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ON SHADOWING PROPERTY FOR INVERSE LIMIT SPACES

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ABSTRACT. We study here the G-shadowing property of the shift map σ on the inverse limit space X_f, generated by an equivariant self-map f on a metric G-space X.

Introduction

The theory of shadowing is a significant part of the qualitative theory of discrete dynamical systems. From numerical point of view, if a dynamical system has the shadowing property, then numerically obtained orbits reflect the real behaviour of trajectories of the systems. Furthermore, we may consider the shadowing property as a weak form of stability of dynamical systems with respect to C^0 perturbations. The present paper concerns the shadowing property for shift maps on inverse limit spaces. A sequence $\{x_n : n \geq 0\}$ is called a *δ-pseudo orbit* of a self continuous map f on X if $d(f(x_n), x_{n+1}) < \delta$, for all $n \geq 0$. Map f is said to have the *shadowing property* if for a given $\epsilon > 0$, there is a $\delta > 0$ such that for every δ-pseudo orbit $\{x_n : n \geq 0\}$ there is a point x in X satisfying $d(f^n(x), x_n) < \epsilon$, for all $n \geq 0$. For a compact metric space X, $X^{\mathbf{Z}}$ is the compact metric space of all two-sided sequences $(x_n)_{n\in\mathbf{Z}}$, endowed with the product topology. Let f be a continuous self-map on X. Then the closed subspace $X_f = \{(x_n) : f(x_n) = x_{n+1} \text{ for all } n \in \mathbf{Z}\}$ of $X^{\mathbf{Z}}$ together with the associated *shift map* $\sigma\colon X_f \to X_f$ defined by $\sigma((x_n)) = (y_n)$, where $y_n = x_{n+1}$ for all $n \in \mathbf{Z}$, is the *inverse limit space* of f.

By a metric G-space X we mean a metric space X on which a topological group G acts continuously by an action θ. For x in X, g in G we denote $\theta(g, x)$

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by gx. The G-*orbit* of a point $x \in X$, denoted by $G(x)$, is given by $\{gx : g \in G\}$. The set X/G of all G-orbits in X endowed with the quotient topology induced by the quotient map $\pi\colon X \to X/G$ defined by $\pi(x) = G(x)$, is called the *orbit space* of X and the map π is called the *orbit map*. A continuous map $f\colon X \to X$ is said to be an *equivariant map* if $f(gx) = gf(x)$, for all $x \in X$ and all $g \in G$. A sequence $\{x_n : n \geq 0\}$ in a G-space X is said to be a δ-G-*pseudo orbit* for f if for each $n \geq 0$, $d(u, x_{n+1}) < \delta$ for some $u \in G(f(x_n))$. A δ-G-pseudo orbit $\{x_n : n \geq 0\}$ for f is said to be ϵ-*shadowed* by a point x of X, if for each $n \geq 0$, $d(v, f^n(x)) < \epsilon$ for some $v \in G(x_n)$. Map f is said to have the G-*shadowing property* (termed as G-pseudo orbit tracing property in [5]) if for each $\epsilon > 0$ there is a $\delta > 0$ such that every δ-G-pseudo orbit for f is ϵ-shadowed by a point of X.

Note that if f is an equivariant map, then the inverse limit space X_f is a G-space under the diagonal action of G.

We study here the G-shadowing property of the shift map σ on the inverse limit space X_f, generated by an equivariant self-map f on a metric G-space X.

G-shadowing for inverse limit space

Theorem 1. *Let f be a surjective equivariant self-map on a compact metric G-space X, where G is compact. If f has the G-shadowing property, then the shift map σ on the inverse limit space X_f has the G-shadowing property.*

Proof. For a given $\epsilon > 0$ uniform continuity of f implies existence of a $\gamma > 0$ satisfying

$$d(x, y) < \gamma \implies d(f^i(x), f^i(y)) < \frac{\epsilon}{8}, \qquad \text{for all } i, \quad 0 \leq i \leq 2N, \qquad \text{(A)}$$

where N is a positive integer satisfying the inequality $0 < \frac{\log \alpha - N \log 2}{\log \epsilon - 3 \log 2} < 1$ and α is the diameter of X. Since f is G-shadowing, there is a $\tau > 0$ such that every τ-G-pseudo orbit for f is γ-shadowed by a point of X. Choose a $\delta > 0$ such that $0 < \delta 2^N < \tau$. In order to show that σ has the G-shadowing property we show that every finite δ-G-pseudo orbit $\big\{(x_i^n) : 0 \leq n \leq k\big\}$ for σ is ϵ-shadowed by a point of X_f. If $\tilde{d}$ denote the usual metric on X_f, then for $0 \leq n \leq k-1$ there is $\tilde{u} \in G(\sigma(x_i^n))$ such that $\delta > \tilde{d}(\tilde{u}, (x_i^{n+1}))$. Now, for some $g_n' \in G$

$$\tilde{d}(\tilde{u}, (x_i^{n+1})) = \sum_{i=-\infty}^{\infty} \frac{d(g_n' f(x_i^n), x_i^{n+1})}{2^{|i|}}$$

and

$$\sum_{i=-\infty}^{\infty} \frac{d(g'_n f(x_i^n), x_i^{n+1})}{2^{|i|}} \geq \frac{d(g'_n f(x_{-N}^n), x_{-N}^{n+1})}{2^N}$$

implies that for $0 \leq n \leq k-1$,

$$d(g'_n f(x_{-N}^n), x_{-N}^{n+1}) < 2^N \delta < \tau.$$

Thus $\{x_{-N}^n : 0 \leq n \leq k\}$ is a τ-G-pseudo orbit for f. Since f has the G-shadowing property, there is a y in X satisfying, for $0 \leq n \leq k$,

$$d(f^n(y), g_n x_{-N}^n) < \gamma \qquad \text{for some} \quad g_n \in G. \tag{$*$}$$

Put $y_{i-N} = f^i(y)$ for $i \geq 0$ and $y_{i-N} \in f^{-1}(y_{i+1-N})$ for $i < 0$, then $\tilde{y} = (y_i) \in X_f$. Using equivariancy of the map f and (A) it is easy to check that $\{(x_i^n) : 0 \leq n \leq k\}$ is ϵ-shadowed by $\tilde{y}$. Observe that the existence of g_n in G is assured by $(*)$. □

The following example shows that in general the G-shadowing of σ on X_f need not imply the G-shadowing of f on X. We first observe the following note.

Note. Let f be a continuous self-map on I and let I_f be the corresponding inverse limit space generated by f. Suppose the shift map $\sigma\colon I_f \to I_f$ has only two fixed points, say, $\tilde{p}$ and $\tilde{q}$ with $\tilde{p} = (p_i)_{i=-\infty}^{\infty}$ and $\tilde{q} = (q_i)_{i=-\infty}^{\infty}$, such that p_0 and q_0 are the end points of $f(I)$. Then the shift map σ has the shadowing property.

Example. Consider the usual $\mathbf{Z}_2$-space I and self-map f on I with $f(0) = \frac{1}{8}$, $f(\frac{1}{4}) = \frac{1}{4}$, $f(\frac{5}{16}) = \frac{1}{2} = f(\frac{11}{16})$, $f(\frac{3}{8}) = \frac{7}{16}$, $f(\frac{5}{8}) = \frac{9}{16}$, $f(\frac{3}{4}) = \frac{3}{4}$ and $f(1) = \frac{7}{8}$ such that f is linear on each of the subintervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{5}{16}]$, $[\frac{5}{16}, \frac{3}{8}]$, $[\frac{3}{8}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{11}{16}]$, $[\frac{11}{16}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$. Then f is an equivariant map with $\operatorname{Fix} f = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. Observe that f is not shadowing as $f\big|_{[0,\frac{1}{2}]}$ is not shadowing. Since the induced map $\hat{f}\colon I/\mathbf{Z}_2 \to I/\mathbf{Z}_2$ is not shadowing, f is not $\mathbf{Z}_2$-shadowing. Further, a point $\tilde{x} = (x_n) \in I_f$ if and only if $x_n \in [\frac{1}{4}, \frac{3}{4}]$ for all $n \in \mathbf{Z}$. Therefore the induced map $\hat{\sigma}\colon I_f/\mathbf{Z}_2 \to I_f/\mathbf{Z}_2$ has only two fixed points. Hence by above note, $\hat{\sigma}$ has the shadowing property. But this implies that σ has the $\mathbf{Z}_2$-shadowing property. Note that f is not a local homeomorphism on I.

In the following theorem we obtain conditions under which G-shadowing of σ on X_f implies the G-shadowing of f on X.

Theorem 2. *Let f be an equivariant local homeomorphism on a compact metric G-space X, where G is compact. If the shift map σ on X_f has the G-shadowing property, then f has the G-shadowing property.*

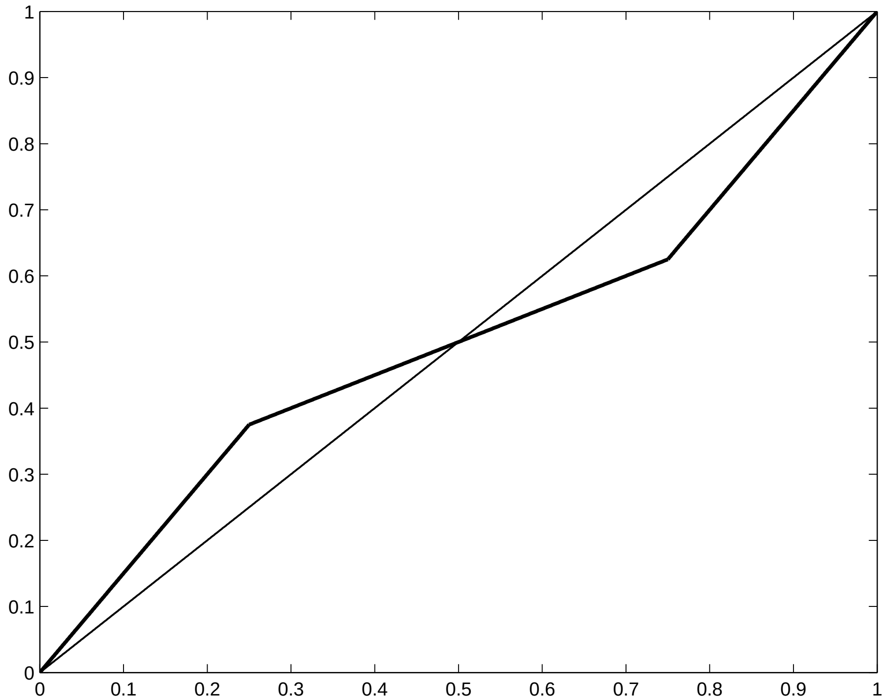
Proof. The G-shadowing property of σ implies that for an $\epsilon > 0$, there is a $\delta > 0$ such that every δ-G-pseudo orbit for σ is ϵ-shadowed by a point of X_f. Choose a positive integer N satisfying the inequality $0 < \frac{\log \alpha - N \log 2}{\log \delta - 3 \log 2} < 1$, where α is the diameter of X. Since f is a local homeomorphism, there exists γ, $0 < \gamma < \frac{\delta}{8}$, such that $f\big|_U : U \to f(U)$ is a homeomorphism, where U is the γ-neighbourhood of x, $x \in X$. Also, f is uniformly continuous therefore there exists an η, $0 < \eta < \gamma$, such that $d(x, y) < \eta$ implies $d(f^j(x), f^j(y)) \leq \frac{\epsilon}{8}$, for all j, $|j| \leq N$. In order to show that f has the G-shadowing property, we show that every η-G-pseudo orbit for f is ϵ-shadowed by a point of X. If $\{x^i : i \geq 0\}$ is an η-G-pseudo orbit for f, then for each $i \geq 0$, there exists a $g_i \in G$ such that $d(g_i f(x^i), x^{i+1}) < \eta$. Construct a δ-G-pseudo orbit $\{(x^i_n) : i \geq 0\}$ for σ in X_f by taking $x^i_0 = x^i$ for all $i \geq 0$. Since σ has the G-shadowing property therefore there exists an (x'_n) in X_f such that for each $i \geq 0$, there exists $k_i \in G$ satisfying $d(f^i(x'_0), k_i x^i_0) = d(f^i(x'_0), k_i x^i) \leq \sum\limits_{n=-\infty}^{\infty} \frac{d(f^i(x'_0), k_i x^i_n)}{2^{|n|}} < \epsilon$. This implies $\{x^i : i \geq 0\}$ is ϵ-shadowed by the point x'_0 of X. Hence f has the G-shadowing property. □

Remark.

(i) From [5, Theorem 3.3] and the fact that the orbit map $\pi : I \to I/\mathbf{Z}_2$ is a covering map we get that for an equivariant self-map f on the $\mathbf{Z}_2$-space I, if the only fixed points of the induced map $\hat{f} : I/\mathbf{Z}_2 \to I/\mathbf{Z}_2$ are the end points of $I/\mathbf{Z}_2$, then f has the $\mathbf{Z}_2$-shadowing property.

(ii) Using the above remark (i), the following map f on I has the $\mathbf{Z}_2$-shadowing property. Therefore from Theorem 1, the shift map on the inverse limit space generated by f has the $\mathbf{Z}_2$-shadowing property.

Using [4, Main Theorem], one can observe that f does not have the shadowing property. Also, the corresponding shift map does not have the shadowing property.



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