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ASYMPTOTIC FORMULAS FOR CERTAIN ARITHMETIC FUNCTIONS

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ABSTRACT. This is an extended summary of a talk given by the last named author at the Czecho-Slovake Number Theory Conference 2005, held at Malenovice in September 2005. It surveys some recent results concerning asymptotics for a class of arithmetic functions, including, e.g., the second moments of the number-of-divisors function $d(n)$ and of the function $r(n)$ which counts the number of ways to write a positive integer as a sum of two squares. For the proofs, reference is made to original articles by the authors published elsewhere.

1. Introduction

At the problem session of the 1991 Czechoslovak number theory conference, Professor A. Schinzel asked the following question:

Let as usual $r(n)$ denote the number of ways to write a positive integer n as a sum of two squares. What is the sharpest error bound at the present state-of-art, in the asymptotics for the quadratic moment

$$\sum_{n\le x}(r(n))^2 = 4x\log x + C\,x + O(???)\,.$$

We shall present a fairly general theorem which will include applications to sums like

$$\sum_n (r(n))^2,\quad \sum_n r(n^3),\quad \sum_n (d(n))^2,\quad \sum_n d(n^3),\quad \sum_n d(n)r(n)\,,$$

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where $d(n)$ is the number-of-divisors function.

The scope of our general result is motivated by the observation that the generating function of $(r(n))^2$ reads, for $\Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{(r(n))^2}{n^s} = 16\,(1+2^{-s})^{-1}\,\frac{(\zeta_{\mathbf{Q}(\mathrm{i})}(s))^2}{\zeta(2s)}\,,$$

where $\zeta_{\mathbf{Q}(\mathrm{i})}(s)$ is the Dedekind zeta-function of the Gaussian field.

Theorem. *Let $0 \le a(n) \ll n^\epsilon$ for every $\epsilon > 0$, with a Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{\prod\limits_{m=1}^{M} \zeta_{\mathbf{K}_m^*}(s)}{\prod\limits_{j=1}^{J} (\zeta_{\mathbf{K}_j}(2s))^{\tau_j}}\, G(s) \qquad (\Re(s) > 1)\,.$$

- *$\zeta_{\mathbf{K}_m^*}$ are Dedekind zeta-functions of number fields $\mathbf{K}_m^*$ of degrees $d_m = [\mathbf{K}_m^* : \mathbf{Q}]$ equal to 1 or 2, with*

$$d_1 + \cdots + d_M = 4\,.$$

- *$\mathbf{K}_1, \ldots, \mathbf{K}_J$ are arbitrary algebraic number fields, $J \ge 0$.*
- *$\tau_1, \ldots, \tau_J$ are fixed real numbers.*
- *$G(s)$ is a "harmless" factor, holomorphic and bounded from above and away from 0 uniformly in a half-plane $\Re(s) \ge \sigma_0$ where $\sigma_0 < \frac{1}{2}$.*

Under these assumptions it follows that

$$\sum_{n \le x} a(n) = H(x) + O\big(x^{1/2}(\log x)^{M+1}(\log\log x)^{|\tau_1|+\cdots+|\tau_J|}\big)\,, \tag{1}$$

with

$$H(x) := \mathrm{Res}_{s=1}\left(F(s)\,\frac{x^s}{s}\right) = x\,P_{M-1}(\log x)\,,$$

$P_{M-1}(\cdot)$ a polynomial of degree $M-1$.

Furthermore, we have the "short interval result"

$$\sum_{x<n\le x+y} a(n) \sim B_0 y (\log x)^{M-1}\,, \tag{2}$$

where B_0 is the leading coefficient of $P_{M-1}(\cdot)$, as long as $y = y(x)$ satisfies

$$\frac{y}{x^{1/2}\log x(\log\log x)^{|\tau_1|+\cdots+|\tau_J|}} \to \infty \qquad \textit{for} \quad x \to \infty\,. \tag{3}$$

Remark. The growth condition on $y = y(x)$ is actually a bit less stringent than what would be immediate from the "long range" asymptotics (1). A direct argument would yield (2) only under the stronger supposition

$$\frac{y}{x^{1/2}(\log x)^2(\log\log x)^{|\tau_1|+\cdots+|\tau_J|}} \to \infty\,.$$

The refinement is effected by an intrinsic device due to A. A. Karatsuba [4], [5].

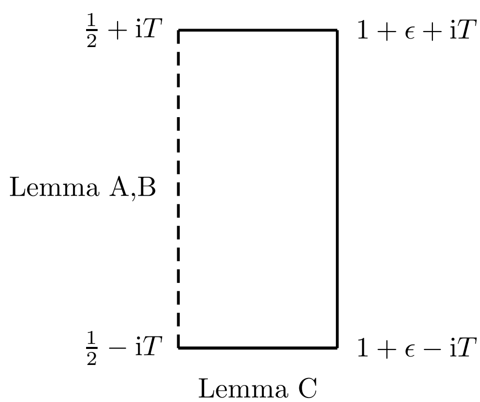
In this note we summarize the essential ideas used in the proof of the theorem and sketch a few applications. For the details of the arguments the reader is referred to the authors' original articles [1], [2], and [6].

2. Outline of proof

A convenient starting point is Perron's formula in the shape

$$\sum_{n\le x} a(n) = \frac{1}{2\pi \mathrm{i}} \int\limits_{1+\epsilon-\mathrm{i}T}^{1+\epsilon+\mathrm{i}T} F(s)\,\frac{x^s}{s}\,\mathrm{d}s + \text{error terms}\,.$$

The path of integration is shifted onto the line $\Re(s) = \frac{1}{2}$, as shown in the figure. To deal with the integral along the critical line, the numerator of the generating function $F(s)$ is estimated by what we will state below as Lemma A, while the denominator is bounded by Lemma B. The treatment of the horizontal segments is in fact the easier part and will be dealt with by Lemma C.



Lemma A. *For number fields $\mathbf{K}_1^*, \dots, \mathbf{K}_M^*$ of degrees $d_1, \dots, d_M \in \{1, 2\}$, such that $d_1 + \dots + d_M = 4$, and T large,*

$$\int_0^T \prod_{m=1}^{M} \left|\zeta_{\mathbf{K}_m^*}\left(\tfrac{1}{2} + \mathrm{i}t\right)\right| \mathrm{d}t \ll T(\log T)^M .$$

P r o o f. By Cauchy's inequality, this is immediate from

$$\int_0^T \left|\zeta\left(\tfrac{1}{2} + \mathrm{i}t\right)\right|^4 \mathrm{d}t \ll T(\log T)^4 ,$$

which is a weak form of A. E. Ingham's classic asymptotics (see, e.g., A. I v i ć's monograph [3, p. 129]), along with the more recent bound

$$\int_0^T \left|\zeta_{\mathbf{K}}\left(\tfrac{1}{2} + \mathrm{i}t\right)\right|^2 \mathrm{d}t \ll T(\log T)^2 ,$$

which is true for any quadratic number field $\mathbf{K}$. This latter estimate has been established by W. M ü l l e r [7], even in the stronger form of an asymptotic formula. □

Lemma B. *Let $\mathbf{K}$ be an arbitrary algebraic number field, $\zeta_{\mathbf{K}}$ its Dedekind zeta-function, and $\epsilon > 0$ fixed. Then there exists a measurable set $\mathscr{A} \subset [10, \infty[$, whose indicator function is denoted by $\mathbf{c}_{\mathscr{A}}(\cdot)$, with the following properties:*

(i) *For $t \geq 10$, $t \notin \mathscr{A}$,*

$$|\zeta_{\mathbf{K}}(1 + \mathrm{i}t)|^{\pm 1} \ll \log\log t .$$

(ii) *For all $T > 10$,*

$$\lambda_{\mathscr{A}}(T) := \int_{10}^{T} \mathbf{c}_{\mathscr{A}}(t) \, \mathrm{d}t \ll T^{\epsilon} .$$

P r o o f. This is based on ideas due to K. R a m a c h a n d r a and A. S a n k a r a n a r a y a n a n who treated the case of the Riemann zeta-function ([8]). □

Remark. The truth of the Riemann Hypothesis (for the particular Dedekind zeta-function involved) would imply the bound $|\zeta_{\mathbf{K}}(1 + \mathrm{i}t)|^{\pm 1} \ll \log\log t$ for all $t \geq 10$. The result says that this is true unconditionally "almost everywhere", i.e., outside of an exceptional set of "small" measure.

Lemma C. *For an arbitrary algebraic number field* $\mathbf{K}$ *with Dedekind zeta-function* $\zeta_{\mathbf{K}}$*, and each fixed* $\epsilon > 0$,

$$|\zeta_{\mathbf{K}}(\sigma + \mathrm{i}t)|^{\pm 1} \ll t^{\epsilon}$$

uniformly in $\sigma \geq 1$, $t \geq 1$. *Furthermore, if* $d = [\mathbf{K} : \mathbf{Q}] = 1$ *or* 2, *then*

$$\zeta_{\mathbf{K}}(\sigma + \mathrm{i}t) \ll 1 + |t|^{(5d/12)(1-\sigma)+\epsilon}$$

uniformly in $\sigma \geq \frac{1}{2}$, $|t| \geq 1$.

Proof. This follows by standard arguments, like the functional equations of zeta-functions, the Phragmén-Lindelöf theorem, and the easy bound $\zeta(\frac{1}{2} + \mathrm{i}t) \ll t^{1/6} \log t$ (see E. C. Titchmarsh [9, Theorem 5.12]). □

3. Examples

3.1. The four-factors divisor function

$$d_4(n) = \#\{(m_1, m_2, m_3, m_4) \in \mathbf{Z}_+^4 : \; m_1 m_2 m_3 m_4 = n\}.$$

Here the generating function has the particularly simple shape

$$F(s) = \sum_{n=1}^{\infty} \frac{d_4(n)}{n^s} = (\zeta(s))^4 \qquad (\Re(s) > 1)$$

and our theorem contains the most classic estimate

$$\sum_{n \leq x} d_4(n) = x\, P_3(\log x) + O\big(x^{1/2} (\log x)^5\big)\,.$$

This was established by Hardy and Ingham already in the 1920's; see also [9, Ch. XII]. However, the corresponding short-interval result reads

$$\sum_{x < n \leq x+y} d_4(n) \sim \frac{1}{6}\, y \log^3 x \qquad \text{if} \quad \frac{y}{x^{1/2} \log x} \to \infty$$

and appears to be new: see [1] for details.

3.2. The second moment of quadratic Dedekind-zeta coefficients

Let $\mathbf{K} = \mathbf{Q}(\sqrt{D})$ a quadratic number field, D its discriminant, and define the arithmetic function $r_{\mathbf{K}}(n)$ by the Dirichlet series

$$\zeta_{\mathbf{K}}(s) = \sum_{n=1}^{\infty} \frac{r_{\mathbf{K}}(n)}{n^s} \qquad (\Re(s) > 1)\,.$$

Considering the quadratic moment of $r_{\mathbf{K}}(n)$ we are lead to the generating function

$$F(s) = \sum_{n=1}^{\infty} \frac{(r_{\mathbf{K}}(n))^2}{n^s} = \frac{\zeta_{\mathbf{K}}^2(s)}{\zeta(2s)} \prod_{p|D} (1+p^{-s})^{-1} \qquad (\Re(s) > 1)\,,$$

as is verified in [6].

Our theorem yields

$$\sum_{n \le x} (r_{\mathbf{K}}(n))^2 = x\,P_1(\log x) + O\big(x^{1/2} (\log x)^3 \log\log x\big)$$

and

$$\sum_{x<n\le x+y} (r_{\mathbf{K}}(n))^2 \sim B_0\, y \log x \qquad \text{if} \quad \frac{y}{x^{1/2} \log x \log\log x} \to \infty\,.$$

Here and throughout the sequel, P_1 is a linear polynomial with leading coefficient B_0, not necessarily the same at different occurrences.

Observe that the special case $\mathbf{K} = \mathbf{Q}(\mathrm{i})$ provides also an answer to A. Schinzel's question mentioned in the introduction.

3.3. Diophantine equations like $U^2 + V^2 = W^3$

In fact, the left-hand side could be replaced by any integral positive definite binary quadratic form of class number 1. To fix notions, we stick to this special equation and count its integer solutions with W ranging up to a large parameter x:

$$\#\big\{(U,V,W) \in \mathbf{Z}^3 :\ U^2 + V^2 = W^3,\ 0 < W \le x\big\} = \sum_{n\le x} r(n^3)\,.$$

As it has been verified in [6], the generating function reads

$$F(s) = \sum_{n=1}^{\infty} \frac{r(n^3)}{n^s} = \frac{\zeta_{\mathbf{Q}(\mathrm{i})}^2(s)}{\zeta(2s)\zeta_{\mathbf{Q}(\mathrm{i})}(2s)}\, G_1(s) \qquad (\Re(s) > 1)\,,$$

where $G_1(s)$ has an absolutely convergent Euler product in $\Re(s) > \frac{1}{3}$.

Our theorem gives

$$\sum_{n\le x} r(n^3) = x\,P_1(\log x) + O\big(x^{1/2} (\log x)^3 (\log\log x)^2\big)$$

and

$$\sum_{x<n\le x+y} r(n^3) \sim B_0\, y \log x \qquad \text{if} \quad \frac{y}{x^{1/2}\log x(\log\log x)^2} \to \infty\,.$$

3.4. The average order of $d(n)r(n)$

In [2] we have computed that

$$\sum_{n=1}^{\infty} \frac{d(n)r(n)}{n^s} = \frac{4\,\zeta^2_{\mathbf{Q}(\mathrm{i})}(s)}{\zeta_{\mathbf{Q}(\mathrm{i})}(2s)\,(\zeta(2s))^{-1}} \qquad (\Re(s)>1)\,.$$

Applying the theorem thus yields

$$\sum_{n\le x} d(n)r(n) = x\,P_1(\log x) + O\big(x^{1/2}(\log x)^3(\log\log x)^2\big)$$

and

$$\sum_{x<n\le x+y} d(n)r(n) \sim B_0\, y \log x \qquad \text{if} \quad \frac{y}{x^{1/2}\log x(\log\log x)^2} \to \infty\,.$$

4. Concluding remark

It is a most natural question to ask for the significance of the very restrictive condition in our theorem that $d_1 + \cdots + d_M = 4$. In the light of the first example, this corresponds to the fact that the divisor problem of dimension 4 is sort of a quite special case: The error bound $O(x^{1/2}(\log x)^5)$ has resisted all attempts to improve it for some 80 years. In view of Ingham's asymptotics for the fourth moment of the zeta-function along the critical line, it is a very "precise" result. On the contrary, the records for error bounds in the divisor problems of dimensions 2 and 3 depend on the progress of exponential sum techniques, while those in dimensions ≥ 5 involve intrinsic order estimates for the zeta-function in the critical strip. See again [3] and [9]. Thus for $d_1 + \cdots + d_M \ne 4$ our particularly fine analysis taking care of log- and $\log\log$-factors would be meaningless. However, it is amazing how many interesting applications (cf. also [2] and [6] for a few more) are contained in the special case considered.

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