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OSCILLATION OF SOLUTIONS OF NEUTRAL PARABOLIC DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

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ABSTRACT. Sufficient conditions are obtained for oscillation of solutions of a class of neutral parabolic differential equations with oscillating coefficients.

1. Introduction

In recent years, several authors (see [1]–[4], [6]–[12]) have studied oscillatory behaviour of solutions of parabolic differential equations. In [6], [7], [8], [12] parabolic equations of neutral type are considered with nonnegative coefficients. In [3], Kusano and Yoshida have studied oscillatory behaviour of solutions of delay parabolic differential equations of the form

$$u_t(x,t) - \left(a(t)\Delta u(x,t) + \sum_{i=1}^{k} b_i(t)\Delta u(x,t-\sigma_i)\right) + c(x,t,u(x,t),u(x,\tau_1(t)),\dots,u(x,\tau_m(t))) = f(x,t)$$

with oscillating coefficients $b_i(t)$.

It seems that no work is done for neutral parabolic differential equations with oscillating coefficients.

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In this paper we consider nonlinear, nonhomogeneous parabolic differential equations of neutral type of the form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[u(x,t)+\sum_{i=1}^{\ell} a_i(t)u(x,t-\tau_i)\right] \\
&-\left[b(t)\Delta u(x,t)+\sum_{j=1}^{m} b_j(t)\Delta u(x,t-\sigma_j)\right] \\
&+c(x,t,u(x,t),u(x,t-\rho_1),\dots,u(x,t-\rho_r))=f(x,t),
\end{aligned}
\tag{1}
$$

$(x,t)\in Q$, where $Q:=\Omega\times(0,\infty)$, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ and Δ is the Laplacian in $\mathbb{R}^n$, along with following boundary conditions

(DBC) $u=\psi$ on $\Gamma\times(0,\infty)$,

(NBC) $\frac{\partial u}{\partial \nu}=\tilde{\psi}$ on $\Gamma\times(0,\infty)$,

where ψ, $\tilde{\psi}$ are real-valued continuous functions on $\Gamma\times(0,\infty)$.

Following assumptions are made for our use in the sequel:

(C_1) Let $\tau_i\ge 0$, $1\le i\le l$, $\sigma_j>0$, $1\le j\le m$ and $\rho_k\ge 0$, $1\le k\le r$, be constants. Let $T_0=\max\{\tau_i,\sigma_j,\rho_k:\ 1\le i\le l,\ 1\le j\le m,\ 1\le k\le r\}$.

(C_2) $f(x,t)$ is a real valued continuous function on $\overline{Q}$ and $a_i,b_j,b\in C([0,\infty),\mathbb{R})$, $1\le i\le l$, $1\le j\le m$ with $b(t)>0$.

(C_3) $c\colon Q\times\mathbb{R}^{r+1}\to\mathbb{R}$ be continuous such that

$$c(x,t,\xi_0,\xi_1,\dots,\xi_r)\ge 0 \qquad \text{for}\quad \xi_k>0,\ \ 0\le k\le r$$

and

$$c(x,t,\xi_0,\xi_1,\dots,\xi_r)\le 0 \qquad \text{for}\quad \xi_k<0,\ \ 0\le k\le r.$$

By a solution of the problem (1), (DBC) (or (NBC)) we mean a real valued continuous function $u(x,t)$ on $Q_{-T_0}\doteq\Omega\times(-T_0,\infty)$ such that

$$\frac{\partial}{\partial t}\left[u(x,t)+\sum_{i=1}^{l} a_i(t)u(x,t-\tau_i)\right]$$

exists, (1) is satisfied identically in Q and (DBC) (or(NBC)) holds.

A solution $u(x,t)$ of the problem (1), (DBC) (or(NBC)) is said to be oscillatory if $u(x,t)$ has a zero in $Q_{t_0}=\Omega\times(t_0,\infty)$ for every $t_0\ge 0$.

It is well-known that the first eigenvalue λ_1 of the eigenvalue problem

$$
\begin{aligned}
-\Delta w &= \lambda w \qquad \text{in}\quad \Omega\\
w &= 0 \qquad\ \ \text{on}\quad \Gamma
\end{aligned}
$$

is positive and the corresponding eigenfunction $\phi(x)$ is of one sign in Ω. We assume that $\phi(x) > 0$ in Ω.

For a solution u of the problem (1), (DBC), we denote

$$U(t) = \int_{\Omega} u(x,t)\phi(x)\,\mathrm{d}x, \qquad t > 0$$

$$\Psi(t) = \int_{\Gamma} \psi(x,t)\frac{\partial\phi(x)}{\partial\nu}\,\mathrm{d}s, \qquad t > 0$$

$$F(t) = \int_{\Omega} f(x,t)\phi(x)\,\mathrm{d}x, \qquad t > 0$$

and for a solution u of the problem (1), (NBC), we denote

$$\tilde{U}(t) = \int_{\Omega} u(x,t)\,\mathrm{d}x, \qquad t > 0$$

$$\tilde{\Psi}(t) = \int_{\Gamma} \tilde{\psi}(x,t)\,\mathrm{d}s, \qquad t > 0$$

$$\tilde{F}(t) = \int_{\Omega} f(x,t)\,\mathrm{d}x, \qquad t > 0.$$

In Section 2, we consider a first-order neutral differential inequality of the form

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t)y(t-\tau_i)\right]' + \sum_{j=1}^{m} b_j(t)y(t-\sigma_j) \le g(t), \qquad t \ge t_0 > 0, \tag{2}$$

where $b_j(t)$ is allowed to change sign. We assume that

(C_4) $a_i, g \in C([t_0,\infty),\mathbb{R})$, $1 \le i \le l$,

(C_5) $\tau_i \ge 0$, $\sigma_j > 0$, $1 \le i \le l$, $1 \le j \le m$

(C_6) $b_j \in C([t_0,\infty),\mathbb{R})$, $j = 1,\dots,m$ and $b_j(t) \ge 0$ on $U_{n=1}^{\infty} I_{n,j}$,

where $I_{n,j} = (t_n - 2\sigma_j, t_n)$ and the sequence $\{t_n\}_{n=1}^{\infty}$ is chosen so that $\{I_{n,j}\}_{n=1}^{\infty}$ are disjoint intervals for each $j = 1,\dots,m$ and $t_n \to \infty$ as $n \to \infty$.

In Section 3, we study the oscillation results of the problem (1), (DBC) and (1), (NBC).

2. Oscillation results for the neutral differential inequality

Lemma 1. *Let* (C_4)–(C_6) *hold. Further, let*

(C_7) $-a_i \le a_i(t) \le 0$, *where* a_i *is a positive constant,* $1 \le i \le l$.

Let us assume that there is a subsequence $\{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ *with the properties that*

(C$_8$) $\lim\limits_{k\to\infty} n_k = \infty$ *and* $1 \le \int\limits_{t_{n_k}-\sigma_{j*}}^{t_{n_k}} b_{j*}(t)\,\mathrm{d}t \le c$, *where* $\sigma_{j^*} = \min\limits_{1\le j\le m}\{\sigma_j\}$ *and* c *is a positive constant,*

(C$_9$) $\lim\limits_{k\to\infty} n_k = \infty$ *and* $\varliminf\limits_{k\to\infty} G(t_{n_k}) = -\infty$ *where*

$$G(t) = \int\limits_{t-\sigma_{j*}}^{t} g(s)\,\mathrm{d}s + \int\limits_{t-\sigma_{j*}}^{t} b_{j*}(s)\left(\int\limits_{s-\sigma_{j*}}^{t-\sigma_{j*}} g(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s.$$

Then (2) *has no eventually positive bounded solution.*

P r o o f. If possible, let $y(t)$ be an eventually positive bounded solution of (2) on $[t_1,\infty)$ for some $t_1 \ge t_0 > 0$. Then $y(t-\tau_i) > 0$, $y(t-\sigma_j) > 0$, $1 \le i \le l$, $1 \le j \le m$ on $[t_2,\infty)$ for some $t_2 > t_1$. We may note that $\lim\limits_{n\to\infty}(t_n - 2\sigma_j) = \infty$ for every j and hence there is an integer $N > 0$ such that $t_n - 2\sigma_j > t_2$ for $n \ge N$ and for every j. Letting $\xi_n = t_n - 2\sigma_{j*}$, we find that $(\xi_n, t_n) \subset (t_n - 2\sigma_j, t_n)$, $j = 1,\dots m$. So $b_j(t) \ge 0$ in (ξ_n, t_n) and $y(t-\tau_i) > 0$, $y(t-\sigma_j) > 0$, $1 \le i \le l$, $1 \le j \le m$, for $t \in (\xi_n, t_n)$ and $n \ge N$. So it follows from (2) that

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t) y(t-\tau_i)\right]' \le g(t)$$

in (ξ_n, t_n). By continuity

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t) y(t-\tau_i)\right]' \le g(t)$$

in $[\xi_n, t_n]$. For any $t \in [t_n - \sigma_{j*}, t_n]$, $[t - \sigma_{j*}, t_n - \sigma_{j*}] \subset [\xi_n, t_n]$ and hence integrating the above inequality we obtain

$$\begin{aligned} &y(t_n - \sigma_{j*}) + \sum_{i=1}^{\ell} a_i(t_n - \sigma_{j*}) y(t_n - \sigma_{j*} - \tau_i) \\ &\quad - y(t - \sigma_{j*}) - \sum_{i=1}^{\ell} a_i(t - \sigma_{j*}) y(t - \sigma_{j*} - \tau_i) \le \int\limits_{t-\sigma_{j*}}^{t_n-\sigma_{j*}} g(s)\,\mathrm{d}s, \end{aligned} \tag{3}$$

that is

$$y(t - \sigma_{j*}) \ge y(t_n - \sigma_{j*}) + \sum_{i=1}^{\ell} a_i(t_n - \sigma_{j*}) y(t_n - \sigma_{j*} - \tau_i) - \int\limits_{t-\sigma_{j*}}^{t_n-\sigma_{j*}} g(s)\,\mathrm{d}s,$$

for $t \in [t_n - \sigma_{j*}, t_n]$. From (2) it follows that

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t) y(t-\tau_i)\right]' + b_{j^*}(t) y(t-\sigma_{j*}) \le g(t),$$

for $t \in [t_n - \sigma_{j*}, t_n]$. Hence

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t) y(t-\tau_i)\right]' + b_{j*}(t) y(t_n - \sigma_{j*})$$
$$+ b_{j*}(t) \sum_{i=1}^{\ell} a_i(t_n - \sigma_{j*}) y(t_n - \sigma_{j*} - \tau_i) \le g(t) + b_{j*}(t) \int\limits_{t-\sigma_{j*}}^{t_n - \sigma_{j*}} g(s)\,\mathrm{d}s.$$

Integrating the above inequality from $t_n - \sigma_{j*}$ to t_n, we get

$$\begin{aligned} & y(t_n) + \sum_{i=1}^{\ell} a_i(t_n) y(t_n - \tau_i) - \sum_{i=1}^{\ell} a_i(t_n - \sigma_{j*}) y(t_n - \sigma_{j*} - \tau_i) \\ & + y(t_n - \sigma_{j*}) \left(\int\limits_{t_n - \sigma_{j*}}^{t_n} b_{j*}(t)\,\mathrm{d}t - 1 \right) \\ & + \left(\sum_{i=1}^{\ell} a_i(t_n - \sigma_{j*}) y(t_n - \sigma_{j*} - \tau_i) \right) \int\limits_{t_n - \sigma_{j*}}^{t_n} b_{j*}(t)\,\mathrm{d}t \\ \le & \int\limits_{t_n - \sigma_{j*}}^{t_n} \left[g(t) + b_{j*}(t) \int\limits_{t-\sigma_{j*}}^{t_n - \sigma_{j*}} g(s)\,\mathrm{d}s \right] \mathrm{d}t. \end{aligned}$$

In particular,

$$\begin{aligned} & y(t_{n_k}) + \sum_{i=1}^{\ell} a_i(t_{n_k}) y(t_{n_k} - \tau_i) \\ & \quad + \left(\sum_{i=1}^{\ell} a_i(t_{n_k} - \sigma_{j*}) y(t_{n_k} - \sigma_{j*} - \tau_i) \right) \int\limits_{t_{n_k} - \sigma_{j*}}^{t_{n_k}} b_{j*}(t)\,\mathrm{d}t \le G(t_{n_k}), \end{aligned}$$

that is

$$\begin{aligned}
y(t_{n_k}) &\le \sum_{i=1}^{\ell} a_i \left[y(t_{n_k} - \tau_i) + y(t_{n_k} - \sigma_{j*} - \tau_i) \int\limits_{t_{n_k}-\sigma_{j*}}^{t_{n_k}} b_{j*}(t)\,\mathrm{d}t \right] + G(t_{n_k}) \\
&\le L \left(\sum_{i=1}^{\ell} a_i \right) \left(1 + \int\limits_{t_{n_k}-\sigma_{j*}}^{t_{n_k}} b_{j*}(s)\,\mathrm{d}s \right) + G(t_{n_k}) \\
&\le L \left(\sum_{i=1}^{\ell} a_i \right) (1+c) G(t_{n_k}),
\end{aligned}$$

where L is the bound of $y(t)$. Taking the limit infimum on both sides we get the contradiction $0 \le \varliminf\limits_{k\to\infty} y(t_{n_k}) < 0$ due to (C_9). Thus the proof is complete. □

Lemma 2. *Suppose that all the conditions of* Lemma 1 *are satisfied except* (C_7) *which is replaced by*

(C_{10}) $0 \le a_i(t) \le a_i$, *where a_i is a positive constant,* $1 \le i \le l$.

Then (2) *has no eventually positive bounded solution.*

Proof. Suppose that $y(t)$ is an eventually positive bounded solution of (2) on $[t_1,\infty)$ for some $t_1 \ge t_0 > 0$. Then $y(t-\tau_i) > 0$, $y(t-\sigma_j) > 0$, $1 \le i \le l$, $1 \le j \le m$ on $[t_2,\infty)$ for some $t_2 > t_1$. Proceeding as in Lemma 1 we get (3), for $t \in [t_n - \sigma_{j^*}, t_n]$, and hence $y(t-\sigma_{j^*}) \ge y(t_n - \sigma_{j*}) - \sum\limits_{i=1}^{\ell} a_i(t-\sigma_{j^*}) y(t - \sigma_{j*} - \tau_i) - \int\limits_{t-\sigma_{j*}}^{t_n-\sigma_{j*}} g(s)\,\mathrm{d}s$.

From (2) it follows that

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t) y(t-\tau_i) \right]' + b_{j*}(t) y(t-\sigma_{j*}) \le g(t)$$

for $t \in [t_n - \sigma_{j*}, t_n]$ and hence

$$\begin{aligned}
&\left[y(t) + \sum_{i=1}^{\ell} a_i(t) y(t-\tau_i) \right]' + b_{j*}(t) y(t_n - \sigma_{j*}) \\
&- b_{j*}(t) \sum_{i=1}^{\ell} a_i(t-\sigma_{j*}) y(t-\sigma_{j*}-\tau_i) \le g(t) + b_{j*}(t) \int\limits_{t-\sigma_{j*}}^{t_n-\sigma_j *} g(s)\,\mathrm{d}s.
\end{aligned}$$

Integrating the above inequality from $t_n - \sigma_{j*}$ to t_n,

$$
\begin{aligned}
y(t_n) - \sum_{i=1}^{\ell} a_i(t_n - \sigma_{j*}) y(t_n - \sigma_{j*} - \tau_i) + y(t_n - \sigma_{j*}) \left[\int_{t_n - \sigma_{j*}}^{t_n} b_{j^*}(t)\,\mathrm{d}t - 1 \right] \\
- \int_{t_n - \sigma_{j*}}^{t_n} b_{j^*}(t) \sum_{i=1}^{\ell} a_i(t - \sigma_{j*}) y(t - \sigma_{j*} - \tau_i)\,\mathrm{d}t \\
\le \int_{t_n - \sigma_{j*}}^{t_n} \left[g(t) + b_{j^*}(t) \int_{t - \sigma_{j*}}^{t_n - \sigma_{j*}} g(s)\,\mathrm{d}s \right] \mathrm{d}t.
\end{aligned}
$$

Thus, in particular,

$$
\begin{aligned}
y(t_{n_k}) - \sum_{i=1}^{\ell} a_i(t_{n_k} - \sigma_{j*}) y(t_{n_k} - \sigma_{j*} - \tau_i) \\
- \int_{t_{n_k} - \sigma_{j*}}^{t_n} b_{j*}(t) \sum_{i=1}^{\ell} a_i(t - \sigma_{j*}) y(t - \sigma_{j*} - \tau_i)\,\mathrm{d}t \le G(t_{n_k}),
\end{aligned}
$$

in view of the condition (C_8), that is,

$$
y(t_{n_k}) - \sum_{i=1}^{\ell} a_i y(t_{n_k} - \sigma_{j*} - \tau_i) - \sum_{i=1}^{\ell} a_i \int_{t_{n_k} - \sigma_{j*}}^{t_{n_k}} b_{j^*}(s) y(s - \sigma_{j*} - \tau_i)\,\mathrm{d}s \le G(t_{n_k}),
$$

that is,

$$
\begin{aligned}
y(t_{n_k}) &\le G(t_{n_k}) + \left(\sum_{i=1}^{\ell} a_i \right) L \left(1 + \int_{t_{n_k} - \sigma_{j*}}^{t_{n_k}} b_{j*}(s)\,\mathrm{d}s \right) \\
&\le G(t_{n_k}) + L \left(\sum_{i=1}^{\ell} a_i \right) (1 + c),
\end{aligned}
$$

where L is the bound of $y(t)$. Taking the limit infimum we get, $0 \le \varliminf_{k \to \infty} y(t_{n_k}) < 0$, a contradiction. Hence the Lemma is proved. $\square$

3. Oscillation results

Theorem 1. *Let* (C_1)–(C_3), (C_6) *and* (C_7) *hold. Then every bounded solution of* (1), (DBC) *oscillates provided that there is a subsequence* $\{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ *with the properties that*

(C_{11}) (i) $\lim\limits_{k\to\infty} n_k = \infty$,

(ii) $1 \le \lambda_1 \int\limits_{t_{n_k}-\sigma_{j*}}^{t_{n_k}} b_{j*}(s)\,\mathrm{d}s \le c$,

where $\sigma_{j*} = \min\limits_{1\le j\le m}\{\sigma_j\}$ *and* c *is a constant,*

(iii) $\varliminf\limits_{k\to\infty} G(t_{n_k}) = -\infty$ *and* $\varlimsup\limits_{k\to\infty} G(t_{n_k}) = \infty$,

where $G(t) = \int\limits_{t-\sigma_{j*}}^{t} g(s)\,\mathrm{d}s + \int\limits_{t-\sigma_{j*}}^{t} b_{j*}(s)\left(\int\limits_{s-\sigma_{j*}}^{t-\sigma_{j*}} g(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s$

and

$$g(t) = F(t) - b(t)\Psi(t) - \sum_{j=1}^{m} b_j(t)\Psi(t-\sigma_j). \tag{4}$$

P r o o f. If possible, let $u(x,t)$ be a bounded nonoscillatory solution of (1), (DBC). Then there exists $at_0 \ge 0$ such that $u(x,t) \ne 0$ in Q_{t_0}. Let $u(x,t) > 0$ in Q_{t_0}. Then multiplying (1) through by $\phi(x)$ and integrating the resulting identity with respect to x over the domain Ω, we get

$$\left[U(t) + \sum_{i=1}^{\ell} a_i(t)U(t-\tau_i)\right]' - \left[b(t)\int_{\Omega} \Delta u(x,t)\phi(x)\,\mathrm{d}x \right.$$
$$\left. + \sum_{j=1}^{m} b_j(t)\int_{\Omega} \Delta u(x,t-\sigma_j)\phi(x)\,\mathrm{d}x\right] \le F(t)$$

for $t \ge t_1 > t_0$. By Green's formula,

$$\begin{aligned}
&\int_{\Omega} \Delta u(x,t)\phi(x)\,\mathrm{d}x \\
&= \int_{\Gamma} \frac{\partial u(x,t)}{\partial \nu}\phi(x)\,\mathrm{d}s - \int_{\Gamma} \frac{\partial \phi(x)}{\partial \nu}u(x,t)\,\mathrm{d}s + \int_{\Omega} u(x,t)\Delta\phi(x)\,\mathrm{d}x \\
&= -\int_{\Gamma} \psi(x,t)\frac{\partial \phi(x)}{\partial \nu}\,\mathrm{d}s - \lambda_1 \int_{\Omega} u(x,t)\phi(x)\,\mathrm{d}x = -\Psi(t) - \lambda_1 U(t).
\end{aligned}$$

Thus we have

$$\left[U(t)+\sum_{i=1}^{\ell} a_i(t)U(t-\tau_i)\right]' + \lambda_1\left[b(t)U(t)+\sum_{j=1}^{m} b_j(t)U(t-\sigma_j)\right]$$
$$\leq F(t)-b(t)\Psi(t)-\sum_{j=1}^{m} b_j(t)\Psi(t-\sigma_j),$$

that is,

$$\left[U(t)+\sum_{i=1}^{\ell} a_i(t)U(t-\tau_i)\right]' + \lambda_1\sum_{j=1}^{m} b_j(t)U(t-\sigma_j)$$
$$\leq F(t)-b(t)\Psi(t)-\sum_{j=1}^{m} b_j(t)\Psi(t-\sigma_j),$$

that is, $U(t)$ is an eventually positive bounded solution of

$$\left[y(t)+\sum_{i=1}^{\ell} a_i(t)y(t-\tau_i)\right]' + \lambda_1\sum_{j=1}^{m} b_j(t)y(t-\sigma_j) \leq g(t),$$

a contradiction to Lemma 1. If $u(x,t)<0$ in Q_{t_0}, then setting $v(x,t)=-u(x,t)$, we get, $v(x,t)>0$ in Q_{t_0} and

$$\frac{\partial}{\partial t}\left[v(x,t)+\sum_{i=1}^{\ell} a_i(t)v(x,t-\tau_i)\right] - \left[b(t)\Delta v(x,t)+\sum_{j=1}^{m} b_j(t)\Delta v(x,t-\sigma_j)\right]$$
$$-c(x,t,-v(x,t),-v(x,t-\rho_1),\ldots,-v(x,t-\rho_r)) = -f(x,t).$$

Proceeding as above we get the required contradiction. Hence the theorem is proved. □

Example 1. Consider the problem

$$\frac{\partial}{\partial t}[u(x,t)-u(x,t-2\pi)] - \left[u_{xx}(x,t)-2\sin 2t u_{xx}\left(x,t-\tfrac{\pi}{4}\right)\right]$$
$$+u(x,t-\pi)+tu(x,t-2\pi) = t\cos t\sin x - 2\sin 2t\cos(t-\tfrac{\pi}{4})\sin x, \quad (5)$$

$(x,t)\in(0,\pi)\times(0,\infty)$ with boundary conditions

$$u(0,t)=0=u(\pi,t). \quad (6)$$

As $a_1(t) = -1$, $b(t) = 1$, $b_1(t) = -2\sin 2t$, $\sigma_1 = \frac{\pi}{4}$, $\phi(x) = \sin x$ and $\lambda_1 = 1$, then

$$\begin{aligned} F(t) &= \int_0^{\pi} \left[t\cos t \sin x - 2\sin 2t \cos(t - \tfrac{\pi}{4}) \sin x\right] \sin x \,\mathrm{d}x \\ &= \frac{\pi}{2}\left(t\cos t - 2\sin 2t\cos(t - \tfrac{\pi}{4})\right) \end{aligned}$$

and hence $g(t) = F(t) - 0 = \frac{\pi}{2} t\cos t - \pi \sin 2t \cos(t - \frac{\pi}{4})$

We notice that $b_j(t) = b_1(t) = -2\sin 2t$ changes sign and > 0 for $t \in \left(t_n - \frac{\pi}{2}, t_n\right) = (n\pi - \pi/2, n\pi)$ and

$$\int_{t_n-\sigma_{j*}}^{t_n} b_{j*}(t)\,\mathrm{d}t = \int_{n\pi-\frac{\pi}{4}}^{n\pi} (-2\sin 2t)\,\mathrm{d}t = \cos 2t\Big|_{n\pi-\frac{\pi}{4}}^{n\pi} = 1, \qquad n = 1, 2, \ldots.$$

Here

$$I_{n,1} = \left(t_n - \frac{\pi}{2}, t_n\right) = \left(n\pi - \frac{\pi}{4}, n\pi\right).$$

Moreover,

$$\begin{aligned} G(t_n) &= \int_{t_n-\sigma_{j*}}^{t_n} g(s)\,\mathrm{d}s + \int_{t_n-\sigma_{j*}}^{t_n} b_{j*}(s)\left(\int_{s-\sigma_{j*}}^{t_n-\sigma_*} g(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\ &= \frac{\pi}{2}\left[\int_{n\pi-\frac{\pi}{4}}^{n\pi} s\cos s\,\mathrm{d}s + \int_{n\pi-\frac{\pi}{4}}^{n\pi} (-2\sin 2s)\left(\int_{s-\frac{\pi}{4}}^{n\pi-\frac{\pi}{4}} \theta\cos\theta\,\mathrm{d}\theta\right)\mathrm{d}s\right] \\ &\quad - \pi\left[\int_{n\pi-\frac{\pi}{4}}^{n\pi} \sin 2s\cos\left(s - \tfrac{\pi}{4}\right)\,\mathrm{d}s\right. \\ &\quad \left. + \int_{n\pi-\frac{\pi}{4}}^{n\pi} (-2\sin 2s)\left(\int_{s-\frac{\pi}{4}}^{n\pi-\frac{\pi}{4}} (\sin 2\theta)\cos\left(\theta - \tfrac{\pi}{4}\right)\mathrm{d}\theta\right)\mathrm{d}s\right] \\ &= \frac{\pi}{2}\left[\cos n\pi + 2\int_{n\pi-\frac{\pi}{4}}^{n\pi} s\sin 2s\sin\left(s - \tfrac{\pi}{4}\right)\mathrm{d}s - \frac{\pi}{2}\int_{n\pi-\frac{\pi}{4}}^{n\pi} \sin 2s\sin\left(s - \tfrac{\pi}{4}\right)\mathrm{d}s\right. \end{aligned}$$

$$+2\int_{n\pi-\frac{\pi}{4}}^{n\pi} \sin 2s\cos\bigl(s-\tfrac{\pi}{4}\bigr)\,\mathrm{d}s\Biggr] - \pi\Biggl[\int_{n\pi-\frac{\pi}{4}}^{n\pi} \sin 2s\cos\bigl(s-\tfrac{\pi}{4}\bigr)\,\mathrm{d}s$$
$$+\frac{\sqrt{2}}{3}\left(\sin^3\bigl(n\pi-\tfrac{\pi}{4}\bigr)-\cos^3\bigl(n\pi-\tfrac{\pi}{4}\bigr)\right)$$
$$-\frac{2\sqrt{2}}{3}\int_{n\pi-\frac{\pi}{4}}^{n\pi} (\sin 2s)\bigl(\cos^3\bigl(s-\tfrac{\pi}{4}\bigr)-\sin^3\bigl(s-\tfrac{\pi}{4}\bigr)\bigr)\,\mathrm{d}s\Biggr].$$

In the above identity all the terms are bounded except

$$\int_{n\pi-\frac{\pi}{4}}^{n\pi} s\sin 2s\sin\bigl(s-\tfrac{\pi}{4}\bigr)\,\mathrm{d}s = \frac{\sqrt{2}}{3}\left[n\pi\cos^3 n\pi - \int_{n\pi-\frac{\pi}{4}}^{n\pi}\sin^3 s\,\mathrm{d}s - \int_{n\pi-\frac{\pi}{4}}^{n\pi}\cos^3 s\,\mathrm{d}s\right].$$

Then $\varliminf\limits_{n\to\infty} G(t_n) = -\infty$ and $\varlimsup\limits_{n\to\infty} G(t_n) = \infty$. So by Theorem 1 all the bounded solutions of (5), (6) oscillate in $(0,\pi)\times(0,\infty)$. In particular, $u(x,t)=\sin x\cos t$ is a bounded oscillatory solution of the problem.

Theorem 2. *Let* (C_1)–(C_3), (C_6), (C_{10}) *and* (C_{11}) *hold. Then every bounded solution of the problem* (1), (DBC) *oscillates.*

The proof is similar to that of Theorem 1 and hence is omitted. In this case Lemma 2 is used.

Example 2. Consider the problem

$$\begin{aligned}&\frac{\partial}{\partial t}[u(x,t)+2u(x,t-\pi)] - \left[u_{xx}(x,t) - 2\sin 2t u_{xx}\left(x,t-\tfrac{\pi}{4}\right)\right]\\ &+(1+t)u(x,t-\pi)+u\bigl(x,t-\tfrac{3\pi}{2}\bigr) = -t\sin t\sin x - 2\sin 2t\sin\bigl(t-\tfrac{\pi}{4}\bigr)\sin x,\end{aligned} \tag{7}$$

$(x,t)\in(0,\pi)\times(0,\infty)$ with boundary conditions

$$u(0,t)=0=u(\pi,t). \tag{8}$$

In this case, $\phi(x)=\sin x$, $\lambda_1=1$, $g(t) = -(\frac{\pi}{2})t\sin t - \pi\sin 2t\sin(t-\frac{\pi}{4})$, $b_{j*}(t) = -2\sin 2t$ and $\sigma_{j*}=\frac{\pi}{4}$. Thus, $\int\limits_{t_n-\sigma_{j*}}^{t_n} b_{j*}(s)\,\mathrm{d}s = \int\limits_{t_n-\frac{\pi}{4}}^{t_n}(-2\sin 2s)\,\mathrm{d}s = 1$, where

$t_n = n\pi$, $n = 1, 2, \dots$, and $I_{n,j^*} = (t_n - \pi/2, t_n)$ and

$$
\begin{aligned}
G(t_n) &= \int_{t_n-\sigma_{j*}}^{t_n} g(s)\,\mathrm{d}s + \int_{t_n-\sigma_{j*}}^{t_n} b_{j*}(s) \left(\int_{s-\sigma_{j*}}^{t_n-\sigma_{j*}} g(\theta)\,\mathrm{d}\theta \right) \mathrm{d}s \\
&= \frac{\pi}{2} \left[\int_{t_n-\frac{\pi}{4}}^{t_n} (-s\sin s)\,\mathrm{d}s + \int_{t_n-\frac{\pi}{4}}^{t_n} (-2\sin 2s) \left(\int_{s-\frac{\pi}{4}}^{t_n-\frac{\pi}{4}} (-\theta\sin\theta)\,\mathrm{d}\theta \right) \mathrm{d}s \right] \\
&\quad - \pi \left[\int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s \sin\left(s - \tfrac{\pi}{4}\right) \mathrm{d}s \right. \\
&\quad \left. + \int_{t_n-\frac{\pi}{4}}^{t_n} (-2\sin 2s) \left(\int_{s-\frac{\pi}{4}}^{t_n-\frac{\pi}{4}} (\sin 2\theta)\sin\left(\theta - \tfrac{\pi}{4}\right) \mathrm{d}\theta \right) \mathrm{d}s \right] \\
&= \frac{\pi}{2} \left[t_n \cos t_n + 2 \int_{t_n-\frac{\pi}{4}}^{t_n} \left(s - \frac{\pi}{4}\right) (\sin 2s) \cos\left(s - \tfrac{\pi}{4}\right) \mathrm{d}s \right. \\
&\quad \left. - 2 \int_{t_n-\frac{\pi}{4}}^{t_n} (\sin 2s) \sin\left(s - \tfrac{\pi}{4}\right) \mathrm{d}s \right] - \pi \left[\int_{t_n-\frac{\pi}{4}}^{t_n} (\sin 2s) \sin\left(s - \tfrac{\pi}{4}\right) \mathrm{d}s \right. \\
&\quad + \frac{2}{3} \left(\sin^3\left(t_n - \tfrac{\pi}{4}\right) + \cos^3\left(t_n - \tfrac{\pi}{4}\right) \right) \\
&\quad \left. + \frac{4}{3} \int_{t_n-\frac{\pi}{4}}^{t_n} (\sin 2s) \left(\sin^3\left(s - \tfrac{\pi}{4}\right) + \cos^3\left(s - \tfrac{\pi}{4}\right) \right) \mathrm{d}s \right] \\
&= \frac{\pi}{2} \left[t_n \cos t_n - \frac{2\sqrt{2}}{3} t_n \cos t_n + \frac{2}{3} t_n \cos^3 t_n - \frac{\pi}{6} \cos^3 t_n \right. \\
&\qquad - \frac{1}{3\sqrt{2}} \sin^3\left(t_n - \tfrac{\pi}{4}\right) - \frac{3}{\sqrt{2}} \sin\left(t_n - \tfrac{\pi}{4}\right) + \frac{3}{\sqrt{2}} \cos t_n \\
&\qquad - \frac{3}{\sqrt{2}} \cos\left(t_n - \tfrac{\pi}{4}\right) - \frac{1}{3\sqrt{2}} \cos 3t_n + \frac{1}{3\sqrt{2}} \cos^3\left(t_n - \tfrac{\pi}{4}\right) \\
&\qquad + \left(\frac{\pi}{\sqrt{2}} - 2\sqrt{2} \right) \frac{\cos^3 t_n}{3} - \left(\frac{\pi}{\sqrt{2}} - 2\sqrt{2} \right) \frac{\cos^3(t_n - \frac{\pi}{4})}{3}
\end{aligned}
$$

$$+\left(\frac{\pi}{\sqrt{2}}+2\sqrt{2}\right)\frac{\sin^3(t_n-\frac{\pi}{4})}{3}\Bigg]$$

$$-\pi\Bigg[\int\limits_{t_n-\frac{\pi}{4}}^{t_n}(\sin 2s)\sin\big(s-\tfrac{\pi}{4}\big)\,\mathrm{d}s+\frac{2}{3}\left(\sin^3\big(t_n-\tfrac{\pi}{4}\big)+\cos^3(t_n-\tfrac{\pi}{4})\right)$$

$$+\frac{4}{3}\int\limits_{t_n-\frac{\pi}{4}}^{t_n}(\sin 2s)\left(\sin^3\left(s-\tfrac{\pi}{4}\right)+\cos^3\left(s-\tfrac{\pi}{4}\right)\right)\,\mathrm{d}s\Bigg].$$

Hence

$$\varliminf_{n\to\infty}G(t_n)=-\infty\qquad\text{and}\qquad\varlimsup_{n\to\infty}G(t_n)=\infty.$$

Thus all bounded solutions of the problem (7), (8) oscillate in $(0,\pi)\times(0,\infty)$. In particular, $u(x,t)=\sin x\sin t$ is a bounded oscillatory solution of the problem.

Theorem 3. *Let* (C_1), (C_2), (C_7) *be satisfied. Let*

(C_{12}) $c\colon Q\times\mathbb{R}^{r+1}\to\mathbb{R}$ *be continuous such that*

$$c(x,t,\xi_0,\xi_1,\dots,\xi_r)\ \begin{aligned}&\geq p(t)\xi_{k^*}, && \xi_k\geq 0,\\ &\leq p(t)\xi_{k^*}, && \xi_k\leq 0\end{aligned}$$

for some $k^*\in\{1,\dots,r\}$, *where* $0\leq k\leq r$, $p(t)\geq 0$ *in* $U_{n=1}^{\infty}I_n$, *where* $I_n=(t_n-2\rho_{k*},t_n)$ *and* $\{t_n\}$ *is a sequence such that* I_n*'s are disjoint intervals, and* $t_n\to\infty$ *as* $n\to\infty$.

Then every bounded solution of the problem (1), (NBC) *oscillates provided that there exists a subsequence*

$$\{t_{n_\alpha}\}_{\alpha=1}^{\infty}\subset\{t_n\}_{n=1}^{\infty}$$

such that

(C_{13}) (i) $\lim\limits_{\alpha\to\infty}n_\alpha=\infty$

(ii) $1\leq\int\limits_{t_{n_\alpha}-\rho_{k*}}^{t_{n_\alpha}}p(t)\,\mathrm{d}t\leq c$

(iii) $\varliminf\limits_{\alpha\to\infty}\tilde{G}(t_{n_\alpha})=-\infty$ *and* $\varlimsup\limits_{\alpha\to\infty}\tilde{G}(t_{n_\alpha})=\infty$

where c *is a constant,*

$$\tilde{G}(t)=\int\limits_{t-\rho_{k*}}^{t}\tilde{g}(s)\,\mathrm{d}s+\int\limits_{t-\rho_{k*}}^{t}p(s)\left(\int\limits_{s-\rho_{k*}}^{t-\rho_{k*}}\tilde{g}(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s$$

and

$$\tilde{g}(t)=\tilde{F}(t)+b(t)\tilde{\Psi}(t)+\sum_{j=1}^{m}b_j(t)\tilde{\Psi}(t-\sigma_j)\tag{9}$$

Proof. If possible, let $u(x,t)$ be a bounded nonoscillatory solution of (1), (NBC). Hence $u(x,t) \neq 0$ in Q_{t_0} for some $t_0 \geq 0$. Let $u(x,t) > 0$ in Q_{t_0}.

Integrating (1) with respect to x and using Green's formula and (C_{12}), we get,

$$\left[\tilde{U}(t) + \sum_{i=1}^{\ell} a_i(t)\tilde{U}(t-\tau_i)\right]' + p(t)\tilde{U}(t-\rho_{k*}) \leq \tilde{g}(t)$$

for $t > t_0 + T_0$, that is, $\tilde{U}(t)$ is an eventually positive bounded solution of

$$\left[y(t) + \sum_{i=1}^{\ell} a_i(t)y(t-\tau_i)\right]' + p(t)y(t-\rho_{k*}) \leq \tilde{g}(t),$$

a contradiction, due to Lemma 1. If $u(x,t) < 0$, then putting $v(x,t) = -u(x,t)$ and proceeding as above we get the required contradiction. Hence the proof of the theorem is complete. $\square$

Example 3. Consider the problem

$$\begin{aligned} &\frac{\partial}{\partial t}[u(x,t) - u(x,t-2\pi)] - [u_{xx}(x,t) + u_{xx}(x,t-\pi)] \\ &\quad + tu(x,t-2\pi) - 2\sin 2t u\big(x, t-\tfrac{\pi}{4}\big) \\ &= -2\sin 2t \sin x \cos\big(t-\tfrac{\pi}{4}\big) + t\sin x\cos t, \end{aligned} \tag{10}$$

$(x,t) \in (0,\pi) \times (0,\infty)$ with boundary conditions

$$-u_x(0,t) = -\cos t = u_x(\pi,t). \tag{11}$$

Thus,

$$\begin{aligned} &\tilde{\Psi}(t) = \tilde{\psi}(\pi,t) - \tilde{\psi}(0,t) = -2\cos t \quad \text{and} \quad \tilde{\Psi}(t-\pi) = 2\cos t, \\ &\tilde{g}(t) = -4\sin 2t\cos\big(t-\tfrac{\pi}{4}\big) + 2t\cos t, \\ &p(t) = -2\sin 2t, \quad \rho_{k*} = \frac{\pi}{4}, \end{aligned}$$

$$\int_{t_n-\rho_{k*}}^{t_n} p(s)\,\mathrm{d}s = \int_{t_n-\frac{\pi}{4}}^{t_n} -2\sin 2s\,\mathrm{d}s = 1,$$

where $t_n = n\pi$, $n = 1, 2, \ldots$, $I_n = (t_n - \frac{\pi}{2}, t_n)$. Furthermore,

$$
\begin{aligned}
\tilde{G}(t_n) &= \int\limits_{t_n-\rho_{k*}}^{t_n} \tilde{g}(s)\,\mathrm{d}s + \int\limits_{t_n-\rho_{k*}}^{t_n} p(s)\Bigg(\int\limits_{s-\rho_{k*}}^{t_n-\rho_{k*}} \tilde{g}(\theta)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&= 2\Bigg[\int\limits_{t_n-\frac{\pi}{4}}^{t_n} s\cos s\,\mathrm{d}s + \int\limits_{t_n-\frac{\pi}{4}}^{t_n}\Bigg(-2\sin 2s\Bigg(\int\limits_{s-\frac{\pi}{4}}^{t_n-\frac{\pi}{4}} \theta\cos\theta\,\mathrm{d}\theta\Bigg)\Bigg)\,\mathrm{d}s\Bigg] \\
&\quad - 4\Bigg[\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\cos\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s \\
&\quad + \int\limits_{t_n-\frac{\pi}{4}}^{t_n}(-2\sin 2s)\Bigg(\int\limits_{s-\frac{\pi}{4}}^{t_n-\frac{\pi}{4}} \sin 2\theta\cos(\theta-\tfrac{\pi}{4})\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&= 2\Bigg[\cos t_n + 2\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \theta\sin 2\theta\sin(\theta-\tfrac{\pi}{4})\,\mathrm{d}\theta - \frac{\pi}{2}\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\sin(s-\tfrac{\pi}{4})\,\mathrm{d}s \\
&\quad + 2\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\cos(s-\tfrac{\pi}{4})\,\mathrm{d}s\Bigg] \\
&\quad - 4\Bigg[\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\cos(s-\tfrac{\pi}{4})\,\mathrm{d}s + \frac{\sqrt{2}}{3}\left(\sin^3\left(t_n-\tfrac{\pi}{4}\right) - \cos^3\left(t_n-\tfrac{\pi}{4}\right)\right) \\
&\quad - \frac{2\sqrt{2}}{3}\Bigg(\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\cos^3\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s - \int\limits_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\sin^3\left(s-\tfrac{\pi}{4}\right)\Bigg)\,\mathrm{d}s\Bigg].
\end{aligned}
$$

In the above identity all the terms are bounded except

$$
\int\limits_{t_n-\frac{\pi}{4}}^{t_n} \theta\sin 2\theta\sin\left(\theta-\tfrac{\pi}{4}\right)\,\mathrm{d}\theta = \frac{\sqrt{2}}{3}\Bigg[t_n\cos^3 t_n - \int\limits_{t_n-\frac{\pi}{4}}^{t_n}\sin^3 s\,\mathrm{d}s - \int\limits_{t_n-\frac{\pi}{4}}^{t_n}\cos^3 s\,\mathrm{d}s\Bigg].
$$

Hence

$$
\varliminf_{n\to\infty} \tilde{G}(t_n) = -\infty \qquad \text{and} \qquad \varlimsup_{n\to\infty} \tilde{G}(t_n) = \infty.
$$

Thus by Theorem 3, bounded solutions of the problem (10), (11) oscillate in Q. In particular, $u(x,t) = \sin x\cos t$ is a bounded oscillatory solution of the problem.

Remark 1. Theorem 3 holds if the condition (C_{12}) is replaced by the following one:

$$c(x,t,\xi_0,\xi_1,\dots,\xi_r)\begin{cases}\geq \sum\limits_{k=0}^{r} p_k(t)\xi_k, & \text{if } \xi_k>0\\ \leq \sum\limits_{k=0}^{r} p_k(t)\xi_k, & \text{if } \xi_k<0,\end{cases}$$

where $0 \leq k \leq r$, $p_0(t) \geq 0$ for t ≥ 0, $p_k(t) \geq 0$ on $\bigcup\limits_{n=1}^{\infty} I_{n,k}$, $I_{n,k} = (t_n - 2\rho_k, t_n)$ for $1 \leq k \leq r$ with a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $I_{n,k}$ are disjoint intervals.

Theorem 4. *Suppose that all the conditions of* Theorem 3 *are satisfied except* (C_7) *which is replaced by* (C_{10}). *Then every bounded solution of* (1), (NBC) *oscillates.*

The proof proceeds in the lines of that of Theorem 3 and makes use of Lemma 2.

Example 4. Consider the problem

$$\begin{aligned}&\frac{\partial}{\partial t}[u(x,t)+2u(x,t-\pi)]-[u_{xx}(x,t)+u_{xx}(x,t-\pi)]\\&\quad+tu(x,t-\pi)+u(x,t-\tfrac{3\pi}{2})-2\sin 2tu\left(x,t-\tfrac{\pi}{4}\right)\\&=-2\sin 2t\sin x\sin(t-\tfrac{\pi}{4})-t\sin x\sin t,\end{aligned}\tag{12}$$

$(x,t) \in (0,\pi)\times(0,\infty)$ with boundary conditions

$$-u_x(0,t) = -\sin t = u_x(\pi,t).\tag{13}$$

In this case, $\tilde{\Psi}(t) = -2\sin t$, $\tilde{\Psi}(t-\pi) = 2\sin t$, $p(t) = -2t\sin 2t$, $\rho_{k*} = \frac{\pi}{4}$ and $\tilde{g}(t) = -4\sin 2t\sin(t-\frac{\pi}{4}) - 2t\sin t$,

$$\int\limits_{t_n-\rho_{k*}}^{t_n} p(s)\,\mathrm{d}s = \int\limits_{t_n-\frac{\pi}{4}}^{t_n} -2\sin 2s\,\mathrm{d}s = 1.$$

Furthermore,

$$
\begin{aligned}
\tilde{G}(t_n) &= \int_{t_n-\rho_{k*}}^{t_n} \tilde{g}(s)\,\mathrm{d}s + \int_{t_n-\rho_{k*}}^{t_n} p(s)\left(\int_{s-\rho_{k*}}^{t_n-\rho_{k*}} \tilde{g}(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\
&= 2\left[\int_{t_n-\frac{\pi}{4}}^{t_n} -s\sin s\,\mathrm{d}s + \int_{t_n-\frac{\pi}{4}}^{t_n} (-2\sin 2s)\left(\int_{s-\frac{\pi}{4}}^{t_n-\frac{\pi}{4}} -\theta\sin\theta\,\mathrm{d}\theta\right)\mathrm{d}s\right] \\
&\quad - 4\left[\int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\sin\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s\right. \\
&\quad \left. + \int_{t_n-\frac{\pi}{4}}^{t_n} (-2\sin 2s)\left(\int_{s-\frac{\pi}{4}}^{t_n-\frac{\pi}{4}} \sin 2\theta\sin\left(\theta-\tfrac{\pi}{4}\right)\,\mathrm{d}\theta\right)\mathrm{d}s\right] \\
&= 2\left[t_n\cos t_n + 2\int_{t_n-\frac{\pi}{4}}^{t_n} s\sin 2s\cos\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s\right. \\
&\quad \left. - \frac{\pi}{2}\int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\cos\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s - 2\int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\sin\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s\right] \\
&\quad - 4\left[\int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\sin\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s + \frac{\sqrt{2}}{3}\left(\sin^3\left(t_n-\tfrac{\pi}{4}\right)+\cos^3\left(t_n-\tfrac{\pi}{4}\right)\right)\right. \\
&\quad \left. + \frac{2\sqrt{2}}{3}\left\{\int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\sin^3\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s + \int_{t_n-\frac{\pi}{4}}^{t_n} \sin 2s\cos^3\left(s-\tfrac{\pi}{4}\right)\,\mathrm{d}s\right\}\right].
\end{aligned}
$$

All the terms in the above identity are bounded except $t_n\cos t_n$ and

$$
\begin{aligned}
&\int_{t_n-\frac{\pi}{4}}^{t_n} \theta\sin 2\theta\cos\left(\theta-\tfrac{\pi}{4}\right)\mathrm{d}\theta \\
&= \frac{\sqrt{2}}{3}\left[-t_n\cos^3 t_n + \frac{1}{\sqrt{2}}t_n\cos^3 t_n - \sqrt{2}\pi\cos^3 t_n\right. \\
&\qquad \left. + \int_{t_n-\frac{\pi}{4}}^{t_n} \cos^3 s\,\mathrm{d}s - \int_{t_n-\frac{\pi}{4}}^{t_n} \sin^3 s\,\mathrm{d}s\right],
\end{aligned}
$$

which implies that $\varliminf_{n\to\infty} \tilde{G}(t_n) = -\infty$ and $\varlimsup_{n\to\infty} \tilde{G}(t_n) = \infty$. Hence, by Theorem 4, the bounded solutions of (12), (13) oscillate. In particular, $u(x,t) = \sin x \sin t$ is a bounded oscillatory solution of the problem.

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