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ON THE ALEXANDROFF DECOMPOSITION THEOREM

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ABSTRACT. We prove an Alexandroff decomposition type theorem, which extends a decomposition theorem proved in [de LUCIA, P.—MORALES, P.: *Decomposition of group-valued measures in orthoalgebras*, Fund. Math. **158** (1998), 109–124].

Introduction

In [D-M1, p. 119] the authors proved an Alexandroff decomposition type theorem (see [A]) for an order bounded inner regular measure μ on a Boolean algebra L with values in a Hausdorff topological lattice group with a strong assumption, i.e. G has a base of neighbourhoods of 0 consisting of sublattices (see also [G-J-M]).

In the present paper we prove that this decomposition theorem is a particular case of a general decomposition theorem (see Theorem 3.9) which holds with a weaker assumption on G, i.e. the positive cone of G is closed, and also holds if L is replaced by a weaker structure, i.e. if L is a D-lattice (= lattice ordered effect algebra) and μ is an exhaustive modular measure on L.

We also prove a Hewitt-Yosida type decomposition theorem (see **3.10**) which extends another result of [D-M1] (see also [H-Y] and [D-N]).

We recall that effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 (see [B-F]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see [B-C], [B-G-L], [D]) and in Mathematical Economics (see [E-Z], [G-M], [B-K]), in particular of orthomodular posets and

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MV-algebras. Therefore the study of measures on effect algebras allows to extend and unify two independent theories: non-commutative measure theory (study of measures on orthomodular lattices) and fuzzy measure theory (study of measures on MV-algebras).

The paper is organized as follows: in Section 2 we prove some preliminary results which we need in what follows and in Section 3 we prove the main decomposition theorem and we compare it with the decomposition theorems of [D-M1].

1. Preliminaries

An *effect algebra* $(L, \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(3) For every $a \in L$, there exists a unique $a^\perp \in L$ such that $a \oplus a^\perp$ is defined and $a \oplus a^\perp = 1$.

(4) If $a \oplus 1$ is defined, then $a = 0$.

In every effect algebra a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a.

Moreover we can define a binary relation on L by $a \leq b$ if and only if there exists $c \in L$ such that $c \oplus a = b$ and $\leq$ is a partial ordering in L, with 0 as the smallest element. We say that two elements $a, b \in L$ are *orthogonal*, and we write $a \perp b$, if $a \oplus b$ exists. Then $a \perp b$ if and only if $a \leq b^\perp$. Moreover, for every $a \in L$, we have $a^\perp = 1 \ominus a$.

Effect algebras are a common generalization of orthomodular posets and MV-algebras. For a study, we refer to [D-P].

If $a_1, \ldots, a_n \in L$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, \ldots, a_n\}$ of L is *orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists.

We say that a subset A of L is *orthogonal* if every finite subset of A is orthogonal. If $A = \{a_\alpha : \alpha \in I\}$ is an orthogonal set in L, we set

$$\bigoplus_{\alpha \in I} a_\alpha = \sup \left\{ \bigoplus_{\alpha \in F} a_\alpha : \ F \subseteq I \ \text{finite} \right\}$$

provided that the right hand side exists.

If $(L, \leq)$ is a lattice, we say that the effect algebra is a *lattice ordered effect algebra* or a *D-lattice.* In this case, we set $a \triangle b = (a \vee b) \ominus (a \wedge b)$ for $a, b \in L$.

If G is a group, a function $\mu \colon L \to G$ is said to be a *measure* if $a \perp b$ implies $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure if and only if $a \leq b$ implies $\mu(b \ominus a) = \mu(b) - \mu(a)$.

A function $\mu \colon L \to G$ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

If G is a topological group, a measure $\mu \colon L \to G$ is said to be *exhaustive* if, for every orthogonal sequence $\{a_n\}$ in L, $\mu(a_n) \to 0$. Moreover μ is said to be *σ-additive* if, for every orthogonal sequence $\{a_n\}$ such that $a = \bigoplus_n a_n$ exists, $\mu(a) = \sum_{n=1}^{\infty} \mu(a_n)$ and μ is said to be *σ-order continuous* (σ-o.c.) if, for every decreasing sequence $\{a_n\}$ in L such that $\inf_n a_n = 0$, $\lim_n \mu(a_n) = 0$.

By [A-B1, 2.4], a measure $\mu \colon L \to G$ is σ-additive if and only if it is σ-o.c.

By [A-B1, 4.2], if G is a topological Abelian group, every modular measure μ generates on L a *D-uniformity* $\mathscr{U}(\mu)$, i.e. a uniformity which makes the lattice operations, as well as $\ominus$ and $\oplus$, uniformly continuous. A base of $\mathscr{U}(\mu)$ is the family consisting of the sets

$$\big\{(a, b) \in L \times L : \ (\forall\, c \leq a \triangle b)(\mu(c) \in W)\big\},$$

where W is a neighbourhood of 0 in G. We write $\mathscr{U}(\mu) = 0$ if $\mathscr{U}(\mu)$ is the trivial uniformity. By [A-V2, 2.7], the set of all D-uniformities on L is a distributive lattice with respect to the usual order between uniformities.

A group G is said to be an *ordered group* if there is an order relation $\leq$ on G with the following property: If $x, y \in G$ and $x \leq y$, then $x + z \leq y + z$ and $z + x \leq z + y$ for every $z \in G$. If $(G, \leq)$ is a lattice, we say that G is a *lattice group* or an *ℓ-group.* If G is a topological group and an ℓ-group and $\vee$, $\wedge$ are continuous, we say that G is a *topological ℓ-group.*

If G is an ℓ-group, we set, for every $x \in G$,

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0,$$
$$|x| = x^+ + x^-.$$

It is known (see e.g. [B, Chap. XIII]) that, for every $x \in G$, we have

$$x = x^+ - x^- \qquad \text{and} \qquad |x| = x \vee (-x).$$

In an ordered group G, a subset A of G is said to be *order convex* if, for every $x, y \in A$ with $x < y$, the *interval* $[x, y] = \{z \in G : \ x \leq z \leq y\}$ is contained in A and *order bounded* if A is contained in an interval of G.

We say that G is *order complete* if, for every non-empty majorized subset D of G, $\sup D$ exists in G and *quasi-order complete* if, for every majorized upward directed subset D of G, $\sup D$ exists in G.

It is known (see Iwasawa theorem in [B]) that every order complete ℓ-group is Abelian.

If $(G, \tau, +)$ is an ordered topological group, we say that G is *locally order convex* if the set of all order convex neighborhoods of 0 in G forms a base for τ and that G is *compatible* if $G^+ = \{x \in G : \ x \geq 0\}$ is closed in τ.

If G is an ℓ-group, a subset A of G is said to be *solid* if, for every $x \in G$ and $y \in A$ with $|x| \leq |y|$, we have $x \in A$. We say that a topological ℓ-group G is *locally solid* if there exists a base of neighbourhoods of 0 consisting of solid sets.

We need the following result ([W, 1.1.8]).

Proposition 1.1. *Let G be a topological ℓ-group. Then G is locally solid if and only if G is a locally order convex topological lattice.*

Following [D-M1], we say that an ordered group G *has the property* (oc) if:

(oc) for every directed upward subset D of G such that $x = \sup D$ exists in G, we have $x \in \overline{D}$,

and a topological ℓ-group G *satisfies the condition* (M) if:

(M) there exists a base of neighbourhoods of 0 consisting of sublattices of G.

We need the following results ([D-M1, 3.1] and [D-M2, 2.2]).

Proposition 1.2. *Let G be a Hausdorff order complete ℓ-group with the property* (oc). *If V is a sublattice of G and D is a non-empty majorized subset of V, then $x = \sup D$ belongs to $\overline{V}$.*

Proposition 1.3. *Let G be an ordered topological group which is locally order convex. Then the following conditions are equivalent:*

(1) *G has the property* (oc).

(2) *For every increasing net $(x_i)_{i \in I}$ in G such that $x = \sup\limits_{i \in I} x_i$ exists in G, we have $x = \lim\limits_{i \in I} x_i$.*

In what follows L *is a D-lattice* and G is a Hausdorff topological Abelian group.

2. The space of all order bounded modular measures

In this Section we suppose that G is ordered.

A measure $\mu \colon L \to G$ is said to be *order bounded* if $\mu(L)$ is order bounded in G. If $\mu \geq 0$, we say that μ is a *positive measure*.

Denote by $B(L, G)$ the set of all order bounded G-valued modular measures on L. It is clear that $B(L, G)$ contains all positive G-valued modular measures on

L and it is an ordered group with respect to the natural order between G-valued functions.

The aim of this Section is to prove that, if G is order complete, then $B(L,G)$ is an ℓ-group (see **2.4**). We need this result in the next Section.

Lemma 2.1. *Let $\mu\colon L \to G$ be a function. Then μ is a modular measure if and only if, for every $a, b \in L$, the following equality holds:*

$$\mu(a) = \mu((a \vee b) \ominus b) + \mu(a \wedge b). \tag{$*$}$$

Proof. If μ is a modular measure, then for every $a, b \in L$, we have $\mu((a \vee b) \ominus b) = \mu(a \vee b) - \mu(b) = \mu(a) - \mu(a \wedge b)$, whence the assertion.

Conversely, if c and d are orthogonal elements in L, applying $(*)$ with $a = c \oplus d$ and $b = d$, we obtain $\mu(c \oplus d) = \mu(c) + \mu((c \oplus d) \wedge d) = \mu(c) + \mu(d)$. Hence μ is a measure and therefore, by $(*)$, we also obtain that μ is modular. □

Lemma 2.2. *Let $\mu\colon L \to G$ be a modular measure and $a, b \in L$. Then, for every $c, d \in L$ with $c \le (a \vee b) \ominus b$ and $d \le a \wedge b$, there exists $e \in L$ such that $e \le a$ and $\mu(e) = \mu(c) + \mu(d)$.*

Proof. Set $s = (a \vee b) \ominus c$.

(i) First we prove that $c = (a \vee s) \ominus s$.

Since $c \le (a \vee b) \ominus b$, we have $s \ge b$. Since $s \le a \vee b$, we obtain $a \vee s = a \vee b$. Then we have $s = (a \vee s) \ominus c$ and $(a \vee s) \ominus s = (a \vee s) \ominus ((a \vee s) \ominus c) = c$.

(ii) Now observe that $a \ominus (a \wedge s) \perp d$ since, by $s \ge b$, we have $a \ominus (a \wedge s) \le a \ominus (a \wedge b)$, whence $a \ominus (a \wedge s) \le (a \wedge b)^{\perp} \le d^{\perp}$. Hence set $e = (a \ominus (a \wedge s)) \oplus d$. We have $e \le (a \ominus (a \wedge b)) \oplus (a \wedge b) = a$ and, since μ is a modular measure, by (i) we obtain $\mu(e) = \mu(a \ominus (a \wedge s)) + \mu(d) = \mu((a \vee s) \ominus s) + \mu(d) = \mu(c) + \mu(d)$. □

Proposition 2.3. *Let $\mu\colon L \to G$ be a modular measure. Set*

$$\mu^+(a) = \sup\{\mu(b) : \; b \in L, \; b \le a\}$$

for $a \in L$. Then μ^+ is a positive modular measure.

Proof. By **2.1**, we have to prove that, for every $a, b \in L$,

$$\mu^+(a) = \mu^+((a \vee b) \ominus b) + \mu^+(a \wedge b). \tag{$*$}$$

Let $p \le a$. Since μ is a modular measure, we have $\mu(p) = \mu((p \vee b) \ominus b) + \mu(p \wedge b) \le \mu^+((a \vee b) \ominus b) + \mu^+(a \wedge b)$, whence $\mu^+(a) \le \mu^+((a \vee b) \ominus b) + \mu^+(a \wedge b)$.

Now let $c \le (a \vee b) \ominus b$ and $d \le a \wedge b$. By **2.2**, we can find $p \le a$ such that $\mu(p) = \mu(c) + \mu(d)$. Then $\mu(c) + \mu(d) \le \mu^+(a)$, whence $\mu^+((a \vee b) \ominus b) + \mu^+(a \wedge b) \le \mu^+(a)$. □

Corollary 1. *Suppose that G is order complete. Then the set $B(L,G)$ is an ℓ-group and, for every $\mu \in B(L,G)$, $\mu \vee 0$ and $(-\mu) \vee 0$ are given, respectively, by the formulae:*

$$\mu^+(a) = \sup\{\mu(b) : \ b \in L, \ \ b \le a\}$$

and

$$\mu^-(a) = -\inf\{\mu(b) : b \in L, \ \ b \le a\}.$$

Proof. By [B, Chap. XIII], it is sufficient to prove that, for every $\mu \in B(L,G)$, $\mu \vee 0$ exists in $B(L,G)$ and equals μ^+. By **2.3**, μ^+ is a modular measure and therefore it is order bounded since it is positive. Trivially μ^+ is an upper bound of μ and 0. Moreover, let $\lambda \in B(L,G)$ be an upper bound of μ and 0. If $a \in L$ and $b \le a$, since λ is positive, we have $\lambda(a) \ge \lambda(b) \ge \mu(b)$, whence $\lambda(a) \ge \mu^+(a)$. Hence $\mu^+ = \mu \vee 0$ in $B(L,G)$. The equality $\mu^- = (-\mu) \vee 0$ follows from [B, Chap. XIII]. □

3. Decomposition theorems

The aim of this Section is to prove that Alexandroff and Hewitt–Yosida decomposition theorems proved in [D-M1] for measures on Boolean algebras with a stronger assumption on G (see [D-M1, pp. 119, 123]) are particular cases of a general decomposition theorem which holds in general for modular measures on D-lattices and with a weaker assumption on G.

If $\mu\colon L \to G$ is a modular measure and $\mathscr{U}$ is a D-uniformity, we write:

- $\mu \ll \mathscr{U}$ if μ is $\mathscr{U}$-continuous,
- $\mu \perp \mathscr{U}$ if, for every neighbourhood W of 0 in G and every neighbourhood U of 0 in $\mathscr{U}$, there exists $a \in L$ such that $\mu(b) \in W$ for every $b \le a$ and $a^\perp \in U$.

If $\lambda\colon L \to G$ is another modular measure, $\mu \ll \lambda$ means $\mu \ll \mathscr{U}(\lambda)$ and $\mu \perp \lambda$ means $\mu \perp \mathscr{U}(\lambda)$.

In [A-V1, 3.5], the following decomposition theorem has been proved.

Theorem 3.1. *Let $\mu\colon L \to G$ be an exhaustive modular measure and $\mathscr{U}$ a D-uniformity on L. Then there exist unique modular measures $\lambda, \nu\colon L \to G$ such that $\mu = \lambda + \nu$, $\lambda \ll \mathscr{U}$ and $\nu \perp \mathscr{U}$. Moreover λ and ν are exhaustive, $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$ and $\mathscr{U}(\mu) = \mathscr{U}(\lambda) \vee \mathscr{U}(\nu)$.*

We want to derive from **3.1** two particular decomposition theorems. We need some definitions.

Definition 3.2. A modular measure $\mu\colon L \to G$ is said to be *purely non σ-additive* if, for every σ-additive modular measure $\lambda\colon L \to G$ such that $\lambda \ll \mu$, we have $\lambda = 0$.

Definition 3.3. Let $\mu\colon L \to G$ be a function such that $\mu(0) = 0$ and $K \subseteq L$. We say that μ is *K-inner regular* if, for every $c \in L$ and every neighbourhood W of 0 in G, there exist $b \in K$ such that $b \le c$ and $\mu(d) \in W$ whenever $d \in L$ and $d \le c \ominus b$.

Recall that, for a modular measure $\mu\colon L \to G$, a base of neighbourhoods of 0 in $\mathscr{U}(\mu)$ is the family consisting of the sets $\big\{a \in L :\ (\forall b \le a)(\mu(b) \in W)\big\}$, where W is a neighbourhood of 0 in G. Therefore it is clear that, if μ is a K-inner regular modular measure and λ is a modular measure such that $\lambda \ll \mu$, then λ is K-inner regular, too.

Definition 3.4. Let $K \subseteq L$ and $\mu\colon L \to G$ be a modular measure. We say that μ is *K-smooth* if, for every decreasing net $(a_i)_{i\in I}$ in K such that $\inf_{i\in I} a_i = 0$ in L, we have that a_i converges to 0 in $\mathscr{U}(\mu)$.

We say that a *D-uniformity $\mathscr{U}$ is K-smooth* if, for every decreasing net $(a_i)_{i\in I}$ in K such that $\inf_{i\in I} a_i = 0$ in L, $(a_i)_{i\in I}$ converges to 0 in $\mathscr{U}$.

Then a modular measure $\mu\colon L \to G$ is K-smooth if and only if $\mathscr{U}(\mu)$ is K-smooth.

Definition 3.5. Let K be a subset of L and $\mu\colon L \to G$ be a modular measure. We say that μ is *K-singular* if, for every K-smooth modular measure $\lambda\colon L \to G$ such that $\lambda \ll \mu$, we have that $\lambda = 0$.

It is clear that, if μ is a K-singular (K-smooth, respectively) modular measure and λ is a modular measure such that $\lambda \ll \mu$, then λ is K-singular (K-smooth, respectively), too.

By [A-V1, 3.3, 3.4, 3.8], the following result holds.

Proposition 3.6. *Let μ be a G-valued modular measure on L and $\mathscr{U}$ a D-uniformity on L. Then:*

(1) *$\mu \ll \mathscr{U}$ if and only if $\mathscr{U}(\mu) \le \mathscr{U}$.*

(2) *$\mu \perp \mathscr{U}$ if and only if $\mathscr{U}(\mu) \wedge \mathscr{U} = 0$.*

(3) *μ is purely non σ-additive if and only if, for every σ-additive modular measure $\lambda\colon L \to G$, we have $\mu \perp \lambda$.*

Lemma 3.7. *Let $\mu\colon L \to G$ be an exhaustive modular measure and $\mathscr{U}$ a D-uniformity such that $\mathscr{U} \le \mathscr{U}(\mu)$. Then there exists a modular measure $\nu\colon L \to G$ such that $\nu \ll \mu$ and $\mathscr{U} = \mathscr{U}(\nu)$. Moreover, if G is a compatible ordered group and μ is positive, we can choose ν such that $0 \le \nu \le \mu$.*

Proof. By **3.1**, we can find two modular measures $\lambda, \nu\colon L \to G$ such that $\mu = \lambda + \nu, \nu \ll \mathscr{U}$ and $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$. Hence, by **3.6**, $\nu \ll \mu$ and, as proved in [A-V1, 3.7], $\mathscr{U} = \mathscr{U}(\nu)$.

Now suppose that μ is positive and G is compatible. In this case, from $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$, we obtain that λ and ν are positive, too, and therefore $\nu \leq \mu$. □

Proposition 3.8. *Let $\mu\colon L \to G$ be an exhaustive modular measure and $K \subseteq L$. Denote by $\mathscr{U}$ the supremum of all K-smooth D-uniformities on L. Then the following conditions are equivalent:*

(1) *μ is K-singular.*

(2) *$\mu \perp \mathscr{U}$.*

(3) *For every K-smooth modular measure $\lambda\colon L \to G$, $\mu \perp \lambda$.*

Proof.

(1) $\Longrightarrow$ (2). By **3.7**, we can find a modular measure $\nu\colon L \to G$ such that $\nu \ll \mu$ and $\mathscr{U}(\nu) = \mathscr{U}(\mu) \wedge \mathscr{U}$. Then ν is K-smooth, since $\mathscr{U}(\nu) \leq \mathscr{U}$. By (1), we obtain $\nu = 0$. Hence $\mathscr{U}(\mu) \wedge \mathscr{U} = 0$. By **3.6**, we have that $\mu \perp \mathscr{U}$.

(2) $\Longrightarrow$ (3). Let $\lambda\colon L \to G$ be a K-smooth modular measure. Then $\mathscr{U}(\lambda) \leq \mathscr{U}$. Hence we have $\mathscr{U}(\mu) \wedge \mathscr{U}(\lambda) = \mathscr{U}(\mu) \wedge \mathscr{U}(\lambda) \wedge \mathscr{U} = 0$ by (2) and **3.6**. Again by **3.6** we obtain that $\mu \perp \lambda$.

(3) $\Longrightarrow$ (1). If $\lambda\colon L \to G$ is a K-smooth modular measure such that $\lambda \ll \mu$, by **3.6** and (3) we have $\mathscr{U}(\lambda) = \mathscr{U}(\lambda) \wedge \mathscr{U}(\mu) = 0$, from which $\lambda = 0$. □

Theorem 3.9 (**Alexandroff decomposition theorem**). *Let $K \subseteq L$ and $\mu\colon L \to G$ be an exhaustive modular measure. Then there exist unique exhaustive modular measures $\lambda, \nu\colon L \to G$ such that:*

(1) *$\mu = \lambda + \nu$.*

(2) *λ is K-smooth.*

(3) *ν is K-singular.*

Moreover:

(4) *If $H \subseteq L$ and μ is H-inner regular, then λ and ν are H-inner regular, too.*

(5) *If G is a compatible ordered group and μ is positive (resp. order bounded), then λ and ν are positive (resp. order bounded), too.*

Proof. Denote by $\mathscr{U}$ the supremum of all K-smooth D-uniformities on L.

By **3.1**, there exist unique modular measures $\lambda, \nu\colon L \to G$ such that $\mu = \lambda + \nu$, $\lambda \ll \mathscr{U}$ and $\nu \perp \mathscr{U}$. Moreover λ and ν are exhaustive, $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$ and $\mathscr{U}(\mu) = \mathscr{U}(\lambda) \vee \mathscr{U}(\nu)$. Therefore λ is K-smooth since, by **3.6**, $\mathscr{U}(\lambda) \leq \mathscr{U}$ and ν is K-singular by **3.8**. Conversely, let $\lambda', \nu'\colon L \to G$ be modular measures with properties (1), (2) and (3). Then, by **3.8**, $\nu' \perp \mathscr{U}$. Moreover we have $\mathscr{U}(\lambda') \leq \mathscr{U}$ and therefore by **3.6**, $\lambda' \ll \mathscr{U}$. Hence the uniqueness follows from **3.1**.

(4) is trivial since $\lambda \ll \mu$ and $\nu \ll \mu$.

(5) follows from $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$, since G is compatible and therefore both the positive cone and the order intervals of G are closed. □

Theorem 3.10 (**Hewitt-Yosida decomposition theorem**). *Let $\mu\colon L \to G$ be an exhaustive modular measure. Then there exist unique modular measures $\lambda\colon L \to G$ and $\nu\colon L \to G$ such that*

(1) $\mu = \lambda + \nu$.

(2) *λ is σ-additive.*

(3) *ν is purely non σ-additive.*

Moreover:

(4) *If $H \subseteq L$ and μ is H-inner regular, then λ and ν are H-inner regular, too.*

(5) *If G is a compatible ordered group and μ is positive (resp. order bounded), then λ and ν are positive (resp. order bounded), too.*

Proof.

(1), (2) and (3) have been proved in [A-V1, 3.9] as a consequence of **3.1**.

(4) and (5) can be proved as in **3.9**. □

Following [D-M1], we give the following definitions:

- A subset K of L is *a paving* if $0 \in K$ and K is closed with respect to finite suprema.
- A subset K of L is said to be a *δ-paving* if K is a paving and every countable subset of K has an infimum in L which belongs to K.
- A *Lindelof space* is a pair $(X, \mathscr{F})$, where X is a non-empty set, $\mathscr{F}$ is a δ-paving containing X in the Boolean algebra of all subsets of X and every covering of X consisting of complements of elements of $\mathscr{F}$ contains a countable subcovering.

  If G is an ordered group, $\mu\colon L \to G$ is a positive modular measure and K is a paving,
- μ is said to be *purely finitely additive* (*purely f.a.*) if, for every σ-additive positive modular measure $\lambda \leq \mu$, we have $\lambda = 0$.
- μ is said to be *K-smooth in the sense of de Lucia-Morales* if, for every decreasing net $(a_i)_{i\in I}$ in K such that $\inf_{i\in I} a_i = 0$ in L, we have $\lim_i \mu(a_i) = 0$ in G.

- μ is said to be *K-singular in the sense of de Lucia-Morales* if μ is K-inner regular and, for every positive K-smooth and K-inner regular modular measure λ such that $\lambda \leq \mu$, we have $\lambda = 0$.

If $\mu\colon L \to G$ is an order bounded measure, μ is said to be *K-singular* (resp. *K-smooth*) *in the sense of de Lucia-Morales* if μ^+ and μ^- are K-singular (resp. K-smooth). In a similar way, μ is said to be *purely f.a.* if μ^+ and μ^- are purely f.a..

In [D-M1, pp. 119, 123] the following decomposition theorems have been proved.

Theorem 3.11. *Let G' be a Hausdorff Abelian ℓ-group which is order complete, locally order convex, has the property* (oc) *and satisfies the condition* (M)*, $\mathscr{A}$ a Boolean algebra and $K \subseteq \mathscr{A}$ a δ-paving. Then, for every order bounded K-inner regular measure $\mu\colon \mathscr{A} \to G'$, there exist unique order bounded K-inner regular measures $\lambda, \mu\colon \mathscr{A} \to G'$ such that:*

(1) $\mu = \lambda + \nu$.

(2) *λ is K-smooth in the sense of de Lucia-Morales.*

(3) *ν is K-singular in the sense of de Lucia-Morales.*

Theorem 3.12. *Let $(X, \mathscr{F})$ be a Lindelof space, $\mathscr{A}$ an algebra of subsets of X which contains $\mathscr{F}$ and G' a Hausdorff Abelian ℓ-group which is order complete, locally order convex, has the property* (oc) *and satisfies the condition* (M)*. Then, for every $\mathscr{F}$-inner regular and order bounded measure $\mu\colon \mathscr{A} \to G'$, there exist unique $\mathscr{F}$-inner regular and order bounded measures $\lambda, \nu\colon \mathscr{A} \to G'$ such that $\mu = \lambda + \nu$, λ is σ-additive and ν is purely finitely additive.*

We want to prove that the decomposition Theorems 3.11 and 3.12 proved in [D-M1] are particular cases of **3.9** and **3.10**.

Proposition 3.13. *Suppose that G is a locally order convex ℓ-group with the property* (M)*. Then G is locally solid (and therefore it is compatible).*

Proof. Let V be a convex neighbourhood of 0 in G. To prove that G is locally solid, it is sufficient to prove that V contains a solid neighbourhood of 0 in G. Set

$$W = \{x \in G :\ [-|x|, |x|] \subseteq V\}.$$

It is clear that $W \subseteq V$. Moreover W is solid since, if $x \in W$ and y is an element of G such that $|y| \leq |x|$, then $[-|y|, |y|] \subseteq [-|x|, |x|]$ and therefore $y \in W$.

It remains to prove that W is a neighbourhood of 0 in G.

Since G satisfies the condition (M), we can find a neighbourhood V' of 0 in G such that $V' \subseteq V$ and V' is a sublattice of G.

We prove that $V' \cap (-V') \subseteq W$, from which we obtain the assertion.

If $x \in V' \cap (-V')$, since V' is a sublattice, we have that $|x| = x \vee (-x)$ and $-|x| = x \wedge (-x)$ belongs to V' and therefore to V. Since V is convex, we obtain that $[-|x|, |x|] \subseteq V$ and therefore $x \in W$.

By **1.1**, we obtain that G is compatible. □

Proposition 3.14. *Suppose that G is an order complete group with the property* (oc)*. Then every order bounded modular measure $\mu\colon L \to G$ is exhaustive.*

Proof. By **2.4** we have $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive. Then it is sufficient to prove the assertion in the case that μ is positive.

Let $\{a_n\}$ be an orthogonal sequence. For each $n \in \mathbb{N}$, set

$$x_n = \sum_{i=0}^{n} \mu(a_i).$$

Since $\mu \geq 0$, $\{x_n\}$ is an increasing sequence in G and $x_n = \mu(\bigoplus_{i=0}^{n} a_i) \leq \mu(1)$. Since G is order complete, there exists $x = \sup_n x_n$ in G. Since G has the property (oc), by **1.3** we have $x = \lim_n x_n$. Since $\mu(a_n) = x_n - x_{n-1}$ (with $x_{-1} = 0$), we obtain $\lim_n \mu(a_n) = 0$. □

In what follows, we use the notations of Section 2. In particular, for $\mu \in B(L,G)$, we set $|\mu| = \mu^+ + \mu^-$.

Proposition 3.15. *Suppose that G is an order complete locally order convex group. Then, if $\mu \in B(L,G)$, $\mathscr{U}(|\mu|) = \mathscr{U}(\mu^+) \vee \mathscr{U}(\mu^-)$.*

Proof. Recall that, for a positive modular measure λ, a base of $\mathscr{U}(\lambda)$ is the family consisting of the sets $\{(a,b) \in L \times L : \lambda(a \triangle b) \in W\}$, where W is a neighbourhood of 0 in G. Moreover, by **2.4**, μ^+ and μ^- are modular measures and therefore $|\mu|$ is a modular measure, too.

Let W be a neighbourhood of 0 in G.

(i) Since G is locally order convex, we can choose a convex neighbourhood V of 0 in G such that $V \subseteq W$ and therefore, since $0 \leq \mu^+ \leq |\mu|$ and $0 \leq \mu^- \leq |\mu|$, we have that $\mu^+(a \triangle b) \in W$ and $\mu^-(a \triangle b) \in W$ whenever $|\mu|(a \triangle b) \in V$.

(ii) Since G is a topological group, we can choose a neighbourhood V' of 0 in G such that $V' + V' \subseteq W$. Then we have $|\mu|(a \triangle b) \in W$ whenever $\mu^+(a \triangle b) \in V'$ and $\mu^-(a \triangle b) \in V'$.

From (i) and (ii), we obtain the assertion. □

Proposition 3.16. *Suppose that G is an order complete locally order convex ℓ-group with the properties* (oc) *and* (M). *Then, for any $\mu \in B(L,G)$, $\mathscr{U}(\mu) = \mathscr{U}(|\mu|)$.*

P r o o f. Let W, V be neighbourhoods of 0 in G such that V is convex and $V - V \subseteq W$. Since, by **3.15**, $\mathscr{U}(|\mu|) = \mathscr{U}(\mu^+) \vee \mathscr{U}(\mu^-)$, we can find a neighbourhood V_0 of 0 in G such that, for every $a, b \in L$, $\mu^+(a \triangle b) \in V$ and $\mu^-(a \triangle b) \in V$ whenever $|\mu|(a \triangle b) \in V_0$. Since $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are monotone and V is convex, we have that, for every $a, b \in L$ and $c \leq a \triangle b$, $\mu(c) \in W$ whenever $|\mu|(a \triangle b) \in V_0$. Therefore $\mathscr{U}(\mu) \leq \mathscr{U}(|\mu|)$.

Moreover, since G satisfies the condition (M), we can find a neighbourhood W_0 of 0 in G such that W_0 is a sublattice of G and $\overline{W_0} \subseteq W$. Therefore, by **1.2**, the condition $\mu(c) \in W_0$ for every $c \leq a \triangle b$ implies $\mu^+(a \triangle b) = \sup\{\mu(c) : c \leq a \triangle b\} \in \overline{W_0} \subseteq W$. Hence $\mathscr{U}(\mu^+) \leq \mathscr{U}(\mu)$. Since $\mu^-(a) = \sup\{-\mu(b) : b \leq a\}$ and G has a base of symmetric neighbourhoods of 0, we obtain in a similar way that $\mathscr{U}(\mu^-) \leq \mathscr{U}(\mu)$. Hence $\mathscr{U}(|\mu|) \leq \mathscr{U}(\mu)$. □

In the next result we use the fact that, by [A-V2, 2.9], the exhaustive D-uniformities on L form a Boolean algebra.

Proposition 3.17. *Let $K \subseteq L$. If λ and μ are G-valued exhaustive K-singular (resp. purely non σ-additive) modular measures on L, then $-\mu$ and $\lambda + \mu$ are K-singular (resp. purely non σ-additive).*

P r o o f. It is clear that $\mathscr{U}(\mu) = \mathscr{U}(-\mu)$ Then, if μ is K-singular (resp. purely non σ-additive), $-\mu$ has the same property.

Moreover, since $\mathscr{U}(\lambda + \mu) \leq \mathscr{U}(\mu) \vee \mathscr{U}(\lambda)$, if ν is a modular measure, by [A-V2, 2.9] we have

$$\mathscr{U}(\lambda+\mu) \wedge \mathscr{U}(\nu) \leq (\mathscr{U}(\lambda) \vee \mathscr{U}(\mu)) \wedge \mathscr{U}(\nu) = (\mathscr{U}(\lambda) \wedge \mathscr{U}(\nu)) \vee (\mathscr{U}(\mu) \wedge \mathscr{U}(\nu)).$$

By **3.7**, we can find modular measures $\nu_1, \nu_2 \colon L \to G$ such that $\nu_1 \ll \lambda$, $\nu_2 \ll \mu$, $\mathscr{U}(\nu_1) = \mathscr{U}(\lambda) \wedge \mathscr{U}(\nu)$ and $\mathscr{U}(\nu_2) = \mathscr{U}(\mu) \wedge \mathscr{U}(\nu)$. Therefore

$$\mathscr{U}(\lambda + \mu) \wedge \mathscr{U}(\nu) \leq \mathscr{U}(\nu_1) \vee \mathscr{U}(\nu_2).$$

(i) Suppose that λ and μ are K-singular and ν is K-smooth. Then we have that ν_1 and ν_2 are K-smooth since $\mathscr{U}(\nu_1) \leq \mathscr{U}(\nu)$ and $\mathscr{U}(\nu_2) \leq \mathscr{U}(\nu)$ and therefore $\nu_1 = \nu_2 = 0$. Hence we obtain $\mathscr{U}(\lambda + \mu) \wedge \mathscr{U}(\nu) = 0$. By **3.8**, we obtain that $\lambda + \mu$ is K-singular.

(ii) Now suppose that λ and μ are purely non σ-additive and ν is σ-additive. Then, since $\mathscr{U}(\nu_1) \leq \mathscr{U}(\nu)$ and $\mathscr{U}(\nu_2) \leq \mathscr{U}(\nu)$, we have, by [A-B1, 2.4], that ν_1 and ν_2 are σ-additive, too. Therefore we have $\nu_1 = \nu_2 = 0$ and then, as before, $\mathscr{U}(\lambda+\mu) \wedge \mathscr{U}(\nu) = 0$. By **3.6**, we obtain that $\lambda+\mu$ is purely non σ-additive. □

Proposition 3.18. *Suppose that G is locally order convex. Let $\mu\colon L \to G$ be a positive modular measure and $K \subseteq L$. Then:*

(1) *If μ is K-inner regular, then μ is K-singular in the sense of de Lucia-Morales if and only if μ is K-singular.*

(2) *μ is K-smooth in the sense of de Lucia-Morales if and only if μ is K-smooth.*

(3) *μ is purely non σ-additive if and only if μ is purely f.a..*

Moreover, if G is an order-complete ℓ-group with the properties (oc) *and* (M), *then the previous equivalences hold for any order bounded modular measure $\mu\colon L \to G$.*

Proof.

(i) First suppose that μ is positive.

In this case, a base of neighbourhoods of 0 in $\mathscr{U}(\mu)$ is the family consisting of the sets $\{a \in L : \mu(a) \in W\}$, where W is a neighbourhood of 0 in G. Then, if $(a_i)_{i\in I}$ is a net in L, we have that $a_i \to 0$ in $\mathscr{U}(\mu)$ if and only if $\mu(a_i) \to 0$ in G. Therefore it is clear that the equivalence in (2) holds.

Moreover, since G is locally order convex, we have that, if $\lambda\colon L \to G$ is a modular measure such that $\lambda \le \mu$, then $\lambda \ll \mu$. Therefore we obtain that:

(a) If μ is K-inner regular and K-singular, then μ is also K-singular in the sense of de Lucia-Morales.

(b) If μ is purely non σ-additive, then μ is purely f.a..

Conversely, if $\lambda\colon L \to G$ is a modular measure such that $\lambda \ll \mu$, by **3.7** we can find a modular measure $\nu : L \to G$ such that $0 \le \nu \le \mu$ and $\mathscr{U}(\nu) = \mathscr{U}(\lambda)$. Hence, if λ is K-smooth (resp. σ-additive), ν is K-smooth (resp. σ-additive), too. Therefore in (1) and in (3) the equivalence holds.

(ii) Now suppose that G is an order-complete ℓ-group with the properties (oc) and (M), and remove the assumption that μ is positive.

Recall that, by **3.14**, μ is exhaustive. Moreover, by **3.16**, we have $\mathscr{U}(\mu) = \mathscr{U}(|\mu|)$. Therefore $|\mu|$ is exhaustive, too, and then μ^+ and μ^- are exhaustive since $\mu^+ \le |\mu|$ and $\mu^- \le |\mu|$. In a similar way, if μ is K-inner regular, we can obtain that μ^+ and μ^- are K-inner regular, too.

(1) and (3): Since $\mu^+ \le |\mu|$ and $\mu^- \le |\mu|$, we have that, if μ is K-singular (resp. purely non σ-additive), then $|\mu|$, μ^+ and μ^- are K-singular (resp. purely non σ-additive), too, and therefore by (i) K-singular in the sense of de Lucia-Morales (resp. purely f.a.). Conversely, if μ is K-singular in the sense of de Lucia-Morales (resp. purely f.a.), we have by (i) that μ^+ and μ^- are K-singular (resp. purely non σ-additive). By **3.17**, $|\mu| = \mu^+ - \mu^-$ is K-singular (resp. purely non σ-additive). By **3.16**, μ is K-singular (resp. purely non σ-additive), too.

(2): It is clear that, if μ is K-smooth, then μ is also K-smooth in the sense of de Lucia-Morales since μ is continuous with respect to $\mathscr{U}(\mu)$. Conversely, suppose that μ is K-smooth in the sense of de Lucia-Morales. Since $|\mu| = \mu^+ + \mu^-$, we have that $|\mu|$ is K-smooth in the sense of de Lucia-Morales and therefore K-smooth by (i). By **3.16**, we have that μ is K-smooth. $\square$

Now we can see that **3.11** and **3.12** are particular cases of **3.9** and **3.10**.

Proof of Theorems 3.11 and 3.12.

By **3.13** G' is compatible.

By **3.14** every order bounded modular measure $\mu\colon \mathscr{A} \to G'$ is exhaustive.

Then, recalling **3.18**, we have that **3.11** follows from **3.9** and **3.12** from **3.10**. $\square$

Remark. In [D-M3, 5.11, 5.14] the following decomposition theorem has been proved:

Suppose that G' is a quasi order-complete locally order convex group with the property (oc), L' is an effect algebra and K, H are pavings in L'. Then, for every positive H-inner regular measure $\mu\colon L' \to G'$, there exist two positive H-inner regular measures $\lambda, \nu\colon L' \to G'$ such that

(1) $\mu = \lambda + \nu$.

(2) λ is K-smooth.

(3) ν is K-singular.

Moreover, if G' is order-complete and L' is a Boolean algebra, the decomposition is unique.

If L' is a D-lattice and μ is modular, this decomposition of μ is not a consequence of **3.9**, since the assumptions of [D-M3] do not imply that G is compatible, as the next example shows. Nevertheless, with a similar proof as in [D-M3] and using the results of the Section 2, it is possible to prove that in this case λ and ν are modular, too, and, if G' is order-complete, the decomposition is unique as in the Boolean case.

The next example has been suggested by Hans Weber.

Example 1. Denote by τ the usual topology in $\mathbb{R}$ and by $\leq$ the usual order in $\mathbb{R}$.

Set $C = \{x \in \mathbb{Q} : x \geq 1\}$. For $a, b \in \mathbb{R}$, define $a \preceq b$ if and only if $b - a \in C$.

We see that $(\mathbb{R}, \preceq, \tau)$ is a quasi order-complete locally order-convex group with the property (oc), but it is not compatible.

It is clear that $(\mathbb{R}, \preceq, \tau)$ is not compatible and, since $a \preceq b$ implies $a \leq b$, it is locally order convex.

Now observe that, if $D \subseteq \mathbb{R}$ is a majorized set with respect to $\preceq$, then D has a maximal element with respect to $\preceq$, otherwise we can construct a sequence $\{d_n\}$ in D such that, for each $n \in \mathbb{N}$, $d_n \geq d_0 + n$, a contradiction with the assumption that D is majorized.

Therefore, if D is a majorized directed upward subset of $\mathbb{R}$ and m is a maximal element of D with respect to $\preceq$, then we have that $m = \max D$ with respect to $\preceq$.

Now it is clear that $(\mathbb{R}, \preceq, \tau)$ is quasi order-complete and has the property (oc).

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