VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-008-0066-3
Math. Slovaca **58** (2008), No. 2, 177–184

GROUPS WITH FEW CONJUGACY CLASSES OF NON-NORMAL SUBGROUPS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. The structure of groups with finitely many non-normal subgroups is well known. In this paper, groups are investigated with finitely many conjugacy classes of non-normal subgroups with a given property. In particular, it is proved that a locally soluble group with finitely many non-trivial conjugacy classes of non-abelian subgroups has finite commutator subgroup. This result generalizes a theorem by Romalis and Sesekin on groups in which every non-abelian subgroup is normal.

1. Introduction

In a famous paper of 1955, B. H. Neumann [9] proved that each subgroup of a group G has finitely many conjugates if and only if the centre $Z(G)$ has finite index, and the same conclusion holds if the restriction is imposed only to conjugacy classes of abelian subgroups (see [6]). This result suggests that the behaviour of conjugacy classes of subgroups with a given property could have a strong influence on the structure of the group. More recently, groups with finitely many conjugacy classes of subgroups with a given property χ have been considered (see for instance [2] and [15]); of course, groups of this type contain only finitely many normal subgroups with the property χ. The aim of this paper is to study groups with finitely many non-trivial conjugacy classes of χ-subgroups, i.e. groups in which non-normal subgroups with the property χ lie into finitely many conjugacy classes. It has been proved in [2] that locally soluble groups with finitely many conjugacy classes of non-normal subgroups actually have only finitely many non-normal subgroups, and such groups have

2000 Mathematics Subject Classification: Primary 20F24.
Keywords: conjugacy class, metahamiltonian group.

been completely described in [8]. Our first theorem deals with the structure of groups with finitely many conjugacy classes of abelian non-normal subgroups.

Theorem A. *An infinite locally soluble group G has finitely many non-trivial conjugacy classes of abelian subgroups if and only if either G has finitely many non-normal subgroups or $G = \langle g, A\rangle$, where A is a non-periodic abelian normal subgroup, A/A^2 and $A[2]$ are finite, $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$.*

A group G is called *metahamiltonian* if every non-abelian subgroup of G is normal. Metahamiltonian groups were introduced and investigated by G. M. Romalis and N. F. Sesekin ([12], [13], [14]; see also [5]), who proved that locally soluble metahamiltonian groups have finite commutator subgroup. Our next theorem improves the main result of Romalis and Sesekin in the following way.

Theorem B. *Let G be a locally (soluble-by-finite) group with finitely many non-trivial conjugacy classes of non-abelian subgroups. Then the commutator subgroup G' of G is finite.*

Corollary B.1. *Let G be a locally (soluble-by-finite) group with finitely many non-trivial conjugacy classes of non-abelian subgroups. Then G contains only finitely many subgroups which are neither abelian nor normal.*

Groups in which every subgroup is either locally nilpotent or normal have been considered by B. Bruno and R. E. Phillips [3], who proved that locally soluble groups with such a property either are locally nilpotent or have finite commutator subgroup; however, the structure of locally nilpotent groups whose non-normal subgroups are nilpotent is not known, and it could be quite complicated. In the case of groups with finitely many conjugacy classes of subgroups which are neither locally nilpotent nor normal, we have the following result.

Theorem C. *Let G be a locally (soluble-by-finite) group with finitely many non-trivial conjugacy classes of non-(locally nilpotent) subgroups. Then either G is locally nilpotent or its commutator subgroup G' is finite.*

The consideration of Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) shows that in all these situations some (generalized) solubility condition must be required.

Most of our notation is standard and can be found in [11].

2. Proof of Theorem A

Our first lemma deals with the case of periodic groups with finitely many conjugacy classes of abelian non-normal subgroups.

Lemma 2.1. *Let G be a locally finite group with finitely many non-trivial conjugacy classes of abelian subgroups. Then G has finitely many non-normal subgroups.*

Proof. Clearly, it can be assumed that G is infinite. Then there exists a positive integer m such that every finite abelian subgroup of G with order larger than m is normal. In particular, all infinite abelian subgroups of G are normal, and hence G is either a Černikov group or a Dedekind group. Suppose that G is a Černikov group, and let N be the subgroup generated by all elements with order at most m. Every subgroup of G which is not contained in N can be generated by elements of order larger than m and hence it is normal in G. As N is finite, it follows that G has only finitely many non-normal subgroups. □

Lemma 2.2. *Let the group $G = \langle g \rangle \ltimes A$ be the semidirect product of an abelian normal subgroup A and a cyclic subgroup $\langle g \rangle$, where $g^2 = 1$ and $a^g = a^{-1}$ for all $a \in A$. If u and v are elements of A, the products gu and gv are conjugate if and only if $uA^2 = vA^2$.*

Proof. If a is any element of A, we have

$$(gv)^{ga} = a^{-1}vga = (gv)^{v^{-1}a}$$

and so gu and gv lie in the same conjugacy class of G if and only if there exists $w \in A$ such that $gu = (gv)^w$. On the other hand,

$$(gv)^w = w^{-1}gvw = gvw^2$$

and hence gu and gv are conjugate if and only if $u = vw^2$ for some $w \in A$, i.e. if and only if $uA^2 = vA^2$. □

Proof of Theorem A. Suppose first that G has finitely many non-trivial conjugacy classes of abelian subgroups, but that G has infinitely many non-normal subgroups. It follows from Lemma 2.1 that G is not periodic; moreover, it can obviously be assumed that G is not abelian. Let A be the Fitting subgroup of G. Clearly, the factor group G/A has finitely many conjugacy classes of cyclic subgroups, so that G/A has only finitely many normal subgroups and hence it is finite, of order m, say. Assume that G contains an element x of infinite order such that $\langle x \rangle$ is not normal, and put $y = x^m$. There exist distinct prime numbers p and q such that $\langle x^p \rangle$ and $\langle x^q \rangle$ are conjugate subgroups of G, so that also the subgroups $\langle y^p \rangle$ and $\langle y^q \rangle$ are conjugate and hence y^q belongs to $\langle y^p \rangle^G$. On the other hand, the coset $y\langle y^p \rangle^G$ has order exactly p (see [1, Lemma 3.2.1]), and this contradiction shows that all infinite cyclic subgroups of G are normal. It follows that A is abelian and so $C_G(A) = A$. Then G/A is isomorphic to a group of power automorphisms of A, so that $|G/A| = 2$ and hence $G = \langle g, A \rangle$, where $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$; in particular, $g^4 = 1$. As $g^2 \in Z(G)$, the factor group $\bar{G} = G/\langle g^2 \rangle$ has finitely many conjugacy classes of cyclic non-normal subgroups. Therefore $\bar{A}/\bar{A}^2$ is finite by Lemma 2.2, and so also A/A^2 is finite. Clearly, $A[2] \leq Z(G)$ and for each element a of $A[2]$ the abelian subgroup $\langle g, a \rangle$ is not normal in G. It follows that there exists a finite subset $\{a_1, \ldots, a_t\}$ of $A[2]$ such that $A[2]$ is contained in $\langle g, a_1, \ldots, a_t \rangle$, and hence $A[2]$ is finite.

Conversely, suppose that $G = \langle g, A \rangle$ has the required structure, and let X be any abelian non-normal subgroup of G. Obviously, X is not contained in A and so $X = \langle g^\varepsilon u, X \cap A \rangle$, where $\varepsilon = \pm 1$ and $u \in A$; moreover, $X \cap A$ is centralized by g and so it is contained in $A[2]$. Put $\bar{G} = G/\langle g^2 \rangle$. As A/A' is finite, it follows from Lemma 2.2 that the elements $\bar{g}\bar{a}$, with $a \in A$, lie into finitely many conjugacy classes $[\bar{g}\bar{a}_1], \ldots, [\bar{g}\bar{a}_t]$. On the other hand, $\bar{g}^\varepsilon \bar{u} = \bar{g}\bar{u}$ and so there exists an element $h \in G$ such that $g^\varepsilon u = (ga_i)^h z$ for some $i = 1, \ldots, t$ and $z \in \{1, g^2\}$. Then $X = \langle g a_i z, X \cap A \rangle^h$ and hence G has finitely many non-trivial conjugacy classes of abelian subgroups. □

3. Proof of Theorems B and C

Also in this section we begin with the case of locally finite groups.

Lemma 3.1. *Let $\mathfrak{X}$ be a subgroup closed group class, and let G be a locally finite group with finitely many non-trivial conjugacy classes of subgroups which are not locally $\mathfrak{X}$. Then either G is locally $\mathfrak{X}$ or its commutator subgroup G' is finite.*

P r o o f. Suppose that the group G is not locally $\mathfrak{X}$, and let $\{X_1, \ldots, X_t\}$ be a set of representatives of conjugacy classes of finite non-normal subgroups of G which are not in $\mathfrak{X}$. Then $X = \langle X_1, \ldots, X_t \rangle$ is a finite subgroup of G, and of course it can be assumed that X is properly contained in G. Consider an element g of $G \setminus X$, and put $N = \langle X, g \rangle$. Let H be any finite subgroup of G containing N. Clearly, H does not belong to $\mathfrak{X}$; moreover, H cannot be conjugate to any X_i and hence it is normal in G. Therefore N is a finite normal subgroup of G and G/N is a Dedekind group, so that G' is finite. □

Recall that the *finite residual* of a group G is the intersection of all (normal) subgroups of finite index of G, and G is *residually finite* if its finite residual is trivial.

Lemma 3.2. *Let G be a group with finitely many non-trivial conjugacy classes of non-(locally nilpotent) subgroups, and let H be a residually finite normal subgroup of G. Then H is a finite extension of a locally nilpotent subgroup.*

Proof. Assume for a contradiction that H is not (locally nilpotent)-by-finite, and let m be the largest index of subgroups of finite index of H which are not normal in G. As H is infinite, it contains a subgroup K of finite index $n > m$. Then all subgroups of finite index of K are normal in G; in particular $\gamma_3(K)$ is contained in the finite residual of K, so that $\gamma_3(K) = \{1\}$ and K is nilpotent. This contradiction shows that H is a finite extension of a locally nilpotent subgroup. □

In our proof we will also need the following result of D. I. Zaicev, for a proof of which we refer to [1, Lemma 4.6.3].

Lemma 3.3. *Let G be a group locally satisfying the maximal condition on subgroups. If X is a subgroup of G such that $X^g \leq X$ for some element g of G, then $X^g = X$.*

If G is any group, we shall denote by $\rho_{\mathfrak{S}}(G)$ the *soluble radical* of G, i.e. the subgroup generated by all soluble normal subgroups of G. In particular, when G is soluble-by-finite, $\rho_{\mathfrak{S}}(G)$ is a soluble subgroup of finite index.

Proof of Theorem C. Assume for a contradiction that G is neither locally nilpotent nor finite-by-abelian, and let E be a finitely generated non-nilpotent subgroup of G. Let $\{X_1, \ldots, X_t\}$ be a set of representatives of conjugacy classes of finitely generated subgroups of G which are neither nilpotent nor normal. Consider the largest integer m among the derived lengths of the soluble radicals of $X_1, \ldots, X_t$, and let n be the maximum of the set

$$\big\{|X_1 : \rho_{\mathfrak{S}}(X_1)|, \ldots, |X_t : \rho_{\mathfrak{S}}(X_t)|\big\}.$$

Suppose first that G contains a finitely generated subgroup L such that either $\rho_{\mathfrak{S}}(\langle E, L\rangle)$ has derived length larger than m or

$$|\langle E, L\rangle : \rho_{\mathfrak{S}}(\langle E, L\rangle)| > n.$$

Then every finitely generated subgroup of G containing $\langle E, L\rangle$ must be normal, so that $G/\langle E, L\rangle$ is a Dedekind group and in particular G is soluble-by-finite. Assume now that for each finitely subgroup U of G the derived length of $\rho_{\mathfrak{S}}(\langle E, U\rangle)$

is at most m and

$$|\langle E,U\rangle : \rho_{\mathfrak{S}}(\langle E,U\rangle)| \leq n.$$

Let V be any finitely generated subgroup of G^n. Clearly, there exists a finitely generated subgroup W of G such that

$$V \leq W^n \leq \langle E,W\rangle^n \leq \rho_{\mathfrak{S}}(\langle E,W\rangle),$$

so that V is soluble with derived length at most m and hence G^n itself is a soluble group. Therefore in any case G is soluble-by-(locally finite).

Among all counterexamples choose one G for which there exists a soluble normal subgroup N with minimal derived length such that G/N is locally finite. By Lemma 3.1 the subgroup N is not abelian; if A is the smallest non-trivial term of the derived series of N, we have that the factor group G/A is either locally nilpotent or finite-by-abelian. In particular, G is abelian-by-(locally nilpotent)-by-finite, and hence all its finitely generated subgroups are residually finite (see [11, Part 2, Theorem 9.51]). It follows from Lemma 3.2 that every finitely generated normal subgroup of G is nilpotent-by-finite and so satisfies the maximal condition on subgroups. Let H be any normal subgroup of G which does not locally satisfy the maximal condition on subgroups, and let K be a finitely generated subgroup of H which does not satisfy the maximal condition. If $h \in H$, the finitely generated subgroup $\langle h,K\rangle$ is neither nilpotent nor normal in G, so that it is conjugate to some X_i and hence $\langle h,K\rangle^G = X_i^G$. Thus

$$H = \big\langle \langle h,K\rangle^G \mid h \in H \big\rangle = X_{i_1}^G \dots X_{i_s}^G$$

for some subset $\{i_1,\dots,i_s\}$ of $\{1,\dots,t\}$. Since there are only finitely many conjugacy classes of subgroups of G which are neither locally nilpotent nor normal, it follows that G has only finitely many conjugacy classes of subgroups which do not have locally the maximal condition, and so G must satisfy locally the maximal condition (see [7, Proposition 3.3]). Application of Lemma 3.3 yields now that G satisfies the minimal condition on subgroups which are neither locally nilpotent nor normal, so that either G is a Černikov group or all non-normal subgroups of G are locally nilpotent (see [10, Theorem B(iii)]). It follows from Lemma 3.1 and from [3, Theorem B] that G is either locally nilpotent or finite-by-abelian, and this contradiction completes the proof of the theorem. □

Theorem B can now be obtained as an easy consequence of Theorem C.

Proof of Theorem B. By Theorem C we may suppose that the group G is locally nilpotent, so that in particular it follows from Lemma 3.3 that G satisfies the minimal condition on non-abelian non-normal subgroups. Thus G is either a Černikov group or metahamiltonian (see [10, Theorem B(iv)]), and hence G' is finite because Lemma 3.1 shows that the statement is true for locally finite groups. □

Proof of Corollary B.1. The commutator subgroup G' of G is finite by Theorem B, so that in particular every finitely generated subgroup of G has only finitely many conjugates. It follows that all but finitely many finitely generated non-abelian subgroup of G are normal. On the other hand, if X is any non-abelian subgroup of G and E is a finitely generated non-abelian subgroup of X, then either X is normal in G or

$$X = \big\langle \langle x, E\rangle \mid x \in X \setminus X_G \big\rangle.$$

Therefore G contains only finitely many non-abelian subgroups which are not normal. □

REFERENCES

[1] AMBERG, B.—FRANCIOSI, S.—DE GIOVANNI, F.: *Products of Groups*. Oxford Math. Monogr., Clarendon Press, Oxford, 1992.

[2] BRANDL, R.—FRANCIOSI, S.—DE GIOVANNI, F.: *Groups with finitely many conjugacy classes of non-normal subgroups*, Proc. Roy. Irish Acad. A **95** (1995), 17–27.

[3] BRUNO, B.—PHILLIPS, R. E.: *Groups with restricted non-normal subgroups*, Math. Z. **176** (1981), 199–221.

[4] ČERNIKOV, S. N.: *Infinite groups with certain prescribed properties of systems of infinite subgroups*, Dokl. Akad. Nauk SSSR **159** (1964), 759–760.

[5] DE MARI, F.—DE GIOVANNI, F.: *Groups with few normalizer subgroups*, Bull. Irish Math. Soc. **56** (2005), 103–113.

[6] EREMIN, I. I.: *Groups with finite classes of conjugate abelian subgroups*, Mat. Sb. **47** (1959), 45–54.

[7] FRANCIOSI, S.—DE GIOVANNI, F.—SYSAK, Y. P.: *Groups with many FC-subgroups*, J. Algebra **218** (1999), 165–182.

[8] HEKSTER, N. S.—LENSTRA, H. W.: *Groups with finitely many non-normal subgroups*, Arch. Math. (Basel) **54** (1990), 225–231.

[9] NEUMANN, B. H.: *Groups with finite classes of conjugate subgroups*, Math. Z. **63** (1955), 76–96.

[10] PHILLIPS, R. E.—WILSON, J. S.: *On certain minimal conditions for infinite group*, J. Algebra **51** (1978), 41–68.

[11] ROBINSON, D. J. S.: *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin 1972.

[12] ROMALIS, G. M.—SESEKIN, N. F.: *Metahamiltonian groups*, Ural. Gos. Univ. Mat. Zap. **5** (1966), 101–106.
[13] ROMALIS, G. M.—SESEKIN, N. F.: *Metahamiltonian groups II*, Ural. Gos. Univ. Mat. Zap. **6** (1968), 52–58.
[14] ROMALIS, G. M.—SESEKIN, N. F.: *Metahamiltonian groups III*, Ural. Gos. Univ. Mat. Zap. **7** (1969/70), 195–199.
[15] SMITH, H.: *Groups with finitely many conjugacy classes of subgroups of large derived length*, Boll. Unione Mat. Ital. Sez. A **9** (1995), 167–175.

Received 6. 2. 2006

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