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COMPLETENESS PROPERTIES OF THE PSEUDOCOMPACT-OPEN TOPOLOGY ON $C(X)$

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Dedicated to Professor Robert A. McCoy

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ABSTRACT. This is a study of the completeness properties of the space $C_{ps}(X)$ of continuous real-valued functions on a Tychonoff space X, where the function space has the pseudocompact-open topology. The properties range from complete metrizability to the Baire space property.

1. Introduction

The set $C(X)$ of all real-valued continuous functions as well as the set $C^*(X)$ of all bounded real-valued continuous functions on a Tychonoff space X has a number of natural topologies. Two commonly used among them are the topology of uniform convergence u and the compact-open topology k. While the topology of uniform convergence on $C(X)$ has been used for more than a century as the proper setting to study the uniform convergence of sequences of functions, the compact-open topology on $C(X)$ made its appearance in 1945 in a paper by Ralph H. Fox, [12] and soon after, it was developed by Richard F. Arens in [3] and by Arens and James Dugundji in [4]. This topology was shown in [17] to be the proper setting to study sequences of functions which converge uniformly on compact subsets. But soon, it also turned out to be a natural and interesting locally convex topology on $C(X)$ from the measure-theoretic viewpoint. In fact, continuous functions and Baire measures

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on Tychonoff spaces are linked by the process of integration. A number of natural locally convex topologies on spaces of continuous functions have been studied in order to clarify this relationship. For more information on these topologies, see [44].

The compact-open topology and the topology of uniform convergence on $C(X)$ (or on $C^*(X)$) are equal if and only if X is compact. Since compactness is such a strong condition, there is a considerable gap between these two topologies. This gap has been especially felt in topological measure theory; consequently in the last five decades, there have been quite a few topologies introduced that lie between k and u, such as the strict topology, the σ-compact-open topology, the topology of uniform convergence on σ-compact subsets and the topology of uniform convergence on bounded subsets. (See for example, [5], [8], [14], [16], [19], [21], [22], [23], [37], [38].)

The pseudocompact-open topology ps is another natural and interesting locally convex topology on $C(X)$, from the viewpoint of both topology and measure theory, though it has not received much attention from the researchers until a formal study of this topology was done in [20]. The spaces $C(X)$, equipped with the point-open topology p, the compact-open topology k and the pseudocompact-open topology ps, are denoted by $C_p(X)$, $C_k(X)$ and $C_{ps}(X)$ respectively.

In [20], in addition to studying some basic properties of $\mathrm{C}_{ps}(X)$ and comparing it with $C_k(X)$, the submetrizability, metrizability and separability of $\mathrm{C}_{ps}(X)$ have been studied. But another important family of properties, the completeness properties of $\mathrm{C}_{ps}(X)$, is yet to be studied. In this paper, we plan to do exactly that. Here we would like to recall that the completeness properties of $C_p(X)$ were studied in [24] while those of $C_k(X)$ were studied in [25]. The situation for $\mathrm{C}_{ps}(X)$ is analogous in some instances to the case for the compact-open topology. But we should keep in mind that while a compact subset in a Tychonoff space is C-embedded, a closed pseudocompact subset need not be so. Also a closed subset of a pseudocompact subset need not be pseudocompact. Often these two facts make the study of $\mathrm{C}_{ps}(X)$, in particular, the study of some completeness properties of $\mathrm{C}_{ps}(X)$, difficult.

In Section 2 of this paper, we begin the study of the completeness properties of $\mathrm{C}_{ps}(X)$ by recalling the definition of the pseudocompact-open topology on $C(X)$ and by having a close look at the uniform completeness of $\mathrm{C}_{ps}(X)$. In Section 3, we study various kinds of completeness of this topology such as complete metrizability, Čech-completeness, local Čech-completeness, sieve-completeness and partition-completeness of $\mathrm{C}_{ps}(X)$. Since $\mathrm{C}_{ps}(X)$ is a locally convex space, we also look at the barreledness of $\mathrm{C}_{ps}(X)$. In Section 3, we also study specially the almost Čech-completeness and pseudocompleteness of $\mathrm{C}_{ps}(X)$. Here we would like to mention that in [25], these two properties for $C_k(X)$ have been studied after studying in detail the compact-open topology on $\mathbb{R}^X$, the set of all real-valued functions on X and subsequently after using several significant results. But here we use the techniques of locally convex spaces in order to have a shorter and more elegant route to the study of almost Čech-completeness and pseudocompleteness of $\mathrm{C}_{ps}(X)$. It suffices to mention that these techniques may be equally used for $C_k(X)$ also.

Throughout this paper, all spaces are Tychonoff spaces. The Stone-Čech-compactification of a space X is denoted by βX and $\mathbb{R}$ denotes the space of real numbers.

2. Uniform Completeness of $\mathrm{C}_{ps}(X)$

There are three ways to consider the pseudocompact-open topology on $C(X)$.

First, one can use as subbase the family $\{[A, V] : \ A$ is a pseudocompact subset of X and V is an open subset of $\mathbb{R}\}$ where $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$. But one can also consider this topology as the topology of uniform convergence on the pseudocompact subsets of X, in which case the basic open sets will be of the form $\langle f, A, \epsilon\rangle = \{g \in C(X) : \ |g(x) - f(x)| < \epsilon$ for all $x \in A\}$, where $f \in C(X)$, A is a pseudocompact subset of X and ϵ is a positive real number.

The third way is to look at the pseudocompact-open topology as a locally convex topology on $C(X)$. For each pseudocompact subset A of X and $\epsilon > 0$, we define the seminorm p_A on $C(X)$ and $V_{A,\epsilon}$ as follows: $p_A(f) = \sup\{|f(x)| \ : x \in A\}$ and $V_{A,\epsilon} = \{f \in C(X) : \ p_A(f) < \epsilon\}$. Let $\mathscr{V} = \{V_{A,\epsilon} : \ A$ is a pseudocompact subset of $X, \ \epsilon > 0\}$. Then for each $f \in C(X)$, $f + \mathscr{V} = \{f + V : \ V \in \mathscr{V}\}$ forms a neighborhood base at f. This topology is locally convex since it is generated by a collections of seminorms and it is same as the pseudocompact-open topology on $C(X)$. It is also easy to see that this topology is Hausdorff.

The topology of uniform convergence on the pseudocompact subsets of X is actually generated by the uniformity of uniform convergence on these subsets. When this uniformity is complete, $C_{ps}(X)$ is said to be *uniformly complete*. This uniform completeness can also be seen as the completeness of a topological group. A topological group E is called complete provided that every Cauchy net in E converges to some element in E, where a net (x_α) in E is Cauchy if for every neighborhood U of 0 in E, there is an α_0 such that $x_{\alpha_1} - x_{\alpha_2} \in U$ for all $\alpha_1, \alpha_2 \geq \alpha_0$ (for E additive). One can check that $\mathrm{C}_{ps}(X)$ is uniformly complete if and only if it is complete as an additive topological group. Also $\mathrm{C}_{ps}(X)$ is completely metrizable if and only if it is uniformly complete and metrizable, (see [7, pp. 34, 36]).

In order to characterize the uniform completeness of $\mathrm{C}_{ps}(X)$, we need to talk about ps-continuous functions and ps_f-spaces.

Definition 2.1. A function $f\colon X \to \mathbb{R}$ is said to be *ps-continuous* if for every pseudocompact subset A of X, there exists a continuous function $g\colon X \to \mathbb{R}$ such that $g|_A = f|_A$. A space X is called a *ps_f-space* if every ps-continuous function on X is continuous.

The following characterization of uniform completeness of $\mathrm{C}_{ps}(X)$ has been proved in a broader context of the $\mathscr{G}$-open topology in [23].

Theorem 2.2. *The space $\mathrm{C}_{ps}(X)$ is uniformly complete if and only if X is a ps_f-space.*

Before investigating which spaces are ps_f-spaces, we would like to give an example of a space which is not a ps_f-space. Let X be an uncountable non-discrete P-space. (See [15, 4N, 9L].) Since X is a P-space, the point-open and pseudocompact-open topologies on $C(X)$ coincide. But since X is not a discrete space, by [26, Corollary 5.1.2(b)], $C_p(X)$ is not uniformly complete. Hence X is not a ps_f-space.

But now we will see that the family of ps_f-spaces is quite large. In particular many well-known spaces are ps_f-spaces. But in order to identify these spaces, we need to talk about pseudo-k-spaces and quasi-k-spaces.

A subset Y of a space X is said to be pseudo-k-closed (quasi-k-closed) in X if $Y \cap A$ is closed in A for every pseudocompact (countably compact) subset A of X. A space X is said to be a pseudo-k-space (quasi-k-space) if every pseudo-k-closed (quasi-k-closed) subset of X is closed in X.

A space X is called locally pseudocompact (locally countably compact) if every point in X has a pseudocompact (countably compact) neighborhood. A space X is a q-space if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. A property stronger that being a q-space is that of being an M-space, which can be characterized as a space that can be mapped onto a metric space by a quasi-perfect map (a continuous closed map in which inverse images of points are countably compact).

It can be easily seen that every quasi-k-space is a pseudo-k-space which, in turn, is a ps_f-space.

The next result for the quasi-k-space was proved in [36]. The analogous result for the pseudo-k-spaces can be proved in a similar manner.

Theorem 2.3. *For a space X, the following assertions are equivalent.*

(a) *X is a pseudo-k-space (quasi-k-space).*

(b) *X is a quotient space of a disjoint topological sum of pseudocompact (countably compact) spaces.*

(c) *X is a quotient space of a locally pseudocompact (locally countably compact) space.*

The next result is a minor modification of the first corollary of [33] for Tychonoff spaces.

Theorem 2.4. *For a space X, the following assertions are equivalent.*

(a) *X is a quasi-k-space.*

(b) *X is a quotient space of a q-space.*

(c) *X is a quotient space of an M-space.*

From the last two results, it is clear that the family of ps_f-spaces includes pseudo-k-spaces, locally pseudocompact spaces and quotient spaces of q-spaces.

3. Complete metrizability and some related completeness properties of $\mathrm{C}_{ps}(X)$

In this section, we study various kinds of completeness of $\mathrm{C}_{ps}(X)$. In particular, here we study the complete metrizability of $\mathrm{C}_{ps}(X)$ in a wider setting, more precisely, in relation to several other completeness properties. So we first recall the definitions of various kinds of completeness.

A space X is called Čech-complete if X is a G_δ-set in βX. A space X is called locally Čech-complete if every point $x \in X$ has a Čech-complete neighborhood. Another completeness property which is implied by Čech-completeness is that of pseudocompleteness, introduced in [35]. This is a space having a sequence of π-bases $\{\mathscr{B}_n : n \in \mathbb{N}\}$ such that whenever $B_n \in \mathscr{B}_n$ for each n and $\overline{B_{n+1}} \subseteq B_n$, then $\bigcap\{B_n : n \in \mathbb{N}\} \neq \emptyset$.

In [1], it has been shown that a space having a dense Čech-complete subspace is pseudocomplete and a pseudocomplete space is a Baire space. Also note that since a locally Baire space is a Baire space, every locally Čech-complete space is a Baire space.

In order to deal with sieve-completeness, partition-completeness and almost Čech-completeness, one needs to recall the definitions of these concepts from [29]. The central idea of all these concepts is that of a complete sequence of subsets of X.

Let $\mathscr{F}$ and $\mathscr{U}$ be two collections of subsets of X. Then $\mathscr{F}$ is said to be controlled by $\mathscr{U}$ if for each $U \in \mathscr{U}$, there exists some $F \in \mathscr{F}$ such that $F \subseteq U$. A sequence (U_n) of subsets of X is said to be complete if every filter base $\mathscr{F}$ on X which is controlled by (U_n) clusters at some $x \in X$. A sequence $(\mathscr{U}_n)$ of collections of subsets of X is called complete if (U_n) is a complete sequence of subsets of X whenever $U_n \in \mathscr{U}_n$ for all n. It has been shown in [13, Theorem 2.8] that the following statements are equivalent for a Tychonoff space X:

(a) X is a G_δ-subset of any Hausdorff space in which it is densely embedded;
(b) X has a complete sequence of open covers;
(c) X is Čech-complete.

From this result, it easily follows that a Tychonoff space X is Čech-complete if and only if X is a G_δ-subset of any Tychonoff space in which it is densely embedded.

For the definitions of sieve, sieve-completeness and partition-completeness, see [9], [29], and [30]. The term "sieve-complete" is due to Michael [27], but the sieve-complete spaces were studied earlier under different names: as λ_b-spaces by Wicke in [45], as spaces satisfying condition $\mathcal{K}$ by Wicke and Worrel Jr. in [46] and as monotonically Čech-complete spaces by Chaber, Čoban and Nagami in [10]. Every space with a complete sequence of open covers is sieve-complete; the converse is generally false, but it is true in paracompact spaces, see [10, Remark 3.9] and [27, Theorem 3.2]. So a Čech-complete space is sieve-complete and a paracompact sieve-complete space is Čech-complete.

We call a collection $\mathscr{U}$ of subsets of X an almost-cover of X if $\bigcup \mathscr{U}$ is dense in X. We call a space almost Čech-complete if X has a complete sequence of open almost-covers. Such a space has been simply called almost complete in [29]. Every almost Čech-complete space is a Baire space, see [29, Proposition 4.5].

The property of being a Baire space is the weakest one among the completeness properties we consider here. Since $\mathrm{C}_{ps}(X)$ is a locally convex space, $\mathrm{C}_{ps}(X)$ is a Baire space if and only if $\mathrm{C}_{ps}(X)$ is of second category in itself. Also since a locally convex Baire space is barreled, first we find a necessary condition for $\mathrm{C}_{ps}(X)$ to be barreled. A locally convex space X is called barreled (tonnelé) if each barrel in X is a neighborhood of 0. A subset E in a locally convex space X is called a barrel (tonneau) if E is closed, convex, balanced and absorbing in X. The absorbing sets are also called absorbent. For details on barreled spaces, see [34].

In order to state the next result, we need the following definition. A subset A of a space X is called bounded or relatively pseudocompact if $f(A)$ is bounded in $\mathbb{R}$ for all $f \in C(X)$. For detailed remarks on bounded subsets, see [20].

Theorem 3.1. *If $\mathrm{C}_{ps}(X)$ is barreled, then every bounded subset of X is contained in a pseudocompact subset of X.*

Proof. Let A be a bounded subset of X and let $W = \{f \in C(X) :\ p_A(f) \leq 1\}$. Then it is routine to check that W is closed, convex, balanced and absorbing, that is, W is a barrel in $\mathrm{C}_{ps}(X)$. Since $\mathrm{C}_{ps}(X)$ is barreled, W is a neighborhood of 0 and consequently there exist a closed pseudocompact subset P of X and $\epsilon > 0$ such that $\langle 0, P, \epsilon\rangle \subseteq W$. We claim that $A \subseteq P$. If not, let $x_0 \in A \setminus P$. So there exists a continuous function $f \colon X \to [0, 2]$ such that $f(x) = 0$ for all $x \in P$ and $f(x_0) = 2$. Clearly $f \in \langle 0, P, \epsilon\rangle$, but $f \notin W$. Hence we must have $A \subseteq P$. □

Example 3.2. If X is realcompact, then every closed pseudocompact subset of X is compact and consequently the pseudocompact-open and compact-open topologies on $C(X)$ coincide. But by the famous Nachbin-Shirota theorem, $C_k(X)$ is barreled if X is realcompact, see [18, Theorem 5, p. 234]. Hence if X is realcompact, then $C_{ps}(X)$ is barreled. In particular, since the Niemytzki plane L is realcompact, (see [11, 3.11.B.(b), p. 219]), $C_{ps}(L)$ is barreled.

But there are spaces X such that $\mathrm{C}_{ps}(X)$ is not barreled. Let D be the Dieudonne plank with the underlying set $[0, \omega_1] \times [0, \omega_0] \setminus \{(\omega_1, \omega_0)\}$ where ω_0 and ω_1 denote respectively the first infinite and the first uncountable ordinal. For the details of this plank, see [40, Example 89] and [23, Example 3.16]. Every pseudocompact subset of D is compact. The subset $C = \{\omega_1\}\times[0, \omega_0)$ is bounded in D, but it is not contained in any pseudocompact subset of D. Hence $C_{ps}(D)$ is not barreled. In [42], Todd constructed an example of a space T which has an infinite bounded subset, but the pseudocompact subsets of which are finite. So $C_{ps}(T)$ is not barreled either.

Theorem 3.3. *For any space X, the following assertions are equivalent.*

(a) $\mathrm{C}_{ps}(X)$ *is completely metrizable.*

(b) $\mathrm{C}_{ps}(X)$ *is Čech-complete.*

(c) $\mathrm{C}_{ps}(X)$ *is locally Čech-complete.*

(d) $\mathrm{C}_{ps}(X)$ *is sieve-complete.*

(e) $\mathrm{C}_{ps}(X)$ *is an open continuous image of a paracompact Čech-complete space.*

(f) $\mathrm{C}_{ps}(X)$ *is an open continuous image of a Čech-complete space.*

(g) $\mathrm{C}_{ps}(X)$ *is partition-complete.*

(h) *X is a hemipseudocompact ps_f-space, (X is called hemipseudocompact if there exists a sequence of pseudocompact sets $\{A_n :\ n \in \mathbb{N}\}$ in X such that for any pseudocompact subset A of X, $A \subseteq A_n$ holds for some n).*

Proof. We have earlier noted that $C_{ps}(X)$ is completely metrizable if and only if it is uniformly complete and metrizable. Hence by [20, Theorem 5.7] and by Theorem 2.2, (a) $\iff$ (h). Note that (a) $\implies$ (b) $\implies$ (c) and (a) $\implies$ (e) $\implies$ (f). By [29, Proposition 4.4], (b) $\implies$ (d) $\implies$ (g). Also (c) $\implies$ (f), see [11, 3.12.19.(d), p. 237].

(f) $\implies$ (a). A Čech-complete space is of pointwise countable type and the property of being pointwise countable type is preserved by open continuous maps. Hence $C_{ps}(X)$ is of pointwise countable type and consequently by [20, Theorem 5.7], $C_{ps}(X)$ is metrizable and hence $C_{ps}(X)$ is paracompact. So by Pasynkov's theorem, (see [11, 5.5.8.(b), p. 341]), $C_{ps}(X)$ is Čech-complete. But a Čech-complete metrizable space is completely metrizable.

(g) $\implies$ (a). If $C_{ps}(X)$ is partition-complete, then by [29, Propositions 4.4 and 4.7], $C_{ps}(X)$ contains a dense Čech-complete subspace. Hence $C_{ps}(X)$ contains a dense subspace of pointwise countable type and consequently by [20, Theorem 5.7], $C_{ps}(X)$ is metrizable. But by [28, Theorem 1.5] and [29, Proposition 2.1], a metrizable space is completely metrizable if and only if it is partition-complete. Hence $C_{ps}(X)$ is completely metrizable. □

Now we would like to extend the list of equivalent completeness properties given in Theorem 3.3, by including the pseudocompleteness and almost Čech-completeness. In order to do this, first we need to embed the space $C_{ps}(X)$ in a larger locally convex function space. Recall that $\mathbb{R}^X$ denotes the set of all real-valued functions defined on X. Let $PC(X) = \{f \in \mathbb{R}^X : f|_A$ is continuous for each pseudocompact subset A of $X\}$. As in case of $C_{ps}(X)$, we can define the pseudocompact-open topology on $PC(X)$ in three different ways. In particular, this is a locally convex Hausdorff topology on $PC(X)$ generated by the family of seminorms $\{p_A\colon A$ is a pseudocompact subset of $X\}$, where for $f \in PC(X)$, $p_A(f)$ is defined as follows: $p_A(f) = \sup\{|f(x)| : x \in A\}$.

We denote the space $PC(X)$ with the pseudocompact-open topology by $PC_{ps}(X)$. It is clear that $C_{ps}(X)$ is a subspace of $PC_{ps}(X)$. Moreover the proof of the following result is immediate.

Theorem 3.4. *If every closed pseudocompact subset of X is C-embedded in X, then $C(X)$ is dense in $PC_{ps}(X)$.*

In the next theorem, the term σ-space refers to a space having a σ-locally finite network. Every metrizable space is a σ-space.

Theorem 3.5. *For a space X, consider the following conditions.*

(a) $C_{ps}(X)$ *is completely metrizable.*

(b) $C_{ps}(X)$ *is a pseudocomplete σ-space.*

(c) $C_{ps}(X)$ *is a pseudocomplete q-space.*

(d) $C_{ps}(X)$ *contains a dense completely metrizable subspace.*

(e) $C_{ps}(X)$ *contains a dense Čech-complete subspace.*

(f) $C_{ps}(X)$ *is almost Čech-complete.*

Then (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (e) $\iff$ (f).

Proof.

(a) $\implies$ (b) and (d) $\implies$ (e). These are immediate.

(b) $\implies$ (c). A Baire space, which is a σ-space as well, has a dense metrizable subspace, see [43]. So if $C_{ps}(X)$ is a pseudocomplete σ-space, then it contains a dense metrizable space. Since every metrizable space is of pointwise countable type, by [20, Theorem 5.7], $C_{ps}(X)$ is a q-space.

(c) $\implies$ (d). If $C_{ps}(X)$ is a q-space, then by [20, Theorem 5.7], $C_{ps}(X)$ is metrizable. But a metrizable space is pseudocomplete if and only if it contains a dense completely metrizable subspace, see [1, Corollary in 2.4].

(e) $\iff$ (f) follows from [29, Propositions 4.4, 4.7]. □

Remark 3.6. If $C_{ps}(X)$ is only assumed to be pseudocomplete, it may not be almost Čech-complete. Consider the non-discrete uncountable P-space S given in [15, [4N]]. It can be easily shown that S is a normal space. Since every pseudocompact subset of S is finite, the pseudocompact-open topology ps on $C(S)$ coincides with the point-open topology p on $C(S)$. Since S is uncountable, $C_p(S)$ is not metrizable. But by [24, Theorem 8.4], $C_p(S)$ is pseudocomplete, since every countable subset in a P-space is closed. But since $C_p(S)$ is not metrizable, by [20, Theorem 5.7], it is not almost Čech-complete either.

Theorem 3.7. *If every closed pseudocompact subset of X is C-embedded in X, then the following assertions are equivalent.*

(a) $C_{ps}(X)$ *is completely metrizable.*

(b) $C_{ps}(X)$ *is almost Čech-complete.*

(c) X *is a hemipseudocompact ps_f-space.*

Proof. We only need to show that (b) $\implies$ (c). If $C_{ps}(X)$ is almost Čech-complete, then $C_{ps}(X)$ contains a dense Čech-complete subspace G. Since every closed pseudocompact subset of X is C-embedded in X, $C(X)$ is dense in

$PC_{ps}(X)$ and consequently G is dense in $PC_{ps}(X)$ also. Now since $PC_{ps}(X)$ contains a dense Baire subspace G, $PC_{ps}(X)$ is itself a Baire space. Also since G is Čech-complete, G is a G_δ-set in $PC_{ps}(X)$.

Note that every ps-continuous function on X is in $PC_{ps}(X)$. In order to show that X is a ps_f-space, we will show that $PC(X) = C(X)$. So let $f \in PC(X)$. Define the map $T_f\colon PC_{ps}(X) \to PC_{ps}(X)$ by $T_f(g) = f + g$ for all $g \in PC(X)$. Since $PC_{ps}(X)$ is a locally convex space, T_f is a homeomorphism and consequently $T_f(G)$ is a dense G_δ-subset of $PC_{ps}(X)$. Since $PC_{ps}(X)$ is a Baire space, $G \cap T_f(G) \neq \emptyset$. Let $h \in G \cap T_f(G)$. Then there exists $g \in G$ such that $h = f + g$. So $f = g - h \in C(X)$. □

Remark 3.8. There is a family of Tychonoff spaces X, known as σ-functionally normal spaces, in which every closed pseudocompact subset of X is C-embedded in X. A space X is called σ-functionally normal if for any two disjoint closed sets A and B in X, there is a sequence (f_n) in $C(X)$ such that if $x \in A$ and $y \in B$, then there exists n such that $f_n(x) \neq f_n(y)$. Obviously a normal space is σ-functionally normal, but the converse need not be true. The Niemytzki plane L and Tychonoff plank are σ-functionally normal, but not normal. Every closed pseudocompact subset in a σ-functionally normal space is C-embedded. For details on σ-functionally normal spaces, see [6].

Whenever X is an M-space, we can add more statements to the list in Theorem 3.3. But in order to do that, we need to recall the following definitions. A continuous map $f\colon X \to Y$ is called p-covering if, given any pseudocompact subset A in Y, there exists a pseudocompact subset C in X such that $A \subseteq \overline{f(C)}$. A space X is called a strongly Baire space if every closed subspace of X is a Baire space.

A space X is hemicountably compact if there exists a sequence of countably compact sets $\{A_n : n \in \mathbb{N}\}$ in X such that for any countably compact subset A of X, $A \subseteq A_n$ holds for some n. Since a closed subspace of an M-space is again an M-space and a pseudocompact M-space is countably compact, (see [39, Theorem 2]), every closed pseudocompact subset of an M-space is countably compact. Consequently, it can be easily seen that an M-space is hemipseudocompact if and only if it is hemicountably compact.

Theorem 3.9. *If X is an M-space, then the following assertions are equivalent.*

(a) $\mathrm{C}_{ps}(X)$ *is completely metrizable.*

(b) $\mathrm{C}_{ps}(X)$ *is metrizable.*

(c) $\mathrm{C}_{ps}(X)$ *is a submetrizable strongly Baire space.*

(d) X *is locally countably compact as well as hemicountably compact.*

Proof.

(b) $\implies$ (a). Since X is an M-space, X is a ps_f space and consequently $\mathrm{C}_{ps}(X)$ is (uniformly) complete. In addition, if $\mathrm{C}_{ps}(X)$ is metrizable, it becomes completely metrizable.

(a) $\implies$ (c). This is immediate.

(c) $\implies$ (d). Since X is an M-space, there exist a metric space M and a quasi-perfect surjection $f\colon X \to M$. Now let A be a pseudocompact, that is, a compact subset of M. Hence by [11, Theorem 3.10.9], $f^{-1}(A)$ is countably compact in X. Since $f(f^{-1}(A)) = A$, f is a p-covering map. Hence by [20, Theorem 4.7], $f^*\colon C_{ps}(M) \to \mathrm{C}_{ps}(X)$ is an embedding. Here $f^*(g) = g \circ f$ for all g in $C(M)$. Since f is a closed continuous surjection, it is a quotient map. Hence by [26, Theorem 2.2.10], $f^*(C(M))$ is a closed subset of $C_p(X)$. Hence $f^*(C(M))$ is also closed in $\mathrm{C}_{ps}(X)$. Since $\mathrm{C}_{ps}(X)$ is a strongly Baire space, $f^*(C(M))$ is a Baire subspace of $C_{ps}(X)$. But since f^* is an embedding, $C_{ps}(M)$ is also a Baire space. Now since M is a metric space, the pseudocompact-open topology on $C(M)$ coincides with the compact-open topology on $C(M)$. Hence $C_k(M)$ is a Baire space.

Now since $\mathrm{C}_{ps}(X)$ is submetrizable, by [20, Theorem 5.4], there exists a sequence $\{A_n:\ n \in \mathbb{N}\}$ of pseudocompact subsets of X such that $\bigcup\limits_{n=1}^{\infty} A_n$ is dense in X. Hence $M = f(X) = \overline{\bigcup\limits_{n=1}^{\infty} f(A_n)}$. Since each $f(A_n)$ is compact, by [25, Theorem 3.1], $C_k(M)$ is submetrizable. Now since $C_k(M)$ is a submetrizable Baire space, by [25, Theorem 7.2], M is a locally compact Lindelöf space. But a locally compact Lindelöf space is hemicompact. Now it is routine to check that X is locally countably compact as well as hemicountably compact.

(d) $\implies$ (a). By Theorem 2.3, a locally countably compact space is a quasi-k-space. But a quasi-k-space is a ps_f-space. $\square$

REFERENCES

[1] AARTS, J. M.—LUTZER, D. J.: *Pseudo-completeness and the product of Baire spaces*, Pacific J. Math. **48** (1973), 1–10.

[2] AARTS, J. M.—LUTZER, D. J.: *Completeness properties designed for recognizing Baire spaces*, Dissertationes Math. (Rozprawy Mat.) **116** (1974), 1–48.

[3] ARENS, R.: *A topology for spaces of transformations*, Ann. of Math. **47** (1946), 480–495.

[4] ARENS, R.—DUGUNDJI, J.: *Topologies for function spaces*, Pacific J. Math. **1** (1951), 5–31.

[5] ARHANGEL'SKIĬ, A. V.: *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. **25** (1982), 852–855.

[6] AULL, C. E.: *Absolute C-embedding in functionally normal spaces and related spaces.* In: Topology. Colloq. Math. Soc. János Bolyai 23, North-Holland, Amsterdam, 1978, pp. 129–136.

[7] BERBERIAN, S. K.: *Lectures in Functional Analysis and Operator Theory.* Grad. Texts in Math. 15, Springer-Verlag, New York, 1974.

[8] BUCK, R. C.: *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.

[9] BUHAGIAR, D.—YOSHIOKA, I.: *Sieves and completeness properties*, Questions Answers Gen. Topology **18** (2000), 143–162.

[10] CHABER, J.—ČOBAN, M. M.—NAGAMI, K.: *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fund. Math. **84** (1974), 107–119.

[11] ENGELKING, R.: *General Topology* (rev. and compl. ed.), Heldermann Verlag, Berlin, 1989.

[12] FOX, R. H.: *On topologies for function spaces*, Bull. Amer. Math. Soc. **51** (1945), 429–432.

[13] FROLÍK, Z.: *Generalizations of the G_δ-property of complete metric spaces*, Czechoslovak Math. J. **10** (1960), 359–379.

[14] GILES, R.: *A generalization of the strict topology*, Trans. Amer. Math. Soc. **161** (1971), 467–474.

[15] GILLMAN, L.—JERISON, M.: *Rings of Continuous Functions*, D. Van Nostrand, Princeton, NJ, 1960.

[16] GULICK, D.: *The σ-compact-open topology and its relatives*, Math. Scand. **30** (1972), 159–176.

[17] JACKSON, J. R.: *Comparison of topologies on function spaces*, Proc. Amer. Math. Soc. **3** (1952), 156–158.

[18] JARCHOW, H.: *Locally Convex Spaces.* Math. Leitfäden [Mathematical Textbooks], B. G. Teubner, Stuttgart, 1981.

[19] KUNDU, S.: $C_b(X)$ revisited: induced map and submetrizability, Quaest. Math. **27** (2004), 47–56.

[20] KUNDU, S.—GARG, P.: *The pseudocompact-open topology on $C(X)$*, Topology Proc. **30** (2006), 279–299.

[21] KUNDU, S.—MCCOY, R. A.: *Topologies between compact and uniform convergence on function spaces*, Int. J. Math. Math. Sci. **16** (1993), 101–109.

[22] KUNDU, S.—MCCOY, R. A.—RAHA, A. B.: *Topologies between compact and uniform convergence on function spaces II*, Real Anal. Exchange **18** (1992/93), 176–189.

[23] KUNDU, S.—RAHA, A. B.: *The bounded-open topology and its relatives*, Rend. Istit. Mat. Univ. Trieste **27** (1995), 61–77.

[24] LUTZER, D. J.—MCCOY, R. A.: *Category in function spaces, I*, Pacific J. Math. **90** (1980), 145–168.

[25] MCCOY, R. A.—NTANTU, I.: *Completeness properties of function spaces*, Topology Appl. **22** (1986), 191–206.

[26] MCCOY, R. A.—NTANTU, I.: *Topological Properties of Spaces of Continuous Functions.* Lecture Notes in Math. 1315, Springer-Verlag, Berlin, 1988.

[27] MICHAEL, E.: *Complete spaces and tri-quotient maps*, Illinois J. Math. **21** (1977), 716–733.

[28] MICHAEL, E.: *A note on completely metrizable spaces*, Proc. Amer. Math. Soc. **96** (1986), 513–522.

[29] MICHAEL, E.: *Almost complete spaces, hypercomplete spaces and related mapping theorems*, Topology Appl. **41** (1991), 113–130.

[30] MICHAEL, E.: *Partition-complete spaces are preserved by tri-quotient maps*, Topology Appl. **44** (1992), 235–240.

[31] MICHAEL, E.: *Partition-complete spaces and their preservation by tri-quotient and related maps*, Topology Appl. **73** (1996), 121–131.

[32] MICHAEL, E.: *Partition-complete paracompact k-spaces are preserved by closed maps*, Topology Appl. **96** (1999), 63–73.

[33] NAGATA, J.: *Quotient and bi-quotient spaces of M-spaces*, Proc. Japan Acad. Ser. A Math. Sci. **45** (1969), 25–29.

[34] NARICI, L.—BECKENSTEIN, E.: *Topological Vector Spaces*, Marcel Dekker, New York, 1985.

[35] OXTOBY, J. C.: *Cartesian products of Baire spaces*, Fund. Math. **49** (1961), 157–166.

[36] SANCHIS, M.: *A note on quasi-k-spaces*, Rend. Istit. Mat. Univ. Trieste **30** (1999), 173–179.

[37] SCHMETS, J.: *Espaces de fonctions continues.* Lecture Notes in Math. 519, Springer-Verlag, New York, 1976.

[38] SENTILLES, F. D.: *Bounded continuous functions on a completely regular space*, Trans. Amer. Math. Soc. **168** (1972), 311–336.

[39] SHIRAKI, T.: *A note on M-spaces*, Proc. Japan Acad. Ser. A Math. Sci. **43** (1967), 873–875.

[40] STEEN, L. A.—SEEBACH, J. A., Jr.: *Counterexamples in topology* (2nd ed.), Springer-Verlag, New York, 1978.

[41] TELGÁRSKY, R.: *On sieve-complete and compact-like spaces*, Topology Appl. **16** (1983), 61–68.

[42] TODD, A. R.: *Pseudocompact sets, absolutely Warner bounded sets and continuous function spaces*, Arch. Math. (Basel) **56** (1991), 474–481.

[43] DOUWEN, E. K. Van: *An unbaireable stratifiable space*, Proc. Amer. Math. Soc. **67** (1977), 324–326.

[44] WHEELER, R. F.: *A survey of Baire measures and strict topologies*, Expo. Math. **1** (1983), 97–190.

[45] WICKE, H. H.: *Open continuous images of certain kinds of M-spaces and completeness of mappings and spaces*, General Topology Appl. **1** (1971), 85–100.

[46] WICKE, H. H.—WORRELL, J. M., Jr.: *On the open continuous images of paracompact Čech complete spaces*, Pacific J. Math. **37** (1971), 265–275.

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