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A NOTE ON THE MULTIPLICITY IN FACTOR RING

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ABSTRACT. Let $(R,m) = k[x_1,\dots,x_n]_{(x_1,\dots,x_n)}$ be a local polynomial ring (k being an algebraically closed field), and $Q := (F_1,\dots,F_r)R$ be a primary ideal in R with respect to a maximal ideal $m \subset R$. In this short note we give a formula for the multiplicity $e_0\left(QR/(F_1)R, R/(F_1)R\right)$.

Let $F_1,\dots,F_r$ be a system of polynomials in the polynomial ring $k[x_1,\dots,x_n]$ over an algebraically closed field k, such that the variety $V(F_1,\dots,F_r)$ contains the null point $\underline{0} = (0,\dots,0) \in E_k^n$. Let R denote the localization of $k[x_1,\dots,x_n]$ with respect to $(x_1,\dots,x_n)$ and m be the maximal ideal of R. Suppose that $Q := (F_1,\dots,F_r)R$ is an m-primary ideal in R.

The aim of this note is to describe a method of calculation of the Samuel multiplicity of the ideal $QR/(F_1)R$ in the factor ring $R/(F_1)R$. Let us remark that the Samuel multiplicity of an ideal Q in a ring R, denoted by $e_0(Q,R)$, is the leading coefficient in the Hilbert-Samuel polynomial $P(n) = l(R/Q^n)$, where $l(R/Q^n)$ is the length of the R-module R/Q^n. In the particular case $r = n$, i.e., if Q is generated by a system of parameters in R, this method describes a way for finding the Samuel multiplicity $e_0(Q,R)$ of Q in R.

Let $W \subset E_k^n$ be the hypersurface defined by the equation $F_1 = 0$. Suppose that W is parametrized by the family of polynomials $u_i(s_1,\dots,s_{n-1}) \in k[s_1,\dots,s_{n-1}]$. Then the parametrization of W is given by

$$\begin{aligned} x_1 &= u_1(s_1,\dots,s_{n-1}),\\ x_2 &= u_2(s_1,\dots,s_{n-1}),\\ &\ \vdots\\ x_n &= u_n(s_1,\dots,s_{n-1}). \end{aligned} \tag{1}$$

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Moreover assume that the ring $k[s_1,\dots,s_{n-1}]$ is a finite $k[W]=k[u_1,\dots,u_n]$ module. Recall that the ring $k[W]$ is the coordinate ring of the variety W, hence $k[W]=k[x_1,\dots,x_n]/(F_1)k[x_1,\dots,x_n]$. For the polynomials F_i we denote by $f_i=F_i(u_1(s_1,\dots,s_{n-1}),\dots,u_n(s_1,\dots,s_{n-1}))$ the polynomials in $k[s_1,\dots,s_{n-1}]$ for $i=2,\dots,n$. Let us put, finally, $S:=k[s_1,\dots,s_{n-1}]_{(s_1,\dots,s_{n-1})}$ and denote by d the dimension of the field $k(s_1,\dots,s_{n-1})$ as the vector space over the field $k(u_1,\dots,u_n)$.

Theorem 1. *With the previous notation and hypothesis we have:*

(a) $$e_0\big(QR/(F_1)R,R/(F_1)R\big)d=e_0\big((f_2,\dots,f_n)S,S\big);$$

(b) *if Q is generated by the system of parameters then*

$$e_0(QR,R)d=e_0\big((f_2,\dots,f_n)S,S\big);$$

(c) *if Q is generated by the system of parameters and W is birationally equivalent to the hyperplane then*

$$e_0(QR,R)=e_0\big((f_2,\dots,f_n)S,S\big).$$

Proof.

(a) Let us construct the following homomorphism

$$\phi\colon k[x_1,\dots,x_n]\to k[s_1,\dots,s_{n-1}]$$
$$x_i\mapsto u_i(s_1,\dots,s_{n-1}).$$

The kernel of ϕ is the ideal $\operatorname{Ker}(\phi)=(F_1)k[x_1,\dots,x_n]$, so there is a monomorphism

$$k[x_1,\dots,x_n]/(F_1)k[x_1,\dots,x_n]\cong k[u_1,\dots,u_n]\hookrightarrow k[s_1,\dots,s_{n-1}],$$

and within the local monomorphism

$$R/(F_1)R\cong k[u_1,\dots,u_n]_{(u_1,\dots,u_n)}\hookrightarrow k[s_1,\dots,s_{n-1}]_{(s_1,\dots,s_{n-1})}.$$

By the assumption $k[s_1,\dots,s_{n-1}]_{(s_1,\dots,s_{n-1})}$ is a finite module over the local ring $k[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$. Now the additivity formula (cf. [2, Theorem 14.7]) applied to the $e_0\big(QR/(F_1)R,R/(F_1)R\big)$ provides

$$e_0\big(QR/(F_1)R,R/(F_1)R\big)\mathrm{length}_{(R/(F_1))_{(0)}}S_{(0)}=e_0\big((f_2,\dots,f_n)S,S\big).$$

The equality $d=\mathrm{length}_{(R/(F_1))_{(0)}}S_{(0)}$ completes the proof of (a).

(b) Suppose that the number of generators of QR in R equals the dimension of R, i.e., $QR=(F_1,\dots,F_n)R$. The associativity formula for the multiplicity $e_0(QR,R)$ ([3, Chap. 7, Theorem 18]) yields the equality

$$\begin{aligned}e_0\big((F_1,\dots,F_n)R,R\big)&=e_0\big((F_1,\dots,F_n)R/(F_1)R,R/(F_1)R\big)\,e_0\big((F_1)R_{(F_1)},R_{(F_1)}\big)\\&=e_0\big((F_1,\dots,F_n)R/(F_1)R,R/(F_1)R\big),\end{aligned}$$

since F_1 is irreducible by the hypothesis.

(c) Let W be birationally equivalent to the hyperplane. Then $k(u_1,\dots,u_n)\cong k(s_1,\dots,s_{n-1})$ and, consequently,

$$e_0(QR,R)=e_0\big((f_2,\dots,f_n)S,S\big).$$

This completes the proof of (c). □

As an application we give the formula for the multiplicity of the polynomial ideal $(x^m-y^az^b,\ y^n-x^cz^d,\ z^l-x^ey^f)$. Let $P=k[x,y,z]$ be the polynomial ring over an algebraically closed field k and $Q=(x^m-y^az^b,\ y^n-x^cz^d,\ z^l-x^ey^f)P$ be an (x,y,z)-primary ideal in P. We prove the formula for the multiplicity of QR in $R=k[x,y,z]_{(x,y,z)}$. To this end we will need the following lemma. In what follows (a,m) denotes the greatest common divisor of a and m.

Lemma 2. *Let W be an irreducible surface in E_k^3 defined by $x^m-y^az^b=0$. Let $\overline{a}=(a,m)$, $\overline{b}=(b,m)$, $a=\alpha\overline{a}$, $m=\mu\overline{a}$, $b=\beta\overline{b}$, $n=\nu\overline{b}$. Then the equations*

$$x=s^\alpha t^\beta,\qquad y=s^\mu,\qquad z=t^\nu$$

form the rational parametrization of W, the polynomial ring $k[s,t]$ is a finite module over the coordinate ring $k(W)=k[s^\alpha t^\beta,\ s^\mu,\ t^\nu]$, and the dimension of $k(s,t)$ as a vector space over the field $k(s^\alpha t^\beta,\ s^\mu,\ t^\nu)$ equals (μ,ν).

Proof. The parametrization of W is easy. Now, both s and t are integral over $k[s^\alpha t^\beta,\ s^\mu,\ t^\nu]$, hence $k[s,t]$ is a finite module over $k[s^\alpha t^\beta,\ s^\mu,t^\nu]$. By the hypothesis $(\alpha,\mu)=(\beta,\nu)=1$. Let $(\mu,\nu)=\xi$. Then $(\mu\beta,\nu)=\xi$ and there are integers p and q with $p\mu\beta+q\nu=\xi$. Therefore $(s^\alpha t^\beta)^{p\mu}(s^\mu)^{-p\alpha}(t^\nu)^q=t^{\pm\xi}$ is an element of $k(s^\alpha t^\beta,\ s^\mu,t^\nu)$. With the same argument one can prove $s^{\pm\xi}\in k(s^\alpha t^\beta,s^\mu,t^\nu)$. Now we have

$$k(s^\alpha t^\beta,s^\mu,t^\nu)=k(s^\alpha t^\beta,s^\xi,t^\xi),$$
$$k(s,t)=k(s^\alpha t^\beta,s^\xi,t^\xi)(t)=k(s^\alpha t^\beta,s^\xi,t^\xi)[t],$$

and the dimension of $k(s,t)=k(s^\alpha t^\beta,s^\xi,t^\xi)[t]$ over $k(s^\alpha t^\beta,s^\xi,t^\xi)$ is equal to the degree of the minimal polynomial $T^\xi-t^\xi$ of t over $k(s^\alpha t^\beta,\ s^\xi,\ t^\xi)$ (cf. [4]). □

Now we can give the formula for the multiplicity $e_0(QR,R)$.

Theorem 3. *Let $m,\ n,\ l,\ a,\ b,\ c,\ d,\ e,\ f$ be positive integers. Let us denote $\overline{a}=(a,m)$, $\overline{b}=(b,m)$, $a=\alpha\overline{a}$, $m=\mu\overline{a}$, $m=\nu\overline{b}$ $b=\beta\overline{b}$, and $(\mu,\nu)=\xi$. Let further $Q=(x^m-y^az^b,\ y^n-x^cz^d,\ z^l-x^ey^f)P$. Then*

$$e_0(QR,R)\xi=\min\{\mu\nu nl,\ \mu\beta(ne+cf)+\nu\alpha(lc+de)+\mu\nu df\}.$$

Proof. Let W be the irreducible surface in E_k^3 defined by $x^m-y^az^b=0$. Then the equations

$$x=s^\alpha t^\beta,\qquad y=s^\mu,\qquad z=t^\nu$$

form a rational parametrization of W. By Lemma 2, the polynomial ring $k[s,t]$ is a finitely generated module over the coordinate ring $k(W)=k[s^\alpha t^\beta,\ s^\mu,\ t^\nu]$, and its dimension as a vector space over the field $k(s^\alpha t^\beta,\, s^\mu,\, t^\nu)$ equals $(\mu,\nu)=\xi$. Thus by Theorem 1 we have

$$e_0(QR,R)\xi = e_0\big((s^{n\mu}-s^{c\alpha}t^{c\beta+d\nu}, t^{l\nu}-s^{e\alpha+f\mu}t^{e\beta}),\, k[s,t]_{(s,t)}\big).$$

If $mn < ac$ (equivalently, $n\mu < \alpha c$), then the multiplicity is equal to

$$\begin{aligned} &e_0\big((s^{n\mu}(1-s^{c\alpha-n\mu}t^{c\beta+d\nu}),\, t^{l\nu}-s^{e\alpha+f\mu}t^{e\beta}),\, k[s,t]_{(s,t)}\big)\\ &\qquad = e_0\big((s^{n\mu}, t^{l\nu}-s^{e\alpha+f\mu}t^{e\beta}),\, k[s,t]_{(s,t)}\big) = \mu\nu nl. \end{aligned}$$

We obtain the same result if $ml < be$ (equivalently, $l\nu < e\beta$).

Let now $mn \geqq ca$ and $ml \geqq be$, (i.e., $n\mu \geqq c\alpha$ and $l\nu \geqq e\beta$). Then we have

$$\begin{aligned} e_0(QR,R)\xi &= e_0\big(s^{c\alpha}(s^{n\mu-c\alpha}-t^{c\beta+d\nu}),\, t^{e\beta}(t^{l\nu-e\beta}-s^{e\alpha+f\mu}),\, k[s,t]_{(s,t)}\big)\\ &= c\alpha e\beta + c\alpha(l\nu-e\beta) + e\beta(n\mu-c\alpha)\\ &\quad + e_0\big((s^{n\mu-c\alpha}-t^{c\beta+d\nu},\, t^{l\nu-e\beta}-s^{e\alpha+f\mu}),\, k[s,t]_{(s,t)}\big)\\ &= c\alpha l\nu + e\beta n\mu - e\beta c\alpha\\ &\quad + \min\{(n\mu-c\alpha)(l\nu-e\beta),\, (c\beta+d\nu)(e\alpha+f\mu)\} \end{aligned}$$

by ([1]). Therefore

$$e_0(QR,R)\xi = \min\{\mu\nu nl,\, \mu\beta(ne+cf)+\nu\alpha(lc+de)+\mu\nu df\}.$$

$\square$

COROLLARY 4. *Let $Q = (x^m - y^a z^b,\, y^n - x^c z^d,\, z^l - x^e y^f)k[x,y,z]$ be an (x,y,z)-primary ideal with $(a,m)=(b,m)=1$. Let $R = k[x,y,z]_{(x,y,z)}$. Then for the multiplicity $e_0(QR,R)$ we have*

$$e_0(QR,R) = \min\{mnl,\, mdf+nbe+lac+ade+bcf\}.$$

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