

WEAK MV-ALGEBRAS

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ABSTRACT. In a recent paper [CHAJDA, I.—KÜHR, J.: *A non-associative generalization of MV-algebras*, Math. Slovaca **57**, (2007), 301–312], authors introduced and studied a non-associative generalization of MV-algebras called NMV-algebras. In contrast to MV-algebras, sections (i.e. principal filters) in NMV-algebras which are proper (i.e. are not MV-algebras), do not admit a structure of an NMV-algebra with respect to the operations defined in a natural way. The aim of the paper is to present a new class of algebras generalizing MV-algebras but sharing the above property.

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1. Introduction

Recall that MV-algebras were introduced in 50'ties by C. C. Chang as an algebraic semantics of the Łukasiewicz many valued propositional logic (see [6], [7]). More precisely, an MV-algebra is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities:

$$(MV1) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

$$(MV2) \quad x \oplus y = y \oplus x$$

$$(MV3) \quad x \oplus 0 = x$$

$$(MV4) \quad \neg\neg x = x$$

$$(MV5) \quad x \oplus 1 = 1 \text{ (where } 1 := \neg 0)$$

$$(MV6) \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

A typical example of an MV-algebra can be obtained as follows: consider any abelian lattice ordered group $(G, +, -, 0, \wedge, \vee)$ and take $0 < u \in G$. Then the interval $[0, u] = \{x \in G : 0 \leq x \leq u\}$ with the operations defined by

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$x \oplus y := (x + y) \wedge u$ and $\neg x := u - x$ becomes an MV-algebra. Denoting such an MV-algebra as $\Gamma(G, u)$, D. Mundici [15] (see also [8]) proved that for every MV-algebra \mathcal{A} there exists an abelian l-group G and $0 < u \in G$ with $\mathcal{A} \cong \Gamma(G, u)$.

Given an MV-algebra \mathcal{A} , the relation \leq defined by

$$(1) \quad x \leq y \iff \neg x \oplus y = 1,$$

is known to be a lattice order on A with $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$, the top or the bottom element of which is 1 or 0, respectively.

Moreover, for any MV-algebra \mathcal{A} and $p \in A$, one can define on the interval $[p, 1]$ (usually called *section*) a structure of an MV-algebra (called *section MV-algebra* on $[p, 1]$) in a natural way as follows:

$$(2) \quad x \oplus_p y = \neg(\neg x \oplus p) \oplus y, \quad \neg_p x = \neg x \oplus p.$$

In the recent years a non-commutative generalization of MV-algebras was introduced and studied by G. Georgescu and A. Iorgulescu [11] as pseudo MV-algebras and independently by J. Rachůnek under the name GMV-algebras ([16]).

In principle, these are algebras with a binary operation \oplus and two unary operations \sim and \neg (negations) coinciding whenever \oplus is commutative. More precisely, given any (not necessarily commutative) l-group G and $0 < u \in G$, then upon defining $x \oplus y := (x + y) \wedge u$, $\neg x := u - x$, $\sim x = -x + u$, the resulting algebra $\Gamma(G, u) = ([0, u], \oplus, \neg, \sim, 0)$ becomes a GMV-algebra.

Similarly as for a commutative case, A. Dvurečenskij [9] proved that all GMV-algebras are of the form $\Gamma(G, u)$ for any l-group G .

Another important approach to generalize MV-algebras by omitting associativity (MV1) but keeping commutativity (MV2) was done by I. Chajda and J. Kühr [3]. More precisely, they considered algebras $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the axioms (MV2)–(MV6), where the axiom (MV1) was substituted by two more axioms

$$(WA) \quad \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$$

(here so-called *weak associativity*)

and

$$(H) \quad \neg x \oplus (x \oplus y) = 1.$$

These algebras are called *NMV-algebras* (non-associative MV-algebras) ([3]). Clearly, every MV-algebra fulfils the axioms (WA) and (H), but the converse is not true.

To clarify the role of the axiom (WA), its validity enables to prove that the relation \leq defined by (1) remains transitive (and hence being the order relation). From a logical point of view, such a property is quite natural since in all reasonable logics the set of truth values should be partially ordered.

We have seen that the sections in MV-algebras form MV-algebras as given by (2). However, this property is not true for NMV-algebras: I. Chajda recently proved that given an NMV-algebra \mathcal{A} , the sections $[p, 1]$ have a structure of an NMV-algebra as defined by (2) iff \oplus is associative. In other words, an NMV-algebra shares the mentioned property iff it is an MV-algebra.

The aim of this paper is to find a new class of generalized MV-algebras admitting the same structure on sections.

2. Weak MV-algebras

DEFINITION 1. An algebra $(A, \oplus, \neg, 0)$ is called a weak MV-algebra of type $(2, 1, 0)$ or *WMV-algebra* briefly if it satisfies the axioms $(1 := \neg 0)$:

$$(WMV1) \quad \neg \neg x = x$$

$$(WMV2) \quad \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$$

$$(WMV3) \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

$$(WMV4) \quad x \oplus 0 = 0 \oplus x = x$$

$$(WMV5) \quad \neg y \oplus (\neg x \oplus y) = 1$$

$$(WMV6) \quad p \leq x \leq y \implies \neg y \oplus p \leq \neg x \oplus p, \text{ where } \leq \text{ is defined by (1).}$$

Note that applying (WMV4) and (WMV5), every WMV-algebra satisfies the identity

$$x \oplus 1 = 1 \oplus x = 1.$$

To show that the relation defined by (1) is a partial order on A for any WMV-algebra \mathcal{A} , we use the same arguments as in the case of NMV-algebras: putting $x = 0$ in (WMV5), with respect to (WMV4), we obtain the identity $\neg y \oplus y = 1$, so \leq is reflexive. The antisymmetry of \leq is guaranteed just by (WMV1), (WMV4) and (WMV3). Finally, if $x \leq y$ and $y \leq z$ then $\neg x \oplus y = \neg y \oplus z = 1$, hence (WMV2) together with (WMV4) entail $\neg x \oplus z = 1$. Altogether, \leq is a partial order on A .

In the sequel we will discuss a structure of sections in WMV-algebras. To this aim, we need the following terminology.

Given a poset (P, \leq) , we denote $L(x, y) = \{a \in P : a \leq x, a \leq y\}$ and $U(x, y) = \{a \in P : a \geq x, a \geq y\}$ for any $x, y \in P$. A poset (P, \leq) is called *upwards (downwards) directed* if $U(x, y) \neq \emptyset$ ($L(x, y) \neq \emptyset$) holds for all $x, y \in P$. Directed poset is both upwards and downwards directed.

In his unpublished thesis [17], V. Snášel introduced the concept of a λ -semilattice as a natural generalization of semilattices:

an algebra (L, \cup) of type (2) is called a λ -*semilattice* if it satisfies the identities

$$(S1) \quad x \cup x = x \text{ (idempotency)}$$

$$(S2) \quad x \cup y = y \cup x \text{ (commutativity)}$$

$$(S3) \quad x \cup ((x \cup y) \cup z) = (x \cup y) \cup z \text{ (weak associativity).}$$

Remark that λ -semilattices were studied by J. Ježek and R. Quackenbush under the name *directoids*, see [12].

Similarly as for lattices, by a λ -lattice ([18]) we mean a structure (L, \cup, \cap) , where (L, \cup) and (L, \cap) are two λ -semilattices connected by absorption laws

$$(Ab) \quad x \cap (x \cup y) = x, \quad x \cup (x \cap y) = x.$$

It is quite easy to show that given a λ -semilattice (L, \cup) , the relation \leq defined by

$$(3) \quad x \leq y \iff x \cup y = y$$

is partial order on L , where $x \cup y \in U(x, y)$ for all $x, y \in L$.

An order relation defined by (3) will be referred as an *induced order*. Thus (L, \leq) is an upwards directed poset.

Remark that given a λ -lattice $\mathcal{L} = (L, \cup, \cap)$, the induced order \leq defined by (3) coincides with that given by $x \leq y \iff x \cap y = x$. Hence we can refer to \leq as the induced order on a λ -lattice \mathcal{L} .

Given a λ -semilattice $\mathcal{L} = (L, \cup)$ with the induced ordering \leq , a mapping $f: L \rightarrow L$ is called an *antitone involution* on \mathcal{L} if

- (i) $f(f(x)) = x$
- (ii) $x \leq y \implies f(x) \geq f(y)$.

A structure (L, \cup, f) is then said to be a λ -semilattice with an antitone involution f .

By a λ -lattice with an antitone involution f we mean a structure (L, \cup, \cap, f) , where (L, \cup, f) is a λ -semilattice with an antitone involution f .

The next statement shows that each WMV-algebra can be viewed as a λ -semilattice of WMV-algebras:

THEOREM 1. *Let $(A, \oplus, \neg, 0)$ be a WMV-algebra, $p \in A$, $x, y \in [p, 1]$. Then upon defining*

$$\begin{aligned} x \oplus_p y &:= \neg(\neg x \oplus p) \oplus y, \\ \neg_p x &:= \neg x \oplus p, \end{aligned}$$

the structure $([p, 1], \oplus_p, \neg_p, p)$ is a WMV-algebra. Moreover, putting

$$\begin{aligned} x \cup y &:= \neg(\neg x \oplus y) \oplus y, \\ x \cap_p y &:= \neg_p(\neg_p x \cup \neg_p y), \end{aligned}$$

the algebra $([p, 1], \cap_p, \cup, \neg_p)$ is a λ -lattice with an antitone involution.

Proof. First we show that (A, \cup) is a λ -semilattice. Indeed, putting $y = z = 0$ in (WMV2) we obtain $\neg x \oplus x = 1$, thus $x \cup x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x$ due to (WMV4). Clearly (WMV3) just means that $x \cup y = y \cup x$. The axiom (WMV2) can be rewritten to $\neg x \oplus ((x \cup y) \cup z) = 1$. Thus

$$\begin{aligned} x \cup ((x \cup y) \cup z) &= \neg(\neg x \oplus ((x \cup y) \cup z)) \oplus ((x \cup y) \cup z) \\ &= \neg 1 \oplus ((x \cup y) \cup z) = 0 \oplus ((x \cup y) \cup z) = (x \cup y) \cup z. \end{aligned}$$

Further, the axiom (WMV5) is equivalent to $y \leq \neg x \oplus y$, hence for each $p \in A$, \cup is a binary operation on $[p, 1]$.

To show that $([p, 1], \cap_p)$ is a λ -semilattice, we compute

$$x \cap_p x = \neg_p(\neg_p x \cup \neg_p x) = \neg_p \neg_p x = \neg(\neg x \oplus p) \oplus p = x \cup p = x$$

for each $x \in [p, 1]$. The commutativity of \cap_p is easily seen from its definition.

Let us prove the identity $x \cap_p ((x \cap_p y) \cap_p z) = (x \cap_p y) \cap_p z$ for all $x, y, z \in [p, 1]$. To simplify expressions, denote by $P = (x \cap_p y) \cap_p z$ and $x \bullet p = \neg x \oplus p$. Then we have $x \cup y = (x \bullet y) \bullet y = (y \bullet x) \bullet x$ and

$$P = (((((x \bullet p) \cup (y \bullet p)) \bullet p) \bullet p) \cup (z \bullet p)) \bullet p = (((((x \bullet p) \cup (y \bullet p)) \cup p) \cup (z \bullet p)) \bullet p).$$

According to (WMV5), $x \bullet p, y \bullet p \geq p$, hence $((x \bullet p) \cup (y \bullet p)) \cup p = (x \bullet p) \cup (y \bullet p)$ and $P = (((x \bullet p) \cup (y \bullet p)) \cup (z \bullet p)) \bullet p$. This yields

$$P \bullet p = (((x \bullet p) \cup (y \bullet p)) \cup (z \bullet p)) \cup p = ((x \bullet p) \cup (y \bullet p)) \cup (z \bullet p) \geq x \bullet p$$

by (WMV2), hence also

$$x \cap_p P = ((x \bullet p) \cup (P \bullet p)) \bullet p = (P \bullet p) \bullet p = P \cup p = P.$$

Finally, we show the validity of absorption laws: we have

$$x \cap_p (x \cup y) = ((x \bullet p) \cup ((x \cup y) \bullet p)) \bullet p = (x \bullet p) \bullet p = x \cup p = x,$$

since $x \bullet p \geq (x \cup y) \bullet p$ by (WMV6). Applying (WMV6) again we obtain

$$x \cap_p y = ((x \bullet p) \cup (y \bullet p)) \bullet p \leq (x \bullet p) \bullet p = x,$$

which gives

$$x \cup (x \cap_p y) = (((x \bullet p) \cup (y \bullet p)) \bullet p) \bullet x = 1 \bullet x = x,$$

and we are done.

Now we show that $([p, 1], \oplus_p, \neg_p, p)$ is a WMV-algebra.

(WMV1) We have $\neg_p \neg_p x = (x \bullet p) \bullet p = x \cup p = x$.

(WMV2) Given $x, y \in [p, 1]$, we compute

$$\neg_p x \oplus_p y = ((x \bullet p) \bullet p) \bullet y = (x \cup p) \bullet y = x \bullet y = \neg x \oplus y.$$

Hence, the validity of (WMV2) for $[p, 1]$ is a conclusion of (WMV2) for \mathcal{A} .

(WMV3) Again, take $x, y \in [p, 1]$. Then

$$\begin{aligned}\neg_p(\neg_p x \oplus_p y) \oplus_p y &= (((((x \bullet p) \bullet p) \bullet y) \bullet p) \bullet p) \bullet y \\ &= (((x \bullet y) \bullet p) \bullet p) \bullet y = ((x \bullet y) \cup p) \bullet y.\end{aligned}$$

Clearly, due to (WMV6), $p \leq y \leq x \bullet y$, hence

$$\neg_p(\neg_p x \oplus_p y) \oplus_p y = x \cup y = y \cup x = \neg_p(\neg_p y \oplus_p x) \oplus_p x.$$

(WMV4) For $x \in [p, 1]$ we derive $x \oplus_p y = (x \bullet p) \bullet p = x \cup p = x$, $p \oplus_p x = (p \bullet p) \bullet x = 1 \bullet x = x$.

(WMV5) If $x, y \in [p, 1]$, then by (WMV2), $\neg_p y \oplus_p (\neg_p x \oplus_p y) = \neg_p y \oplus_p (x \bullet y) = ((y \bullet p) \bullet p) \bullet (x \bullet y) = (y \cup p) \bullet (x \bullet y) = y \bullet (x \bullet y) = 1$.

(WMV6) Let $p \leq q \leq x \leq y$. Then we conclude $\neg_p y \oplus_p p = \neg y \oplus p$, $\neg_p x \oplus_p p = \neg x \oplus p$. Hence $\neg y \oplus p \leq \neg x \oplus p$ by (WMV6); moreover, denoting \leq_p the order on $[p, 1]$ given by $x \leq_p y$ iff $\neg_p x \oplus_p y = 1$, we have $\neg_p x \oplus_p y = \neg x \oplus y$, thus \leq is the same as \leq_p and we are done. \square

3. λ -semilattices with section antitone involutions

By a λ -semilattice with section antitone involutions we mean a λ -semilattice (S, \cup) with the top element 1, where every section $[p, 1]$ has an antitone involution $p: x \mapsto x^p$. If, moreover, (S, \cup) has a least element 0, we speak about a *bounded* λ -semilattice with section antitone involutions.

Thus a λ -semilattice with section antitone involutions is a structure of type $(S, \cup, ({}^a)_{a \in S}, 1)$.

In order to overcome the difficulties with many partial unary operations p , one can define a total binary operation \bullet on S by

$$(4) \quad x \bullet y := (x \cup y)^y.$$

Clearly $x \bullet y$ is well defined since $x \cup y \in [y, 1]$.

The following easy lemma shows that λ -semilattices with section antitone involutions can be axiomatized by identities:

LEMMA 1. *A λ -semilattice (S, \cup) with the top element 1 is a λ -semilattice with section antitone involutions iff there is a binary operation \bullet on S having the properties:*

- (i) $((x \cup p) \cup y) \bullet p) \bullet ((x \cup p) \bullet p) = 1$,
- (ii) $x \cup y = (x \bullet y) \bullet y$,
- (iii) $((x \bullet y) \bullet y) \bullet y = x \bullet y$.

PROOF. Let (S, \cup) be a λ -semilattice with section antitone involutions with the induced order \leq and let \bullet be defined by (4). Observe that $x \leq y$ iff $x \bullet y = 1$.

Now given $x, y, p \in S$, clearly $\alpha = (x \cup p) \cup y \geq x \cup p \geq p$. Since the involution p on $[p, 1]$ is antitone, we have $\alpha^p \leq (x \cup p)^p$, thus $\alpha^p \bullet (x \cup p)^p = 1$, which is just (i).

To prove (ii), we compute $(x \bullet y) \bullet y = ((x \cup y)^y \cup y)^y = (x \cup y)^{yy} = x \cup y$. Finally, applying (i) we get $((x \bullet y) \bullet y) \bullet y = (x \cup y) \bullet y = ((x \cup y) \cup y)^y = (x \cup y)^y = x \bullet y$.

Conversely, assume that \bullet satisfies the conditions (i)–(iii). For $x \in [p, 1]$, define $x^p := x \bullet p$. By (ii) and (iii), $(x \bullet p) \cup p = ((x \bullet p) \bullet p) \bullet p = x \bullet p$, thus $x^p = x \bullet p \geq p$ and so $x^p \in [p, 1]$. Further, $x^{pp} = (x \bullet p) \bullet p = x \cup p = x$ by (ii), hence $x \mapsto x^p$ is an involution on $[p, 1]$. To show that $x \mapsto x^p$ is antitone, assume $p \leq x \leq y$. Then $(x \cup p) \cup y = x \cup y = y$, $x \cup p = x$, which due to (i) gives $y^p \leq x^p$ as desired. Moreover, in view of (ii) and (iii) $x \bullet y = ((x \bullet y) \bullet y) \bullet y = (x \cup y) \bullet y = (x \cup y)^y$. \square

Lemma 1 shows that λ -semilattices with section antitone involutions can be considered as algebras $(S, \cup, 1, \bullet)$ of type $(2, 0, 2)$, where $(S, \cup, 1)$ is a λ -semilattice with a top element 1 satisfying the identities (i)–(iii).

By a λ -lattice with section antitone involutions we mean a structure $(A, \cup, \cap, 1, \bullet)$, where $(A, \cup, \cap, 1)$ is λ -lattice with a top element 1 and $(A, \cup, 1, \bullet)$ is a λ -semilattice with section antitone involutions.

We show that λ -lattices with section antitone involutions fulfill from a congruence point of view as much as we can hope:

THEOREM 2. *The variety of all λ -lattices with sectional antitone involutions is regular and arithmetical.*

PROOF. Let \mathcal{V} be the variety of λ -lattices with section antitone involutions.

\mathcal{V} is regular: Let

$$\begin{aligned} t_1(x, y, z) &= ((x \bullet y) \cap (y \bullet x)) \cap z, \\ t_2(x, y, z) &= ((x \bullet y) \bullet z) \cup ((y \bullet x) \bullet z). \end{aligned}$$

We show that $t_1(x, y, z) = t_2(x, y, z) = z$ iff $x = y$.

Obviously, $t_1(x, x, z) = t_2(x, x, z) = z$. Conversely, assume that $t_1(x, y, z) = t_2(x, y, z) = z$. Then $z \leq x \bullet y, y \bullet x$ and $z \geq (x \bullet y) \bullet z, (y \bullet x) \bullet z$. But by

Lemma 1(iii) we have $(x \bullet y) \bullet z, (y \bullet x) \bullet z \geq z$, so that $(x \bullet y) \bullet z = z = (y \bullet x) \bullet z$, whence $x \bullet y = (x \bullet y) \cup z = ((x \bullet y) \bullet z) \bullet z = z \bullet z = 1$, so $x \leq y$. Similarly $y \leq x$, and hence $x = y$.

\mathcal{V} is arithmetical: Let

$$m(x, y, z) = (((x \bullet y) \bullet z) \cap ((z \bullet y) \bullet x)) \cap (x \cup z).$$

Prove that $m(x, y, y) = m(x, y, x) = m(y, y, x) = x$. We have

$$\begin{aligned} m(x, y, y) &= (((x \bullet y) \bullet y) \cap ((y \bullet y) \bullet x)) \cap (x \cup y) = ((x \cup y) \cap x) \cap (x \cup y) = x, \\ m(x, y, x) &= (((x \bullet y) \bullet x) \cap ((x \bullet y) \bullet x)) \cap (x \cup x) = ((x \bullet y) \bullet x) \cap x = x \end{aligned}$$

since $(x \bullet y) \bullet x \geq x$ by Lemma 1(ii) and (iii), and $m(y, y, x) = (((y \bullet y) \bullet x) \cap ((x \bullet y) \bullet y)) \cap (y \cup x) = (x \cap (x \cup y)) \cap (y \cup x) = x$. \square

We shall prove that there is a 1–1 correspondence between WMV-algebras and bounded λ -semilattices with section antitone involutions:

THEOREM 3.

(a) Let $(A, \oplus, \neg, 0)$ be a WMV-algebra. Define

$$x \cup y := \neg(\neg x \oplus y) \oplus y \text{ and}$$

$$x \bullet y := \neg x \oplus y.$$

Then $\Phi(A) = (A, \cup, \bullet, 0, 1)$ is a bounded λ -semilattice with section antitone involutions.

(b) Let $(S, \cup, \bullet, 0, 1)$ be a bounded λ -semilattice with section antitone involutions. If we define

$$x \oplus y := (x \bullet 0) \bullet y,$$

$$\neg x := x \bullet 0,$$

then $\Psi(S) = (S, \oplus, \neg, 0)$ is a WMV-algebra.

(c) Moreover, we have $\Psi(\Phi(A)) = A$ and $\Phi(\Psi(S)) = S$.

Proof.

(a) We already know from Theorem 1 that (A, \cup) is a bounded λ -semilattice. We show that the conditions (i)–(iii) of Lemma 1 are satisfied. It is immediately seen from the definition of induced order \leq on A that

$$(5) \quad x \leq y \iff \neg x \oplus y = 1 \iff x \bullet y = 1.$$

Thus given $x, y, p \in A$, we have $p \leq x \cup p \leq (x \cup p) \cup y$, which due to (WMV6) yields

$$((x \cup p) \cup y) \bullet p = \neg((x \cup p) \cup y) \oplus p \leq \neg(x \cup p) \oplus p = (x \cup p) \bullet p.$$

This inequality is with respect to (5) just (i). The identity (ii) is just the transcription of the axiom (WMV3). Finally, to prove (iii) we compute

$$((x \bullet y) \bullet y) \bullet y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = (\neg x \oplus y) \cup y = \neg x \oplus y = x \bullet y,$$

since $y \leq \neg x \oplus y$ by (WMV5).

(b) Assume conversely that $(S, \cup, \bullet, 0, 1)$ is a bounded λ -semilattice with section antitone involutions.

Observe that by (ii), $\neg x \oplus y = ((x \bullet 0) \bullet 0) \bullet y = (x \cup 0) \bullet y = x \bullet y$.

(WMV1) $\neg \neg x = (x \bullet 0) \bullet 0 = x \cup 0 = x$ by (ii),

(WMV2) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = x \bullet (((x \bullet y) \bullet y) \bullet z) \bullet z = x \bullet ((x \cup y) \cup z) = 1$ since $x \leq (x \cup y) \cup z$.

(WMV3) $\neg(\neg x \oplus y) \oplus y = (x \bullet y) \bullet y = x \cup y = y \cup x = \neg(\neg y \oplus x) \oplus x$ by (ii),

(WMV4) $x \oplus 0 = (x \bullet 0) \bullet 0 = x \cup 0 = x$, $0 \oplus x = (0 \bullet x) \bullet x = 1 \bullet x = 1^x = x$ as for each $x \in A$ the involution x on $[x, 1]$ exchanges the elements $x \leftrightarrow 1$.

(WMV5) $\neg x \oplus y = x \bullet y \geq y$, hence $\neg y \oplus (\neg x \oplus y) = 1$.

(WMV6) Rewriting (i) in terms of \oplus and \neg , we get

$$\neg((x \cup p) \cup y) \oplus p \leq \neg(x \cup y) \oplus p,$$

which is just (WMV6).

Now, let $(A, \oplus, \neg, 0)$ be a WMV-algebra and denote $\Phi(A) = (A, \cup, \bullet, 0, 1)$, $\Psi(\Phi(A)) = (A, \oplus', \neg', 0)$. We have $x \oplus' y = (x \bullet 0) \bullet y = \neg(\neg x \oplus 0) \oplus y = x \oplus y$, $\neg' x = x \bullet 0 = \neg x \oplus 0 = \neg x$. Hence $\Psi(\Phi(A)) = A$.

Conversely, let $(S, \cup, \bullet, 0, 1)$ be a bounded λ -semilattice with section antitone involutions. Denote $\Psi(S) = (S, \oplus, \neg, 0)$ and $\Phi(\Psi(S)) = (S, \cup', \bullet', 0, 1')$. Clearly $x \cup' y = \neg(\neg x \oplus y) \oplus y = (x \bullet y) \bullet y = x \cup y$, $x \bullet' y = \neg x \oplus y = x \bullet y$ and $1' = \neg 0 = 0 \bullet 0 = 1$, thus $\Phi(\Psi(S)) = S$. \square

COROLLARY. *The variety of all WMV-algebras is regular and arithmetical.*

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