb{F}) = n_1$, when $\mathbb{F} \in \{\mathbb{H}, \mathbb{R}\}$.

The result for the signature is well known. When the signature does not vanish, then the best upper bounds for the stable span which one may obtain with the results of the proposition are obtained by the signature. When the signature vanishes, we use in the case of complex flag manifolds formula (iii) to establish upper bounds for stable span. For any integer q the symbol $\alpha(q)$ denotes the number of ones in the dyadic expansion of q.

Corollary 1. *For the positive integers* $n_1, \dots, n_s$ *let* $j = \alpha(n_1) + \dots + \alpha(n_s) - \alpha(n_1 + \dots + n_s)$.

(i) *When at most one of the* n_i *is odd, then* stable span $G(n_1, \dots, n_s) \le a'_j$ *for all three types of flag manifolds.*

(ii) *When at most two of the integers* $n_1, \dots, n_s$ *are odd, then*

$$\operatorname{stable\,span} G_{\mathbb{C}}(n_1, \dots, n_s) \le 2j.$$

In the case of real and quaternionic flag manifolds formula (ii) of the preceding proposition is used to find upper bounds for stable span.

Corollary 2. *Let* $G(n_1, \dots, n_s) \in \{G_{\mathbb{R}}(2n_1, \dots, 2n_s), G_{\mathbb{H}}(n_1, \dots, n_s)\}$, *where the number of odd integers in* $n_1, \dots, n_s$ *is two, and let* $j = 1 + \alpha(n_1) + \dots + \alpha(n_s) - \alpha(n_1 + \dots + n_s)$.

(i) *For the real flag manifolds is* stable span $G_{\mathbb{R}}(2n_1, \dots, 2n_s) \le a_j$.

(ii) *For the quaternionic flag manifolds is* stable span $G_{\mathbb{H}}(n_1, \dots, n_s) \le b_j$.

Remark 1. All assertions of the corollaries, which were obtained by using the signature, were well known. More results were obtained by using Stiefel-Whitney classes (see e.g. [8] for more details).

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