VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0058-8
Math. Slovaca **58** (2008), No. 1, 101–114

BOUNDEDNESS, ONE-POINT EXTENSIONS AND B-EXTENSIONS

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(*Communicated by Ľubica Holá*)

ABSTRACT. The general concept of boundedness in a topological space generalizes both metric boundedness and relative compactness. A one-point extension $o(\mathscr{F}_X)$ of the space X is naturally associated to each boundedness $\mathscr{F}_X$ and every Hausdorff one-point extensions of a space X can be obtained in this way. Imitating this construction, it is possible to define a much more general class of Hausdorff extensions of a locally bounded space with respect to a given boundedness, the so-called B-extensions. In this paper we study separation properties and metrizability of this kind of extension.

1. Introduction

Following [8], we say that a nonempty family $\mathscr{F}_X$ of subsets of a topological space X is a *boundedness* in X in case

a) every subset of a member of $\mathscr{F}_X$ is in $\mathscr{F}_X$;

b) finite unions of members of $\mathscr{F}_X$ are in $\mathscr{F}_X$.

The family $\mathscr{C}_X$ of relatively compact subsets of a space X is a boundedness in X. The construction of the one-point compactification can be generalized replacing $\mathscr{C}_X$ by any boundedness $\mathscr{F}_X$, that is, endowing $X \cup \{p\}$ with the topology

$$\mathscr{T}_X \cup \big\{\{p\} \cup (X \setminus F): \ F = \mathrm{Cl}_X(F) \in \mathscr{F}_X\big\},$$

where $\mathscr{T}_X$ is the topology of X. We denote this extension by $o(\mathscr{F}_X)$. Every extension $X \cup \{p\}$ where $\{p\}$ is closed can be defined in this way.

In Sections 2 and 3 of this paper, after some basic definitions and preliminary results, we will show that some properties of a boundedness in X are equivalent to separation properties of the corresponding one-point extension. In particular we characterize Tychonoff and perfectly normal one-point extensions.

2000 Mathematics Subject Classification: Primary 54D35, 54D10, 54D20.
Keywords: boundedness, one-point extension, B-extension.

In Section 4 we consider the class of the so called B-extensions of a Hausdorff space X, locally bounded with respect to a given boundedness (see [4]). As we will see, many important examples, for instance, the Moore-Niemytzki plane, can be obtained as B-extension. We will extend some results in Section 3 to B-extensions with compact remainders. We also generalize a result by Beer ([2]) on the metrizability of one-point extensions.

2. Basic definitions. T_1 and Hausdorff one-point extensions

Let X be any topological space. We recall some definitions given in [4], [8].

If $\mathscr{F}_X$ is a boundedness on X, we will say that a subset $F \subset X$ is *bounded* if $F \in \mathscr{F}_X$, *unbounded* otherwise. $\mathscr{F}_X$ is said to be nontrivial if X is unbounded.

Clearly, $o(\mathscr{F}_X)$ is a dense extension if and only if $\mathscr{F}_X$ is nontrivial.

Definition 2.1. We say that a subcollection $\mathscr{B}$ of a boundedness $\mathscr{F}_X$ is a *basis* for $\mathscr{F}_X$, and that $\mathscr{B}$ *generates* $\mathscr{F}_X$, if every element of $\mathscr{F}_X$ is contained in some element of $\mathscr{B}$.

For every nonempty family $\mathscr{C}$ of subsets of X, the collection of finite unions of members of $\mathscr{C}$ is a basis for a boundedness in X.

Definition 2.2. A boundedness $\mathscr{F}_X$ is said to be *closed* if $F \in \mathscr{F}_X$ implies $\mathrm{Cl}_X(F) \in \mathscr{F}_X$; $\mathscr{F}_X$ is said to be *open* if every bounded set of X is contained in an open bounded set. A boundedness is said to be *proper* if it is both closed and open.

Clearly $\mathscr{F}_X$ is closed (open) if and only if there is a basis of $\mathscr{F}_X$ consisting of closed (resp. open) subsets.

Definition 2.3. A boundedness $\mathscr{F}_X$ is said to be a *bornology* if, for every $x \in X$, $\{x\}$ is bounded. X is said to be *locally bounded* with respect to a boundedness $\mathscr{F}_X$ if every $x \in X$ has a bounded neighborhood.

A boundedness $\mathscr{F}_X$ is a bornology if and only if $\bigcup_{F \in \mathscr{F}_X} F = X$. Clearly, if X is locally bounded with respect to $\mathscr{F}_X$, then $\mathscr{F}_X$ is a bornology. The converse holds in case $\mathscr{F}_X$ is open.

Remark 2.4. If X is locally bounded, then the family of bounded neighborhoods of x is a local basis. If $\mathscr{F}_X$ is open and $E \in \mathscr{F}_X$, then, for every open subset U of X containing E there is an open bounded set V such that $E \subset V \subset U$.

Examples 2.5. For every space X, $\mathscr{F}_X = \{F \subset X : F \text{ is finite}\}$ is the smallest bornology in X. If X is T_1 then $\mathscr{F}_X$ is closed and it is open if and only if X is discrete.

The family $\mathscr{C}_X$ of relatively compact subsets of a space X is a closed bornology. $\mathscr{C}_X$ is open if and only if X is locally compact.

If (X, ϱ) is a metric space, then the usual boundedness induced by ϱ is proper and X is locally bounded with respect to it. The family of totally bounded subsets of X is a closed bornology which is not open in general.

It is easy to see the following:

Proposition 2.6. *If X is a T_1 space and $\mathscr{F}_X$ is any boundedness in X, then $o(\mathscr{F}_X)$ is T_1 if and only if $\mathscr{F}_X$ is a bornology.*

Proposition 2.7. *Let X be a Hausdorff space and let $\mathscr{F}_X$ be a boundedness in X. Then the following are equivalent:*

(a) *Every $x \in X$ has a bounded closed neighborhood.*

(b) *$\mathscr{F}_X$ contains a closed boundedness $\mathscr{G}_X$ and X is locally bounded with respect to $\mathscr{G}_X$.*

(c) *$o(\mathscr{F}_X)$ is Hausdorff.*

Proof.

(a) $\Longrightarrow$ (b): We can put $\mathscr{G}_X = \{F \in \mathscr{F}_X : \ \mathrm{Cl}_X(F) \in \mathscr{F}_X\}$. The rest of the proof is easy. $\square$

Example 2.8. Let $X = (\mathbb{R}, \mathscr{T})$ where $\mathscr{T}$ is generated by the union of the usual topology and the family $\mathscr{A} = \{(-a, a) \setminus \{1/n\}_{n\in\mathbb{N}} : \ a \in \mathbb{R}^+\}$ (X is a classical example of a non-regular Hausdorff space). The family

$$\mathscr{B} = \Big\{(-a, a) \setminus \{\tfrac{1}{n}\}_{n \geq k} : \ a \in \mathbb{R}^+, \ \ k \in \mathbb{N}\Big\}$$

is closed with respect to finite unions. Let $\mathscr{F}_X$ be the boundedness generated by $\mathscr{B}$. Clearly $\mathscr{F}_X$ is an open boundedness and X is locally bounded with respect to $\mathscr{F}_X$. The point 0 has no bounded closed neighborhood, hence $o(\mathscr{F}_X)$ is not Hausdorff.

As usual, two extensions aX and bX of X are said to be *equivalent* if there is a homeomorphism $f\colon aX \to bX$ whose restriction to X is the identity map.

A natural closed boundedness in X associated to a given extension aX of X is defined by

$$\mathscr{H}_X(aX) = \{A \subset X : \ \mathrm{Cl}_X(A) = \mathrm{Cl}_{aX}(A)\}.$$

If $aX = X \cup \{p\}$ is a one-point extension, then $\mathscr{H}_X(aX)$ consists of the complements in aX of the neighborhoods of p and is generated by the complements of the open neighborhoods.

If $\mathscr{F}_X$ is any closed boundedness, then $\mathscr{H}_X(o(\mathscr{F}_X)) = \mathscr{F}_X$. Furthermore, if $aX = X \cup \{p\}$ is a one-point extension and X is open in aX, then $o(\mathscr{H}_X(aX))$ is equivalent to aX. So one has

Proposition 2.9. *If X is a T_1-space, then the map $\mathscr{F}_X \mapsto o(\mathscr{F}_X)$ is a bijection between the closed bornologies in X and the one-point T_1-extensions of X (up to equivalence).*

If X is Hausdorff, then the extension $o(\mathscr{F}_X)$, corresponding to a closed boundedness $\mathscr{F}_X$, is Hausdorff if and only if X is locally bounded.

If $\mathscr{F}_X$, $\mathscr{G}_X$ are closed and $\mathscr{F}_X \subset \mathscr{G}_X$, then the natural bijection from $o(\mathscr{G}_X)$ onto $o(\mathscr{F}_X)$ is continuous (that is, the topology of $o(\mathscr{G}_X)$ is "finer" than the one of $o(\mathscr{F}_X)$).

3. Tychonoff and perfectly normal one-point extensions

From now on all spaces will be Hausdorff and all boundednesses will be closed.

The following result is known (see [13, Theorem 1.1]).

Proposition 3.1. *Let X be a T_3-space (T_4-space), and let $\mathscr{F}_X$ be a closed boundedness in X. Then $o(\mathscr{F}_X)$ is T_3 (respectively T_4) if and only if $\mathscr{F}_X$ is open and X is locally bounded.*

In the following, $\mathbf{I}$ will denote the real unit interval and $C(X, \mathbf{I})$ the family of all continuous functions from X to $\mathbf{I}$.

Definition 3.2. We say that a boundedness $\mathscr{F}_X$ is *functionally open* if for every $F \in \mathscr{F}_X$ there exists a map $f \in C(X, \mathbf{I})$ such that $F \subset f^{-1}(0)$ and $f^{-1}([0,1)) \in \mathscr{F}_X$. This is clearly equivalent to the following condition: for every $F \in \mathscr{F}_X$ there is an open $G \in \mathscr{F}_X$ such that F and $X \setminus G$ are completely separated.

A functionally open boundedness is trivially open and closed.

Now we will characterize one-point Tychonoff extensions.

Theorem 3.3. *Let X be a Tychonoff space, locally bounded with respect to a closed boundedness $\mathscr{F}_X$. Then $o(\mathscr{F}_X) = X \cup \{p\}$ is Tychonoff if and only if $\mathscr{F}_X$ is functionally open.*

P r o o f. Let $\mathscr{F}_X$ be functionally open and let $x \notin F = \mathrm{Cl}_{o(\mathscr{F}_X)} F$. First suppose that $x \in X$. Then there is an open bounded subset U of X such that $x \in U \subset X \setminus F$ and there exists $f \in C(X, \mathbf{I})$ with $f(x) = 1$ and $f(X \setminus U) = 0$. If we define $\tilde{f}\colon o(\mathscr{F}_X) \to \mathbf{I}$ by $\tilde{f}(p) = 0$ and $\tilde{f}|X = f$, then $\tilde{f}$ is a continuous extension of f with $\tilde{f}(x) = 1$ and $\tilde{f}(F) = 0$. Now suppose $x = p$. Since F is closed in $o(\mathscr{F}_X)$ then F is bounded in X. Choose $f \in C(X, \mathbf{I})$ such that $F \subset f^{-1}(0) \subset f^{-1}([0,1)) \in \mathscr{F}_X$. Again $\tilde{f}\colon o(\mathscr{F}_X) \to \mathbf{I}$, where $\tilde{f}|X = f$ and $\tilde{f}(p) = 1$, is a continuous extension of f. In fact, $W = o(\mathscr{F}_X) \setminus \mathrm{Cl}_X\big(f^{-1}([0,1))\big)$ is an open neighborhood of p such that $\tilde{f}(W) = 1$. Moreover $\tilde{f}(x) = 1$ and $\tilde{f}(F) = 0$.

Conversely, assume that $o(\mathscr{F}_X)$ is Tychonoff and let $F \in \mathscr{F}_X$. Then $\mathrm{Cl}_X(F)$ is closed in $o(\mathscr{F}_X)$. Let $h \in C(o(\mathscr{F}_X), \mathbf{I})$ such that $h(\mathrm{Cl}_X(F)) = 0$ and $h(p) = 1$ and let $f = h|X$. Since $U = h^{-1}\big((\frac{1}{2}, 1]\big)$ is an open subset of $o(\mathscr{F}_X)$ containing p, then $o(\mathscr{F}_X) \setminus U = f^{-1}\big([0, \frac{1}{2}]\big)$ is bounded. Therefore $f^{-1}\big([0, \frac{1}{2})\big) \in \mathscr{F}_X$ and $F \subset f^{-1}(0)$ is completely separated from $X \setminus f^{-1}\big([0, \frac{1}{2})\big)$. This proves that is $\mathscr{F}_X$ is functionally open. □

Clearly, a closed boundedness $\mathscr{F}_X$ in a T_4 space X is functionally open if and only if it is open.

Let X be a Tychonoff space, locally bounded with respect to a boundedness $\mathscr{F}_X$. Put

$$\mathscr{Z}\mathscr{F}_X = \big\{F \in \mathscr{F}_X : \ \exists f \in C(X, \mathbf{I}) \text{ such that } F \subset f^{-1}(0) \subset f^{-1}([0,1)) \in \mathscr{F}_X\big\}.$$

$\mathscr{Z}\mathscr{F}_X$ is a functionally open boundedness such that X is locally bounded.

Clearly $\mathscr{F}_X$ is functionally open if and only if $\mathscr{F}_X = \mathscr{Z}\mathscr{F}_X$.

Examples 3.4. We can use $\mathscr{Z}\mathscr{F}_X$, for a suitable $\mathscr{F}_X$, to obtain a Tychonoff one-point extension of X having a specific topological property. This has been done for Lindelöfness ([11]) and sequential compactness ([13]), without an explicit definition of functionally open boundednesses. Let us consider the following closed boundednesses in X:

$$\begin{aligned}
\mathscr{L}_X &= \{E \subset X : \ \mathrm{Cl}_X(E) \text{ is Lindelöf}\},\\
\omega\text{-}\mathscr{C}_X &= \big\{E \subset X : \ \mathrm{Cl}_X(E) \text{ is countably compact}\big\},\\
\mathscr{S}C_X &= \{E \subset X : \ \mathrm{Cl}_X(E) \text{ is sequentially compact}\}.
\end{aligned}$$

If X is Tychonoff and locally bounded with respect to $\mathscr{F}_X$, where $\mathscr{F}_X = \mathscr{L}_X$, (or $\mathscr{F}_X = \omega\text{-}\mathscr{C}_X$ or $\mathscr{F}_X = \mathscr{S}C_X$), then $o(\mathscr{F}_X)$ is Lindelöf (resp. countably compact, sequentially compact), but can fail to be Tychonoff. Since $o(\mathscr{Z}\mathscr{F}_X)$ is a continuous image of $o(\mathscr{F}_X)$, then it is a Tychonoff extension with the required property.

The following example shows that there exist proper not functionally open boundednesses.

Example 3.5. In [6, Example 1.5.9], a regular non-Tychonoff space M is described. One has $M = M_0 \cup \{z\}$ where $M_0 = \mathbb{R} \times \big(\mathbb{R}^+ \cup \{0\}\big)$ and $z = (0, -1)$. M is endowed with a topology such that the points of M_0 have a local basis consisting of clopen sets and z has a countable local basis. The subspace M_0 is T_3 and 0-dimensional, so it is Tychonoff. Put $X = M_0$ and $aX = M$. Then, by 3.1 and 3.3, the closed boundedness $\mathscr{H}_X(aX)$ is open but is not functionally open.

For sake of completeness we give a characterization of the closed boundednesses such that the corresponding one-point extensions are perfectly normal, even though this is a particular case of Proposition 4.6 in the next section.

Proposition 3.6. *Let $\mathscr{F}_X$ be a closed boundedness in a perfectly normal space X. Then $o(\mathscr{F}_X) = X \cup \{p\}$ is perfectly normal if and only if X is locally bounded, $\mathscr{F}_X$ is open and $X = \bigcup_{n\in\mathbb{N}} L_n$ with $L_n \in \mathscr{F}_X$.*

P r o o f. If $o(\mathscr{F}_X)$ is perfectly normal then $X = \bigcup_{n\in\mathbb{N}} L_n$ where L_n is closed in $o(\mathscr{F}_X)$, hence bounded. The other properties follows from Proposition 3.1.

Conversely, $o(\mathscr{F}_X)$ is T_4 by 3.1. Since $\mathscr{F}_X$ is closed, one has $X = \bigcup_{n\in\mathbb{N}} L_n$, where $L_n = \mathrm{Cl}_X(L_n) \in \mathscr{F}_X$. This implies that $\{p\} = \bigcap_{n\in\mathbb{N}} U_n$, where $U_n = o(\mathscr{F}_X) \setminus L_n$ is open in $o(\mathscr{F}_X)$.

We will prove that every closed subset A of $o(\mathscr{F}_X)$ is a G_δ. The case $A \subset X$ is trivial. Suppose $p \in A$. One has $A \cap X = \bigcap_{n\in\mathbb{N}} W_n$, where every W_n open in X, hence in $o(\mathscr{F}_X)$. Then one has

$$A = \left(\bigcap_{n\in\mathbb{N}} U_n\right) \cup \left(\bigcap_{m\in\mathbb{N}} W_m\right) = \bigcap_{n,m\in\mathbb{N}} (U_n \cup W_m).$$

□

Remark 3.7. Let X be a non-Lindelöf locally Lindelöf space. If X is T_4, then $\mathscr{L}_X$ is open (see [4, Prop. 2.4]), hence $o(\mathscr{L}_X)$ is a T_4 not perfectly normal space. In fact $X = \bigcup_{n\in\mathbb{N}} L_n$, where $L_n = \mathrm{Cl}_X(L_n) \in \mathscr{L}_X$ would imply that X is Lindelöf. Similarly, if X is Tychonoff, $o(\mathscr{Z}L_X)$ is T_4, but not perfectly normal.

Remark 3.8. Let X be any T_3 space. The condition in Proposition 3.6, that is:

(H) $\mathscr{F}_X$ is proper, X is locally bounded and $X = \bigcup_{n\in\mathbb{N}} L_n$ with $L_n \in \mathscr{F}_X$,

is generally weaker than the following:

(M) X is locally bounded, $\mathscr{F}_X$ is proper and has a countable basis.

In both cases $o(\mathscr{F}_X) = X \cup \{p\}$ is T_3, but (H) implies that p is the intersection of a countable family of neighborhoods, while (M) implies that p has a countable local basis. (H) and (M) are equivalent for $\mathscr{F}_X = \mathscr{C}_X$.

Example 3.9. Let X be the subspace $\mathbb{N} \times \mathbb{R}$ of the Euclidean plane and let $\mathscr{F}_X$ be the boundedness consisting of the sets B such that $B \cap (\{j\} \times \mathbb{R})$ is bounded, in the usual sense, for every $j \in \mathbb{N}$. Then, clearly, $\mathscr{F}_X$ satisfies (H) but not (M).

As we have already observed, if X is a metrizable space, then there is a natural boundedness associated to every compatible metric ϱ. It consists of all subsets of X having a finite diameter. We denote it by $\varrho\mathscr{F}_X$. A boundedness $\mathscr{G}_X$ which can be induced by a metric is said to be a *metric boundedness*. It is easy to prove that a metric boundedness always satisfies (M).

It was proved by Hu that the converse is also true, that is, if X is metrizable and $\mathscr{G}_X$ satisfies (M), then $\mathscr{G}_X$ is a metric boundedness ([9, Theorem 5.11]).

The author found a function $f\colon X \to \mathbb{R}$ such that $A \in \mathscr{G}_X$ if and only if $\sup\{f(x) : x \in A\} < \infty$ (a function with this property will be later called a *forcing function*). Then he proved that, for every compatible metric d of X,

$$\varrho(x,y) = \min\{d(x,y), 1\} + |f(x) - f(y)|$$

defines a compatible metric such that $\varrho\mathscr{F}_X = \mathscr{G}_X$.

We recall that a metrizable space is said to be *boundedly compact* if there is a compatible metric ϱ on X such that the family of compact subsets coincides with the family of closed bounded subsets (that is $\mathscr{C}_X = \varrho\mathscr{F}_X$). From the above result by Hu, we can deduce an easy proof of the following known result (see [12]).

3.10. *A metrizable space X is boundedly compact if and only if it is locally compact and hemicompact (or, equivalently, σ-compact or second countable).*

Proof. One has $\mathscr{C}_X = \varrho\mathscr{F}_X$, for some compatible metric ϱ on X, if and only if $\mathscr{C}_X$ satisfies (M), that is, if X is locally compact and hemicompact. □

We will say that the boundedness $\mathscr{F}_X$ is a *M-boundedness* if it satisfies (M).

A non-metrizable space X can admit a M-boundedness $\mathscr{F}_X$. If X is perfectly normal, then $o(\mathscr{F}_X)$ will be perfectly normal. If X is Tychonoff but is not normal then $o(\mathscr{F}_X)$ can fail to be Tychonoff.

Examples 3.11. Let L be the Sorgenfrey line, and let $\mathscr{F}_L$ be the boundedness generated by $\{(-n,n)\}_{n\in\mathbb{N}}$. Clearly $\mathscr{F}_L$ is a M-boundedness and so $o(\mathscr{F}_L)$ is perfectly normal. If $X = M_0$, and $aX = M$ are the spaces considered in Example 3.5, then $\mathscr{H}_X(aX)$ satisfies (M), but aX is not Tychonoff.

In [1], [2], G. Beer studied the "dual" concept of a M-boundedness, that is the so-called *metric modes of convergence to infinity.* They are equivalence classes of decreasing sequences $\langle F_k\rangle$ of closed subsets of a space X satisfying some conditions, which are equivalent to $\{X \setminus F_k\}_{n\in\mathbb{N}}$ being a basis of a nontrivial M-boundedness in X, denoted by $\mathscr{B}(\langle F_k\rangle)$. One has $\mathscr{B}(\langle G_k\rangle) = \mathscr{B}(\langle F_k\rangle)$ if and only if $\langle F_k\rangle$ and $\langle G_k\rangle$ are equivalent. Then, for a given space X, the map $\langle F_k\rangle \mapsto \mathscr{B}(\langle F_k\rangle)$ is a bijection from the family of the metric modes of convergence to infinity onto the family of nontrivial M-boundednesses.

A non-metric M-boundedness may not admit any forcing function. In [2, Theorem 3.2], it is proved that, for a metric mode of convergence to infinity $\langle F_k\rangle$ defined on a T_3 space X, $\mathscr{B}(\langle F_k\rangle)$ admits a forcing function if and only if:

(2) For each $n \in \mathbb{N}$ there is $j > n$ such that $X \setminus \mathrm{Int}(F_n)$ and F_j are functionally separated.

Clearly a M-boundedness $\mathscr{F}_X = \mathscr{B}(\langle F_k\rangle)$ satisfies the above condition if and only if it is functionally open.

We can rephrase a result in [2, Theorem 4.3], in the following way:

Theorem 3.12. (G. Beer) *If X is a metrizable space and $\mathscr{F}_X$ is a M-boundedness, then $o(\mathscr{F}_X)$ is metrizable.*

By Theorem 3.6 and Remark 3.8, the converse is also true, so one has:

Corollary 3.13. *If X is a metrizable space and $\mathscr{F}_X$ is a boundedness in X, then $o(\mathscr{F}_X)$ is metrizable if and only if $\mathscr{F}_X$ is a M-boundedness.*

Example 3.9 shows that, in Theorem 3.12 and Corollary 3.13, condition (M) cannot be replaced by (H).

4. B-extensions with compact remainders

We want to extend some results in the previous section to a larger class of extensions.

The following construction was given in [4], as a generalization of ESH-compactifications (ESH is an abbreviation for "essential semilattice homomorphism", see [3]).

Definition 4.1. Let $\mathscr{F}_X$ be a nontrivial closed boundedness on the space X and let $\mathscr{B}$ be an open basis for a space Y, closed with respect to finite unions.

A map $\pi = \pi(\mathscr{B}, \mathscr{F}_X)\colon \mathscr{B} \to (\mathscr{T}_X \setminus \mathscr{F}_X) \cup \{\emptyset\}$, with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$, is said to be a *B-map* if the following conditions are satisfied:

B1) If $\{U_i\}_{i \in A} \subset \mathscr{B}$ is a cover of Y, then $X \setminus \Big[\bigcup_{i\in A} \pi(U_i)\Big] \in \mathscr{F}_X$;

B2) If $U, V \in \mathscr{B}$ then $\pi(U \cup V)\,\Delta\,[\pi(U) \cup \pi(V)] \in \mathscr{F}_X$;

B3) If $U, V \in \mathscr{B}$ and $\mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset$ then $\pi(U) \cap \pi(V) \in \mathscr{F}_X$.

If $\emptyset \in \mathscr{B}$ then $\pi(\emptyset) \in \mathscr{F}_X$. We can always add $\emptyset$ to $\mathscr{B}$, putting $\pi(\emptyset) = \emptyset$.

If Y is compact, then $Y \in \mathscr{B}$ and B1) can be replaced by:

B1′) $X \setminus \pi(Y) \in \mathscr{F}_X$.

Let X be a (Hausdorff) space, locally bounded and unbounded with respect to a closed boundedness $\mathscr{F}_X$. Putting on the disjoint union $X \cup Y$ the topology generated by

$$\mathscr{T}_X \cup \big\{U \cup (\pi(U) \setminus F):\ U \in \mathscr{B},\ \ F = \mathrm{Cl}_X(F) \in \mathscr{F}_X\big\},$$

we obtain a Hausdorff dense extension of X, denoted by $X \cup_\pi Y$. An extension which can be constructed in this way is said to be a *B-extension*.

The axiom B1) implies that, for every basic cover $\{U_i\}$ of Y, the family $\{U_i \cup \pi(U_i)\}$ covers all of $X \cup_\pi Y$ except a negligible subset and has the same cardinality as $\{U_i\}$.

B2) means that the family of the basic open subsets of $X \cup_\pi Y$ which meet Y essentially has the same semilattice structure of $\mathscr{B}$.

By $B3$) we obtain that two points of Y, separated by disjoint members of $\mathscr{B}$, are also separated in $X \cup_\pi Y$ (outside of some closed bounded set).

All n-point Hausdorff extensions are B-extensions. Moreover, a T_4-extension of X such that $aX \setminus X$ is 0-dimensional is a B-extension ([4, 1.5]).

For instance, the Franklin-Rajagopalan space (see [6, 3.12.17(d)]), is a B-extension of $\mathbb{N}$ with respect to the boundedness of finite subsets.

We need the following known result (see [4, Proposition 1.1]):

Lemma 4.2. *Let $aX = X \cup_\pi Y$ be a B-extension of X and let F be a closed subset of X. If F is bounded, then F is closed in aX. If Y is compact, then F is closed in aX if and only if it is bounded.*

Theorem 4.3. *Let $aX = X \cup_\pi Y$ be a B-extension of X, where $\pi = \pi(\mathscr{B}, \mathscr{F}_X)$. Suppose X is a T_3-space and Y is compact. Then $X \cup_\pi Y$ is T_3 if and only if $\mathscr{F}_X$ is open.*

P r o o f. Suppose $\mathscr{F}_X$ is open and let $x \in X$. Then x has a local basis consisting of closed bounded neighborhoods in X, hence a local basis of closed neighborhoods in aX. Now, let $x \in Y$ and let A be a closed subset of aX with $x \notin A$. Let U', U'' be disjoint open subsets of Y such that $x \in U'$ and $K = A \cap Y \subset U''$. There is $U \in \mathscr{B}$ such that $x \in U \subset U'$. For every $y \in K$, there is $V_y \in \mathscr{B}$ such that $y \in V_y \subset \mathrm{Cl}_Y(V_y) \subset U''$. Let $\{V_{y_k} :\ 1 \le k \le n\}$ be a subcover of K and

$$V = \bigcup_{k=1}^{n} V_{y_k} \in \mathscr{B}.$$

One has $A \cap X \subset V \subset \mathrm{Cl}_Y(V) \subset U''$, hence $\mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset$. Then, by B3),

$$\pi(U) \cap \pi(V) = G \in \mathscr{F}_X.$$

Put $G_1 = \mathrm{Cl}_X(G) \in \mathscr{F}_X$ and $H = A \setminus (V \cup \pi(V))$. Then H is a subset of X which is closed in aX, so it is bounded. Let W be a bounded neighborhood of H in X and $H_1 = \mathrm{Cl}_X(W) \in \mathscr{F}_X$. Then $(V \cup \pi(V)) \cup W$ is an open subset of aX containing A and it is disjoint from $U \cup [\pi(U) \setminus (G_1 \cup H_1)]$, which is a basic neighborhood of x in aX.

Conversely, suppose aX is T_3 and let $A \in \mathscr{F}_X$. Then $\mathrm{Cl}_X(A)$ is bounded, so it is closed in aX. Since Y is compact, there exist disjoint open subsets U and V of aX which contain $\mathrm{Cl}_X(A)$ and Y respectively. Then $\mathrm{Cl}_X(A) \subset U \subset aX \setminus V \subset X$ and $aX \setminus V$ is closed in aX, hence bounded. Therefore A is contained in an open member of $\mathscr{F}_X$. □

Theorem 4.4. *Let $aX = X \cup_\pi Y$ be a B-extension of X, where $\pi = \pi(\mathscr{B}, \mathscr{F}_X)$. Suppose X is T_4 and Y is compact. Then $X \cup_\pi Y$ is T_4 if and only if $\mathscr{F}_X$ is open.*

P r o o f. Suppose $\mathscr{F}_X$ is open and let A, B be disjoint closed subset of aX. Let U', U'' be disjoint open subsets of Y such that $A \cap Y \subset U'$ and $B \cap Y \subset U''$. As in the the proof of Theorem 4.3, we can find $U, V \in \mathscr{B}$, such that $A \cap Y \subset U \subset \mathrm{Cl}_Y(U) \subset U'$ and $B \cap Y \subset V \subset \mathrm{Cl}_Y(V) \subset U''$. Then $\mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset$ and, by B3), one has

$$H = \pi(V) \cap \pi(U) \in \mathscr{F}_X.$$

Let $H_1 = \mathrm{Cl}_X(H)$. The subsets of X

$$A_1 = A \setminus [U \cup (\pi(U) \setminus H_1)], \qquad B_1 = B \setminus [V \cup (\pi(V) \setminus H_1)]$$

are closed in $X \cup_\pi Y$, hence they are bounded. We can find two disjoint bounded open subsets F, G of X containing A_1 and B_1 respectively (see Remark 2.4). Clearly we can choose F disjoint from B and G disjoint from A. Let F_1, G_1 be open in X with $A_1 \subset F_1 \subset \mathrm{Cl}_X(F_1) \subset F$ and $B_1 \subset G_1 \subset \mathrm{Cl}_X(G_1) \subset G$. If we put

$$W = [U \cup (\pi(U) \setminus (H_1 \cup \mathrm{Cl}_X(G_1)))] \cup F_1, \quad W' = [V \cup (\pi(V) \setminus (H_1 \cup \mathrm{Cl}_X(F_1)))] \cup G_1,$$

then one has

$$A \subset W, \quad B \subset W', \qquad W \cap W' = \emptyset.$$

The converse follows from Theorem 4.3. □

Example 4.5. The Moore-Niemytzki plane is a union of two normal spaces which is not normal. We will show that it can be obtained as B-extension with respect to a proper boundedness.

Let X be the upper half plane, defined by $\{(x,y): \ y>0\}$, endowed with the usual topology, and Y be the x-axis with the discrete topology. For every $z \in Y$ and $r \in \mathbb{R}^+$, let $D(z,r)$ be the closed disk, of radius r, tangent to Y at z, and let $S(z,r)$ be the interior (with the usual meaning) of $D(z,r)$. For $r_1 > r_2$, put $A(z,r_1,r_2) = D(z,r_1) \setminus (S(z,r_2) \cup \{z\})$. We denote by d be the Euclidean metric on the plane $\mathbb{R}^2$. Let

$$\begin{aligned} \mathscr{A}_1 &= \{F \subset X: \ F = \mathrm{Cl}_X(F), \ d(F,Y) > 0\}, \\ \mathscr{A}_2 &= \{A(z,r_1,r_2): \ r_1, r_2 \in \mathbb{R}, \ r_1 > r_2, \ z \in Y\}. \end{aligned}$$

Both $\mathscr{A}_1$ and $\mathscr{A}_2$ are collection of closed subsets of X. Let $\mathscr{F}_X$ be the (closed) boundedness in X generated by the set of finite unions of members of $\mathscr{A}_1 \cup \mathscr{A}_2$. Clearly, every element of $\mathscr{A}_1$ is contained in a bounded open subset of X. For every $A(z,r_1,r_2) \in \mathscr{A}_2$, let $r_3 > r_1 > r_2 > r_4 > 0$. One has

$$A(z,r_1,r_2) \subset S(z,r_3) \setminus D(z,r_4) \subset A(z,r_3,r_4) \in \mathscr{A}_2.$$

$S(z,r_3) \setminus D(z,r_4)$ is a bounded open subset of X containing $A(z,r_1,r_2)$. Therefore $\mathscr{F}_X$ is a proper boundedness. Note that every $S(z,r)$ is an unbounded open subset of X.

Let $\mathscr{B}$ be the collection of finite subsets of Y and let $\pi: \mathscr{B} \to \mathscr{T}_X \setminus \mathscr{F}_X$ be defined by $B \mapsto \bigcup_{z \in B} S(z,1)$. It is easy to see that π is a B-map. Now we will prove that $X \cup_\pi Y$ ha the same topology as the Moore-Niemytzki plane $(X \cup Y, \mathscr{T})$. Let $U = \{z\} \cup S(z,r)$ be a basic open neighborhood of $z \in Y$ with respect to $\mathscr{T}$. We can suppose $r < 1$. Then $S(z,r) = S(z,1) \setminus (D(z,1) \setminus S(z,r))$, hence $U = \{z\} \cup (\pi(\{z\}) \setminus A(z,1,r))$ which is open in $X \cup_\pi Y$. Conversely, we have only to prove that every set of the form $\{z\} \cup (S(z,1) \setminus A)$, where $z \in Y$ and $A = A_1 \cup \cdots \cup A_n$, $A_i \in \mathscr{A}_1 \cup \mathscr{A}_2$, contains a basic neighborhood of z with respect to $\mathscr{T}$. If A_i is of the form $A(z,r_1,r_2)$, then put $r(i) = r_2/2$. Otherwise, put $r(i) = d_i/2$, where $d_i = d(z,A_i)$. Let $r = \min\{r(1),\ldots,r(n)\}$. Then $\{z\} \cup S(z,r)$ is the required neighborhood.

Therefore, the hypothesis that X and Y are T_4 is not sufficient to ensure that a B-extension $X \cup_\pi Y$ with respect to a proper boundedness is T_4.

Proposition 4.6. *Let $aX = X \cup_\pi Y$ be a B-extension of X with respect to the boundedness $\mathscr{F}_X$ and suppose that X and Y are perfectly normal and Y is compact. Then aX is perfectly normal if and only if $\mathscr{F}_X$ satisfies* (H).

Proof. Using Theorem 4.4, the proof is similar to the one of Proposition 3.6. We will only prove that, assuming (H), every closed subset A of aX is a G_δ. $A \cap Y$ is a compact G_δ in Y. For every open $V \in \mathscr{T}_Y$ containing $A \cap Y$ there is $U \in \mathscr{B}$ such that $A \cap Y \subset U \subset V$. Then $A \cap Y = \bigcap_{n\in\mathbb{N}} U_n$ with $U_n \in \mathscr{B}$. By hypothesis, $X = \bigcup_{n\in\mathbb{N}} L_n$, where $L_n = \mathrm{Cl}_X(L_n) \in \mathscr{F}_X$ and $\{L_n\}_{n\in\mathbb{N}}$ is increasing. One has

$$A \cap Y = \bigcap_{n\in\mathbb{N}} [U_n \cup (\pi(U_n) \setminus L_n)].$$

Since X is also perfectly normal, $A \cap X$ is the intersection of a decreasing sequence $\{W_n\}_{n\in\mathbb{N}}$ of members of $\mathscr{T}_X \subset \mathscr{T}_{aX}$. It is easy to see that

$$A = \bigcap_{n\in\mathbb{N}} [U_n \cup (\pi(U_n) \setminus L_n) \cup W_n].$$

□

In order to obtain a condition ensuring the metrizability of a B-extension, we need the following lemma.

Lemma 4.7. *Let $aX = X \cup_\pi Y$ be a B-extension, where $\pi = \pi(\mathscr{B}, \mathscr{F}_X)$. Suppose $\mathscr{B}_1 \subset \mathscr{B}$ is a basis for the open sets of Y which is closed with respect to finite unions. Then $\pi_1 = \pi|_{\mathscr{B}_1}$ is a B-map and $X \cup_{\pi_1} Y$ has the same topology as aX.*

Proof. Clearly π_1 is a B-map. Let $U \in \mathscr{B}$. We only need to prove that $W = U \cup \pi(U)$ is open in $X \cup_{\pi_1} Y$. Clearly W is a neighborhood, in $X \cup_{\pi_1} Y$, for each point in $\pi(U)$. Let $x \in U$ and let $V \in \mathscr{B}_1$ such that $x \in V \subset U$. Put $A = \pi(V) \setminus \pi(U)$. One has $A = [\pi(U) \cup \pi(V)] \Delta [\pi(U \cup V)]$ which is bounded by B2). Then $x \in V \cup [\pi_1(V) \setminus \mathrm{Cl}_X(A)] \subset W$. □

Theorem 4.8. *Let X, Y be metrizable spaces and suppose Y is compact. Let $\pi = \pi(\mathscr{B}, \mathscr{F}_X)$ be a B-map, where $\mathscr{B}$ is a basis for Y. Then $aX = X \cup_\pi Y$ is metrizable if and only if $\mathscr{F}_X$ is an M-boundedness.*

Proof. Suppose $\mathscr{F}_X$ is an M-boundedness. By Theorem 4.3, aX is T_3. We will prove that aX has a σ-locally finite basis.

By hypothesis, $\mathscr{F}_X$ admits a countable basis $\{M_k\}_{k\in\mathbb{N}}$, where M_k is open and $\mathrm{Cl}_X(M_k)$ is bounded. One has $\bigcup_{k\in\mathbb{N}} M_k = X$. Let $\mathscr{C} = \bigcup_{n\in\mathbb{N}} \mathscr{C}_n$ be a basis for the open subsets of X, where every $\mathscr{C}_n$ is a locally finite family. Put

$$\mathscr{C}_n^k = \{C \cap M_k : C \in \mathscr{C}_n\}, \qquad n, k \in \mathbb{N}.$$

Since Y is second countable, $\mathscr{B}$ contains a countable basis, closed with respect to finite unions. Therefore, by the above lemma, we can suppose that $\mathscr{B}$ is countable, $\mathscr{B} = \{U_n\}_{n\in\mathbb{N}}$. For every $n, k \in \mathbb{N}$, let $\mathscr{E}_n^k$ be the family consisting of the the single element $U_n \cup [\pi(U_n) \setminus \mathrm{Cl}_X(M_k)]$. We claim that

$$\mathscr{S} = \Big(\bigcup_{n,k\in\mathbb{N}} \mathscr{C}_n^k\Big) \cup \Big(\bigcup_{n,k\in\mathbb{N}} \mathscr{E}_n^k\Big)$$

is a σ-locally finite basis for aX. Let W be an open subset of aX and let x be in W. If $x \in X$, then $x \in M_k$ for some k and there is $C \in \mathscr{C}_n$, for some n, such that $x \in C \subset W$. Then $x \in C \cap M_k \subset W$, with $C \cap M_k \in \mathscr{C}_n^k$. If $x \in Y$, then there is $U_n \in \mathscr{B}$ and $F = \mathrm{Cl}_X(F) \in \mathscr{F}_X$ such that $x \in U_n \cup (\pi(U_n) \setminus F) \subset W$. One has $F \subset M_k$ for some k, hence

$$x \in U_n \cup [\pi(U_n) \setminus \mathrm{Cl}_X(M_k)] \subset U_n \cup (\pi(U_n) \setminus F) \subset W.$$

Now, we need show that $\mathscr{S}$ is σ-locally finite. Since every $\mathscr{E}_n^k$ consists of one element, we have only to show that every $\mathscr{C}_n^k$ is locally finite. Let $x \in X$. Since $\mathscr{C}_n$ is locally finite, there is a neighborhood of x that meets only finitely many members of $\mathscr{C}_n$, hence of $\mathscr{C}_n^k$. If $x \in Y$, let $U_n \in \mathscr{B}$ such that $x \in U_n$. Then the basic neighborhood $U_n \cup [\pi(U_n) \setminus \mathrm{Cl}_X(M_k)]$ meets no member of $\mathscr{C}_n^k$. We have proved that aX is metrizable.

Conversely, by Theorem 4.4, $\mathscr{F}_X$ is proper. We need to prove that $\mathscr{F}_X$ has a countable basis. Let ϱ be a compatible metric on aX and put $M_n = \{x \in X : \varrho(x, Y) \geq 1/n\}$. Every M_n is closed in aX, hence bounded. Let F be a bounded closed subset of X. Since Y is compact one has $\varrho(F, Y) = d > 0$. If $1/n < d$ then $F \subset M_n$. Since $\mathscr{F}_X$ is closed, every $A \in \mathscr{F}_X$ is a subset of some M_n, that is, $\{M_n\}_{n\in\mathbb{N}}$ is a countable basis of $\mathscr{F}_X$. □

Remark 4.9. If (X, ϱ) is a metric space, Y is compact and metrizable and $X \cup_\pi Y$ is a B-extension with respect to $\varrho\mathscr{F}_X$, then every unbounded subset of X, with respect to ϱ, has some accumulation point in Y (see 4.2). Therefore, for every unbounded sequence in X, there is a subsequence which converges to a point of Y.

Example 4.10. The Mrówka space, denoted by Ψ in [7], is the union $\mathbb{N}\cup\{x_A\}_{A\in\mathscr{A}}$, where $\mathscr{A}$ is a maximal almost disjoint family of infinite subsets of $\mathbb{N}$. All subsets of $\mathbb{N}$ are open in Ψ and a local basis for x_A consists of the sets of the form $\{x_A\} \cup (A \setminus F)$, where F is a finite subset of $\mathbb{N}$. Let $M = \{x_A\}_{A\in\mathscr{A}}$, endowed with the discrete topology. Clearly, $\Psi = \mathbb{N} \cup_{\pi(\mathscr{B},\mathscr{F}_\mathbb{N})} M$, where $\mathscr{B} = \{E \subset \{x_A\}_{A\in\mathscr{A}} : E \text{ is finite}\}$, $\mathscr{F}_\mathbb{N}$ is the boundedness consisting of finite subsets of $\mathbb{N}$ and $\pi(E) = \bigcup_{x_A\in E} A$ for each E. Ψ is locally compact, hence Tychonoff, but is not normal, although $\mathbb{N}$ and M are both discrete and $\mathscr{F}_\mathbb{N}$ is an M-boundedness. Then Proposition 4.6 and Theorem 4.8 may not hold if we drop the hypothesis that Y is compact.

We do not know whether the regularity of a B-extension $X \cup_\pi Y$, where X and Y are T_3, can be proved without the hypothesis that Y is compact. However, this can be done provided $X \cup_\pi Y$ belongs to a particular class of B-extensions, the so-called B-singular extensions, defined in [4] (see also [5]).

Let X be unbounded and locally bounded with respect to a closed boundedness $\mathscr{F}_X$. A continuous mapping from X to any (Hausdorff) space Y is said to be *B-singular* (with respect to $\mathscr{F}_X$) if $f^{-1}(U) \notin \mathscr{F}_X$ for every nonempty open subset U of Y. If f is B-singular, then the map

$$\pi\colon \mathscr{T}_Y \to (\mathscr{T}_X \setminus \mathscr{F}_X) \cup \{\emptyset\}, \qquad \pi(U) = f^{-1}(U),$$

is a B-map. The B-extension induced by π is denoted by $X \cup_f Y$ and is said to be *B-singular*.

THEOREM 4.11. *Let $aX = X \cup_f Y$ be a B-singular extension of X with respect to an open boundedness $\mathscr{F}_X$. If X, Y are T_3, then aX is also T_3.*

Proof. The proof of the case $x \in X$ is the same as in Theorem 4.3.

Now, let $x \in Y$ and let $U \cup (f^{-1}(U) \setminus F)$ be a basic neighborhood of x, where $U \in \mathscr{T}_Y$ and $F = \mathrm{Cl}_X(F) \in \mathscr{F}_X$. There exists $W \in \mathscr{T}_Y$ such that $x \in W \subset \mathrm{Cl}_Y(W) \subset U$. Then

$$W \cup f^{-1}(W) \subset \mathrm{Cl}_Y(W) \cup f^{-1}(\mathrm{Cl}_Y(W)) \subset U \cup f^{-1}(U).$$

Note that $\mathrm{Cl}_Y(W) \cup f^{-1}(\mathrm{Cl}_Y(W)) = aX \setminus [(Y \setminus \mathrm{Cl}_Y(W)) \cup f^{-1}(Y \setminus \mathrm{Cl}_Y(W))]$ is closed in aX.

Since $\mathscr{F}_X$ is proper, there is an open subset A of X such that $F \subset A \subset \mathrm{Cl}_X(A) \in \mathscr{F}_X$. One has

$$x \in W \cup [f^{-1}(W) \setminus \mathrm{Cl}_X(A)] \subset [\mathrm{Cl}_Y(W) \cup f^{-1}(\mathrm{Cl}_Y(W))] \setminus A \subset U \cup (f^{-1}(U) \setminus F),$$

that is, $U \cup (f^{-1}(U) \setminus F)$ contains a closed neighborhood of x. □

The B-extensions in Examples 4.5 and 4.10 are not B-singular. We cannot provide any example of a non-normal B-singular extension $X \cup_f Y$, with respect to a proper boundedness, where X and Y are T_4. Therefore the problem of the normality of such an extension remains open.

Acknowledgement. The authors are grateful to Professor J. Beer for the interesting conversations about this subject and for asking the question answered in Example 3.5.

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Received 16. 3. 2006
Revised 4. 8. 2006

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