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CARATHÉODORY OUTER MEASURE ON IF-SETS

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ABSTRACT. In this paper an outer measure on IF-sets is studied as a mapping to the set of all compact subintervals of the unit interval. We characterize the properties of the outer measure by the help of the properties of functions given by the edges of the intervals. Then there are defined measurable elements and there is proved that the family of measurable elements is a lattice. Finally the outer measure induced by a measure is constructed.

1. Introduction

We shall consider the set $\mathscr{F} = \{(f_A, g_A)\colon\ f_A, g_A\colon \Omega \to \langle 0,1\rangle,\ \ f_A + g_A \le 1\}$ as a partially ordered set with the ordering

$$A = (f_A, g_A) \le B = (f_B, g_B) \iff f_A \le f_B,\ \ g_A \ge g_B.$$

With respect to this ordering, $\mathscr{F}$ is a lattice with the operations

$$(f_A, g_A) \vee (f_B, g_B) = (f_A \vee f_B, g_A \wedge g_B)$$

$$(f_A, g_A) \wedge (f_B, g_B) = (f_A \wedge f_B, g_A \vee g_B)$$

and the least element $(0,1)$ and the greatest element $(1,0)$. In the paper [6] there have been introduced the binary operations $\hat{+}$, $\hat{-}$ on $\mathbb{R}\times\mathbb{R}$ by the formulas:

$$A \hat{+} B = (f_A, g_A)\ \hat{+}\ (f_B, g_B) = (f_A + f_B,\ g_A + g_B - 1)$$

$$A \hat{-} B = (f_A, g_A)\ \hat{-}\ (f_B, g_B) = (f_A - f_B,\ g_A - g_B + 1).$$

It is not difficult to prove that $(\mathbb{R}\times\mathbb{R}, \le, \hat{+})$ is a lattice ordered group and $\mathscr{F} \subset \{(f_A, g_A)\colon\ (0,1) \le (f_A, g_A) \le (1,0)\}$.

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There are also defined Lukasiewicz operations on $\mathscr{F}$

$$A \oplus B = (f_A, g_A) \ \oplus \ (f_B, g_B) = (f_A \oplus f_B, \ g_A \odot g_B)$$

$$A \odot B = (f_A, g_A) \ \odot \ (f_B, g_B) = (f_A \odot f_B, \ g_A \oplus g_B)$$

where

$$f_A \oplus f_B = (f_A + f_B) \wedge 1, \qquad f_A \odot f_B = (f_A + f_B - 1) \vee 0.$$

In Section 2 we show that an outer measure λ^* on IF-sets can be expressed by two fuzzy events and we show their properties. Using this expression there is proved that the measurable elements form a lattice. In Section 3 the notion of a (finitely additive) measure is introduced and the induced outer measure is constructed. As a corollary a measure extension theorem is formulated. Similar results have been achieved in [3] and [4].

2. Carathéodory outer measure

Remark 2.1. Since outer measure we consider is a function from $\mathscr{F}$ to the set $\mathscr{J}$ of all compact subintervals of $\langle 0, 1\rangle$, we recall that

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$$

and

$$\langle a, b\rangle \leq \langle c, d\rangle \iff a \leq c \ \ \textit{and} \ \ b \leq d.$$

Definition 2.2. Let $\mathscr{F}$ be a set of IF-sets. By an *outer measure* we mean a function $\lambda^* \colon \mathscr{F} \to \mathscr{J}$ satisfying the following conditions:

1. $\lambda^*((0, 1)) = \langle 0, 0\rangle = \{0\}$;
2. $\lambda^*((1, 0)) = \langle 1, 1\rangle = \{1\}$;
3. if $A \odot B = (0, 1)$, then $\lambda^*(A \oplus B) \leq \lambda^*(A) + \lambda^*(B)$;
4. if $A \subset B$, then $\lambda^*(A) \leq \lambda^*(B)$.

Theorem 2.3. *Let $\lambda^* \colon \mathscr{F} \to \mathscr{J}$ be expressed as $\lambda^*((f, g)) = \langle \mu^*(f), 1 - \nu^*(g)\rangle$, where μ^*, ν^* are defined on the set of all functions from Ω to $\langle 0, 1\rangle$ with values in the closed interval $\langle 0, 1\rangle$. Then λ^* is an outer measure if and only if μ^*, ν^* satisfy the following properties:*

1. $\mu^*(0) = 0$;
2. $\mu^*(1) = 1$;
3. *if* $f_A \odot f_B = 0$, *then* $\mu^*(f_A \oplus f_B) \leq \mu^*(f_A) + \mu^*(f_B)$;
4. *if* $f_A \leq f_B$, *then* $\mu^*(f_A) \leq \mu^*(f_B)$;
5. $\nu^*(0) = 0$;

6. $\nu^*(1) = 1$;
7. *if* $g_A \oplus g_B = 1$, *then* $\nu^*(g_A \odot g_B) \geq \nu^*(g_A) + \nu^*(g_B) - 1$;
8. *if* $g_A \leq g_B$, *then* $\nu^*(g_A) \leq \nu^*(g_B)$.

Remark 2.4. Let $A \odot B = (0, 1)$; then $f_A \odot f_B = (f_A + f_B - 1) \vee 0 = 0$, i.e. $f_A + f_B \leq 1$ and $f_A \oplus f_B = f_A + f_B$ and also $g_A \oplus g_B = (g_A + g_B) \wedge 1 = 1$, i.e. $g_A + g_B - 1 \geq 0$ and $g_A \odot g_B = g_A + g_B - 1$.

Therefore condition 3. can be written as:

$$f_A + f_B \leq 1 \implies \mu^*(f_A + f_B) \leq \mu^*(f_A) + \mu^*(f_B)$$

and condition 7.:

$$g_A + g_B - 1 \geq 0 \implies \nu^*(g_A + g_B - 1) \geq \nu^*(g_A) + \nu^*(g_B) - 1.$$

Proof.

$\Longrightarrow$: Let $\lambda^*((0, 1)) = \langle 0, 0\rangle$; then $\lambda^*(0, 1) = \langle \mu^*(0), 1 - \nu^*(1)\rangle = \langle 0, 0\rangle$ therefore $\mu^*(0) = 0$ and $\nu^*(1) = 1$.

Let $\lambda^*((1, 0)) = \langle 1, 1\rangle$; then $\lambda^*(1, 0) = \langle \mu^*(1), 1 - \nu^*(0)\rangle = \langle 1, 1\rangle$, therefore $\mu^*(1) = 1$ and $\nu^*(0) = 0$.

Let $A \odot B = (0, 1)$ (i.e. $f_A + f_B \leq 1$ and $g_A + g_B - 1 \geq 0$) and $\lambda^*(A \oplus B) \leq \lambda^*(A) + \lambda^*(B)$; then

$$\lambda^*(f_A + f_B,\ g_A + g_B - 1) \leq \lambda^*(f_A, g_A) + \lambda^*(f_B, g_B),$$

therefore

$$\langle \mu^*(f_A + f_B), 1 - \nu^*(g_A + g_B - 1)\rangle \leq \langle \mu^*(f_A), 1 - \nu^*(g_A)\rangle + \langle \mu^*(f_B), 1 - \nu^*(g_B)\rangle$$

and

$$\mu^*(f_A + f_B) \leq \mu^*(f_A) + \mu^*(f_B)$$
$$\nu^*(g_A + g_B - 1) \geq \nu^*(g_A) + \nu^*(g_B) - 1.$$

Let $A \subset B \implies \lambda^*(A) \leq \lambda^*(B)$.

Let $f_A \leq f_B$. Put $A = (f_A, 1 - f_A)$, $B = (f_B, 1 - f_B)$. Then $A \subset B$, hence

$$\mu^*(f_A) \leq \mu^*(f_B).$$

Similarly, if $g_A \leq g_B$, define $C = (1 - g_B, g_B)$, $D = (1 - g_A, g_A)$. Then again $C \subset D$ hence

$$\nu^*(g_B) \leq \nu^*(g_A).$$

$\Longleftarrow$: Let $\mu^*(0) = 0$ and $\nu^*(1) = 1$; then

$$\lambda^*((0, 1)) = \langle \mu^*(0), 1 - \nu^*(1)\rangle = \langle 0, 0\rangle = \{0\}.$$

Let $\mu^*(1) = 1$ and $\nu^*(0) = 0$; then

$$\lambda^*((1, 0)) = \langle \mu^*(1), 1 - \nu^*(0)\rangle = \langle 1, 1\rangle = \{1\}$$

Let $\mu^*(f_A + f_B) \leq \mu^*(f_A) + \mu^*(f_B)$, $f_A + f_B \leq 1$ and $\nu^*(g_A + g_B - 1) \geq \nu^*(g_A) + \nu^*(g_B) - 1$, $g_A + g_B - 1 \geq 0$; then

$$\begin{aligned}\lambda^*(A \oplus B) &= \lambda^*(f_A + f_B,\ g_A + g_B - 1)\\ &= \langle \mu^*(f_A + f_B), 1 - \nu^*(g_A + g_B - 1)\rangle\\ &\leq \langle \mu^*(f_A) + \mu^*(f_B), 1 - \nu^*(g_A) + 1 - \nu^*(g_B)\rangle\\ &= \langle \mu^*(f_A), 1 - \nu^*(g_A)\rangle + \langle \mu^*(f_B), 1 - \nu^*(g_B)\rangle = \lambda^*(A) + \lambda^*(B)\end{aligned}$$

and $A \odot B = (0, 1)$.

Let $f_A \leq f_B \implies \mu^*(f_A) \leq \mu^*(f_B)$ and $g_A \leq g_B \implies \nu^*(g_A) \leq \nu^*(g_B)$, then if $f_A \leq f_B$, $g_A \geq g_B$ (i.e. $A \subset B$), then

$$\lambda^*(A) = \langle \mu^*(f_A), 1 - \nu^*(g_A)\rangle \leq \langle \mu^*(f_B), 1 - \nu^*(g_B)\rangle = \lambda^*(B).$$

□

Definition 2.5. Let $\lambda^* \colon \mathscr{F} \to \mathscr{J}$ be an outer measure. An element $A \in \mathscr{F}$ is called *measurable* if

$$\lambda^*(H) = \lambda^*(H \wedge A) + \lambda^*(H \hat{-} (H \wedge A)).$$

for each $H \in \mathscr{F}$.

Remark 2.6. In the following operations, $\wedge$ and $\vee$ take precedence over the operations $+$, $-$; thus $(H \hat{-} H \wedge A)$ denotes $(H \hat{-} (H \wedge A))$.

Theorem 2.7. *An element $A = (f_A, g_A)$ is measurable if and only if for each $H = (f_H, g_H)$ there hold*

$$\mu^*(f_H) = \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A)$$

and

$$\nu^*(g_H) = \nu^*(g_H \vee g_A) + \nu^*(g_H - g_H \vee g_A + 1) - 1.$$

Proof.

$\implies$: Let $A, H \in \mathscr{F}$. Then

$$\lambda^*(H) = \lambda^*(f_H, g_H) = \langle \mu^*(f_H), 1 - \nu^*(g_H)\rangle.$$

And

$$\begin{aligned}&\lambda^*(H \wedge A) + \lambda^*(H \hat{-} H \wedge A)\\ &= \lambda^*(f_H \wedge f_A, g_H \vee g_A) + \lambda^*((f_H, g_H) - (f_H \wedge f_A, g_H \vee g_A))\\ &= \lambda^*(f_H \wedge f_A, g_H \vee g_A) + \lambda^*(f_H - f_H \wedge f_A, g_H - g_H \vee g_A + 1)\\ &= \langle \mu^*(f_H \wedge f_A), 1 - \nu^*(g_H \vee g_A)\rangle\\ &\quad + \langle \mu^*(f_H - f_H \wedge f_A), 1 - \nu^*(g_H - g_H \vee g_A + 1)\rangle.\end{aligned}$$

Since

$$\lambda^*(H) = \lambda^*(H \wedge A) + \lambda^*(H \hat{-} (H \wedge A)),$$

therefore

$$\mu^*(f_H) = \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A)$$

and

$$\nu^*(g_H) = \nu^*(g_H \vee g_A) + \nu^*(g_H - g_H \vee g_A + 1) - 1$$

$\Longleftarrow$: Let

$$\begin{aligned} &\mu^*(f_H) = \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A), \\ &\nu^*(g_H) = \nu^*(g_H \vee g_A) + \nu^*(g_H - g_H \vee g_A + 1) - 1, \\ &\lambda^*(f, g) = \langle \mu^*(f), 1 - \nu^*(g) \rangle. \end{aligned}$$

Then

$$\begin{aligned} \lambda^*(H) &= \lambda^*(f_H, g_H) = \langle \mu^*(f_H), 1 - \nu^*(g_H) \rangle \\ &= \langle \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A), 1 - (\nu^*(g_H \vee g_A) \\ &\qquad + \nu^*(g_H - g_H \vee g_A + 1) - 1) \rangle \\ &= \langle \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A), 1 - \nu^*(g_H \vee g_A) \\ &\qquad + 1 - \nu^*(g_H - g_H \vee g_A + 1) \rangle \\ &= \langle \mu^*(f_H \wedge f_A), 1 - \nu^*(g_H \vee g_A) \rangle \\ &\qquad + \langle \mu^*(f_H - f_H \wedge f_A), 1 - \nu^*(g_H - g_H \vee g_A + 1) \rangle \\ &= \lambda^*(f_H \wedge f_A, g_H \vee g_A) + \lambda^*(f_H - f_H \wedge f_A, g_H - g_H \vee g_A) \\ &= \lambda^*(H \wedge A) + \lambda^*(H \hat{-} (H \wedge A)). \end{aligned}$$

$\square$

Remark 2.8. If equation $\mu^*(f_H) = \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A)$ holds, then function f_A is called 1-measurable function.

If equation $\nu^*(g_H) = \nu^*(g_H \vee g_A) + \nu^*(g_H - g_H \vee g_A + 1) - 1$ holds, then function g_A is called 2-measurable function.

Theorem 2.9. *The measurable elements of $\mathscr{F}$ form a lattice.*

Proof.

(i) We show that if A, B are measurable elements, then $A \wedge B$ is also a measurable element. We recall that

$$A \wedge B = (f_A, g_A) \wedge (f_B, g_B) = (f_A \wedge f_B, g_A \vee g_B).$$

We need to show that

$$\mu^*(f_H) = \mu^*(f_H \wedge f_A \wedge f_B) + \mu^*(f_H - f_H \wedge f_A \wedge f_B)$$

and

$$\nu^*(g_H) = \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H - g_H \vee g_A \vee g_B + 1) - 1.$$

Because μ^* is subadditive it is sufficient to show an inequality

$$\mu^*(f_H) \geq \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A).$$

Let f_A, f_B be the 1-measurable functions. Then for any f_H

$$\mu^*(f_H) = \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A)$$

and

$$\mu^*(f_H \wedge f_A) = \mu^*(f_H \wedge f_A \wedge f_B) + \mu^*(f_H \wedge f_A - f_H \wedge f_A \wedge f_B).$$

Then

$$\begin{aligned}
\mu^*(f_H) &= \mu^*(f_H \wedge f_A \wedge f_B) + \mu^*(f_H \wedge f_A - f_H \wedge f_A \wedge f_B) + \mu^*(f_H - f_H \wedge f_A)\\
&\geq \mu^*(f_H \wedge f_A \wedge f_B) + \mu^*(f_H \wedge f_A - f_H \wedge f_A \wedge f_B + f_H - f_H \wedge f_A)\\
&= \mu^*(f_H \wedge f_A \wedge f_B) + \mu^*(f_H - f_H \wedge f_A \wedge f_B).
\end{aligned}$$

It proves that $f_A \wedge f_B$ is the 1-measurable function.

We know that holds the inequality

$$\nu^*(g_C + g_D - 1) \geq \nu^*(g_C) + \nu^*(g_D) - 1.$$

Let $g_C = g_H \vee g_A \vee g_B$ and $g_D = g_H - g_H \vee g_A \vee g_B + 1$.

Then $g_H = g_C + g_D - 1$ and

$$\begin{aligned}
\nu^*(g_H) &= \nu^*\big((g_H \vee g_A \vee g_B) + (g_H - g_H \vee g_A \vee g_B + 1) - 1\big)\\
&\geq \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H - g_H \vee g_A \vee g_B + 1) - 1
\end{aligned}$$

and therefore it is suffice to show that

$$\nu^*(g_H) \leq \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H - g_H \vee g_A \vee g_B + 1) - 1.$$

Let g_A, g_B be the 2-measurable functions. Then for any g_H

$$\nu^*(g_H) = \nu^*(g_H \vee g_A) + \nu^*(g_H - g_H \vee g_A + 1) - 1$$

and

$$\nu^*(g_H \vee g_A) = \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H \vee g_A - g_H \vee g_A \vee g_B + 1) - 1.$$

Then

$$\begin{aligned}
\nu^*(g_H) &= \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H \vee g_A - g_H \vee g_A \vee g_B + 1) - 1\\
&\quad + \nu^*(g_H - g_H \vee g_A + 1) - 1\\
&\leq \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H \vee g_A - g_H \vee g_A \vee g_B + 1\\
&\quad + g_H - g_H \vee g_A) - 1\\
&= \nu^*(g_H \vee g_A \vee g_B) + \nu^*(g_H - g_H \vee g_A \vee g_B + 1) - 1.
\end{aligned}$$

It proves that $g_A \vee g_B$ is the 2-measurable function. And hence $A \wedge B$ is the measurable element.

(ii) We show that if A, B are measurable elements, then $A \vee B$ is also a measurable element. We recall that

$$A \vee B = (f_A, g_A) \vee (f_B, g_B) = (f_A \vee f_B, g_A \wedge g_B).$$

We need to show that

$$\mu^*(f_H) = \mu^*(f_H \wedge (f_A \vee f_B)) + \mu^*(f_H - f_H \wedge (f_A \vee f_B))$$

and

$$\nu^*(g_H) = \nu^*(g_H \vee (g_A \wedge g_B)) + \nu^*(g_H - g_H \vee (g_A \wedge g_B) + 1) - 1.$$

Since $f_H - f_H \wedge f_A = f_H \vee f_A - f_A$, we have

$$\mu^*(f_H - f_H \wedge f_A) = \mu^*(f_H \vee f_A - f_A).$$

Therefore if f_A is the 1-measurable function, then for any f_H

$$\mu^*(f_H) - \mu^*(f_H \wedge f_A) = \mu^*(f_H \vee f_A) - \mu^*(f_A)$$

or

$$\mu^*(f_H \wedge f_A) = \mu^*(f_H) + \mu^*(f_A) - \mu^*(f_H \vee f_A).$$

Let $f_A, f_B, f_A \wedge f_B$ be the 1-measurable functions; then

$$\begin{aligned}\mu^*(f_H \wedge f_A \wedge f_B) &= \mu^*((f_H \wedge f_A) \wedge (f_H \wedge f_B)) \\ &= \mu^*(f_H \wedge f_A) + \mu^*(f_H \wedge f_B) - \mu^*(f_H \wedge (f_A \vee f_B))\end{aligned}$$

and also

$$\begin{aligned}&\mu^*(f_H - f_H \wedge f_A \wedge f_B) \\ &= \mu^*((f_H - f_H \wedge f_A) \vee (f_H - f_H \wedge f_B)) \\ &= \mu^*(f_H - f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_B) \\ &\quad - \mu^*((f_H - f_H \wedge f_A) \wedge (f_H - f_H \wedge f_B)) \\ &= \mu^*(f_H - f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_B) - \mu^*(f_H - f_H \wedge (f_A \vee f_B)).\end{aligned}$$

Therefore

$$\begin{aligned}&\mu^*(f_H \wedge (f_A \vee f_B)) + \mu^*(f_H - f_H \wedge (f_A \vee f_B)) \\ &= \mu^*(f_H \wedge f_A) + \mu^*(f_H - f_H \wedge f_A) + \mu^*(f_H \wedge f_B) + \mu^*(f_H - f_H \wedge f_B) \\ &\quad - \mu^*(f_H \wedge f_A \wedge f_B) - \mu^*(f_H - f_H \wedge f_A \wedge f_B) \\ &= \mu^*(f_H) + \mu^*(f_H) - \mu^*(f_H) = \mu^*(f_H).\end{aligned}$$

It proves that $f_A \vee f_B$ is the 1-measurable function.

Since $g_H - g_H \wedge g_A = g_H \vee g_A - g_A$, we have

$$\nu^*(g_H - g_H \wedge g_A) = \nu^*(g_H \vee g_A - g_A).$$

Therefore if g_A is the 2-measurable function, then for any g_H

$$\nu^*(g_H) - \nu^*(g_H \wedge g_A) = \nu^*(g_H \vee g_A) - \nu^*(g_A)$$

or

$$\nu^*(g_H \vee g_A) = \nu^*(g_H) + \nu^*(g_A) - \nu^*(g_H \wedge g_A).$$

Let g_A, g_B, $g_A \vee g_B$ be the 2-measurable functions; then

$$\begin{aligned}\nu^*(g_H \vee g_A \vee g_B) &= \nu^*((g_H \vee g_A) \vee (g_H \vee g_B)) \\ &= \nu^*(g_H \vee g_A) + \nu^*(g_H \vee g_B) - \nu^*(g_H \vee (g_A \wedge g_B))\end{aligned}$$

and also

$$\begin{aligned}&\nu^*(g_H - g_H \vee g_A \vee g_B + 1) \\ =&\nu^*((g_H - g_H \vee g_A + 1) \wedge (g_H - g_H \vee g_B + 1)) \\ =&\nu^*(g_H - g_H \vee g_A + 1) + \nu^*(g_H - g_H \vee g_B + 1) - \nu^*(g_H - g_H \vee (g_A \wedge g_B) + 1).\end{aligned}$$

Therefore

$$\begin{aligned}&\nu^*(g_H \vee (g_A \wedge g_B)) + \nu^*(g_H - g_H \vee (g_A \wedge g_B) + 1) - 1 \\ =&\nu^*(g_H \vee g_A) + \nu^*(g_H - g_H \vee g_A + 1) - 1 \\ &+ \nu^*(g_H \vee g_B) + \nu^*(g_H - g_H \vee g_B + 1) - 1 \\ &- \nu^*(g_H \vee g_A \vee g_B) - \nu^*(g_H - g_H \vee g_A \vee g_B + 1) + 1 \\ =&\nu^*(g_H) + \nu^*(g_H) - \nu^*(g_H) = \nu^*(g_H).\end{aligned}$$

It proves that $g_A \vee g_B$ is the 2-measurable function and all measurable elements of $\mathscr{F}$ form a lattice. □

3. Induced outer measure

Definition 3.1. Let H_0 be a set of real functions $f\colon \Omega \to \langle 0, 1\rangle$ satisfying the following conditions:

1. if $f_A, f_B \in H_0$, then $f_A \vee f_B \in H_0$;
2. if $f_A, f_B \in H_0$, then $f_A \wedge f_B \in H_0$;
3. if $f_A \geq f_B$, then $f_A - f_B \in H_0$.

We shall consider the set $\mathscr{F}_0 = \big\{(f, g) : \quad f, g \in H_0; \;\; f + g \leq 1\big\}$.

Definition 3.2. Let $\mathscr{F}_0$ be a subset of $\mathscr{F}$ closed under operations $\oplus$, $\odot$. A measure on $\mathscr{F}_0$ is a mapping $\lambda\colon \mathscr{F}_0 \to \mathscr{J}$ satisfying the following conditions:

1. $\lambda((f, g)) = \langle \mu(f), 1 - \nu(g)\rangle$;
2. $\lambda((0, 1)) = \langle 0, 0\rangle$, $\lambda((1, 0)) = \langle 1, 1\rangle$;
3. if $(f_A, g_A) \odot (f_B, g_B) = (0, 1)$, then
$$\lambda((f_A, g_A)) + \lambda((f_B, g_B)) = \lambda((f_A, g_A) \oplus (f_B, g_B)).$$

THEOREM 3.3. *Let $\lambda((f,g)) = \langle \mu(f), 1-\nu(g)\rangle$ and μ, ν be the mappings from H_0 to $\langle 0,1\rangle$. Then there are satisfied the following conditions:*

1. $\mu(0) = 0,\ \mu(1) = 1$;
2. $\nu(0) = 0,\ \nu(1) = 1$;
3. *if* $f_A \odot f_B = 0$, *then* $\mu(f_A) + \mu(f_B) = \mu(f_A \oplus f_B)$;
4. *if* $g_A \oplus g_B = 1$, *then* $\nu(g_A) + \nu(g_B) = \nu(g_A \odot g_B) + 1$.

P r o o f. Since

$$\lambda((f,g)) = \langle \mu(f), 1-\nu(g)\rangle$$

and

$$\lambda((0,1)) = \langle 0,0\rangle, \qquad \lambda((1,0)) = \langle 1,1\rangle,$$

then automatically

$$\mu(0) = 0, \qquad \mu(1) = 1$$

and

$$\nu(0) = 0, \qquad \nu(1) = 1.$$

At the beginning of the paper we said that

$$(f_A, g_A) \oplus (f_B, g_B) = (f_A \oplus f_B,\ g_A \odot g_B)$$

and

$$(f_A, g_A) \odot (f_B, g_B) = (f_A \odot f_B,\ g_A \oplus g_B).$$

Let $(f_A, g_A) \odot (f_B, g_B) = (0,1)$ (i.e. $f_A \odot f_B = 0$ and $g_A \oplus g_B = 1$). Then

$$\begin{aligned} \lambda((f_A, g_A)) + \lambda((f_B, g_B)) &= \lambda((f_A, g_A) \oplus (f_B, g_B)) \\ &= \lambda((f_A \oplus f_B,\ g_A \odot g_B)) \\ &= \langle \mu(f_A \oplus f_B), 1 - \nu(g_A \odot g_B)\rangle \end{aligned}$$

and also

$$\begin{aligned} \lambda((f_A, g_A)) + \lambda((f_B, g_B)) &= \langle \mu(f_A), 1-\nu(g_A)\rangle + \langle \mu(f_B), 1-\nu(g_B)\rangle \\ &= \langle \mu(f_A) + \mu(f_B), 1-\nu(g_A) + 1 - \nu(g_B)\rangle. \end{aligned}$$

Therefore

$$\mu(f_A) + \mu(f_B) = \mu(f_A \oplus f_B)$$

and

$$\nu(g_A) + \nu(g_B) = 1 + \nu(g_A \odot g_B).$$

□

Proposition 3.1. *Let $A, B, C \in \mathscr{F}_0$, $A = B \oplus C$; then $\lambda(A) = \lambda(B) + \lambda(C)$.*

P r o o f. Since

$$A = (f_A, g_A) = (f_B + f_C, g_B + g_C - 1) = B \oplus C,$$

we have

$$f_B + f_C \leq 1, \qquad g_B + g_C - 1 \geq 0.$$

Therefore

$$B \odot C = (f_B \odot f_C, g_B \oplus g_C) = ((f_B + f_C - 1) \vee 0, (g_B + g_C) \wedge 1) = (0, 1).$$

We obtain

$$\lambda(B) + \lambda(C) = \lambda(B \odot C) + \lambda(B \oplus C) = \lambda((0,1)) + \lambda(A) = \lambda(A).$$

□

Proposition 3.2. *If $A \leq B \oplus C$, then $\lambda(A) \leq \lambda(B) + \lambda(C)$.*

P r o o f. Let $A \leq B \oplus C = A \oplus ((B \oplus C) \hat{-} A)$; then Proposition 3.1 implies:

$$\lambda(B) + \lambda(C) = \lambda(A) + \lambda((B \oplus C) \hat{-} A) \geq \lambda(A).$$

□

Theorem 3.4. *Let μ, ν be finite additive submeasures on H_0. Define*

$$\lambda^*((f, g)) = \langle \mu^*(f), 1 - \nu^*(g) \rangle,$$

where

$$\mu^*(f) = \bigwedge \big\{ \mu(h) : \ f \leq h; \ \ h \in H_0 \big\},$$

$$\nu^*(g) = \bigvee \big\{ \nu(k) : \ g \geq k; \ \ k \in H_0 \big\}.$$

Then λ^* is an outer measure.

P r o o f. We show, that λ^* satisfied conditions of Theorem 2.3.

1. $\mu^*(0) = \bigwedge \big\{ \mu(h) : \ 0 \leq h; \ \ h \in H_0 \big\} \leq \mu(0) = 0$;
2. Evidently $\mu^*(1) \leq \mu(1) = 1$.

If $1 \leq h$, then $1 = \mu(1) \leq h$. Therefore $1 \leq \mu^*(1)$.

3. Let $h_A, h_B \in H_0$, $f_A \leq h_A$, $f_B \leq h_B$, $f_A + f_B \leq 1$, $h_A + h_B \leq 1$.

Then

$$f_A + f_B \leq h_A + h_B$$

and

$$\mu^*(f_A + f_B) \leq \mu(h_A + h_B) = \mu(h_A) + \mu(h_B).$$

Now we fix for a moment h_A. Since the preceding inequality holds for any h_B, we obtain:

$$\mu^*(f_A + f_B) - \mu(h_A) \leq \mu(h_B),$$

hence
$$\mu^*(f_A + f_B) - \mu(h_A) \le \mu^*(f_B).$$
Similarly the relation
$$\mu^*(f_A + f_B) - \mu^*(f_B) \le \mu(h_A)$$
for any $h_A \ge f_A$ implies
$$\mu^*(f_A + f_B) - \mu^*(f_B) \le \mu^*(f_A).$$
Therefore
$$\mu^*(f_A + f_B) \le \mu^*(f_A) + \mu^*(f_B).$$

4. Let $h_A, h_B \in H_0$, $f_A \le f_B$, $f_A \le h_A$, $f_B \le h_B$. Then it holds that
$$\{\mu(h_A) : \ f_A \le h_A\} \supset \{\mu(h_B) : \ f_B \le h_B\}.$$
We can see that $\mu^*(f_A)$ is the infimum of the larger set, so it is also a lower bound of the smaller set. Therefore $\mu^*(f_A) \le \mu^*(f_B)$.

The conditions of submeasure ν^* can be proved by the similar way. □

Theorem 3.5. *Let $\mathscr{M}$ be the set of all measurable elements with respect to λ^*. Then $\lambda^*\big|_{\mathscr{M}}$ is an additive measure and $\mathscr{F}_0 \subset \mathscr{M}$.*

Proof.

- Firstly we show that $\mu^*\big|_{\mathscr{M}}$ is a measure.

1. We show that $(0,1) \in \mathscr{M}$.

Let $f_H \in H_0$ and $f_A = 0$; then
$$\mu^*(0) = 0,$$
$$\mu^*(0) + \mu^*(f_H) = \mu^*(f_H),$$
$$\mu^*(f_H \wedge 0) + \mu^*(f_H - f_H \wedge 0) = \mu^*(f_H).$$
The last equality implies that $f_A = 0$ is 1-measurable function.

Let $g_H \in H_0$ and $g_A = 1$; then
$$\nu^*(1) = 1$$
$$\nu^*(1) + \nu^*(g_H) - 1 = \nu^*(g_H)$$
$$\nu^*(g_H \vee 1) + \nu^*(g_H - g_H \vee 1 + 1) - 1 = \nu^*(g_H).$$
Therefore $g_A = 1$ is 2-measurable function and $(0,1) \in \mathscr{M}$.

2. We show that $(1,0) \in \mathscr{M}$.

Let $f_H \in H_0$, $f_A = 1$; then
$$\mu^*(f_H \wedge 1) + \mu^*(f_H - f_H \wedge 1) = \mu^*(f_H).$$
Let $g_H \in H_0$, $g_A = 0$; then
$$\nu^*(g_H \vee 0) + \nu^*(g_H - g_H \vee 0 + 1) - 1 = \nu^*(g_H).$$
Therefore $(1,0) \in \mathscr{M}$.

3. Let f_A, f_B be 1-measurable functions, $f_A + f_B \leq 1$. Then $\mu^*(f_A \odot f_B) = 0$ and
$$\mu^*(f_A \oplus f_B) = \mu^*(f_A + f_B).$$
Since f_A is 1-measurable function,
$$\begin{aligned}\mu^*(f_A + f_B) &= \mu^*((f_A + f_B) \wedge f_A) + \mu^*((f_A + f_B) - (f_A + f_B) \wedge f_A)\\ &= \mu^*(f_A) + \mu^*(f_B).\end{aligned}$$
Let g_A, g_B be 2-measurable functions $g_A + g_B - 1 \leq 1$. Then $\nu^*(g_A \oplus g_B) = 1$ and
$$\nu^*(g_A \odot g_B) = \nu^*(g_A + g_B - 1).$$
Since g_A is 2-measurable function,
$$\begin{aligned}&\nu^*(g_A + g_B - 1)\\ =&\nu^*((g_A + g_B - 1) \vee g_A) + \nu^*((g_A + g_B - 1) - (g_A + g_B - 1) \vee g_A + 1) - 1\\ =&\nu^*(g_A) + \nu^*(g_B) - 1.\end{aligned}$$

- Secondly we show that $\mathscr{F}_0 \subset \mathscr{M}$.

An induced outer measure was defined as:
$$\lambda^*((f_A, f_B)) = \langle \mu^*(f_A), 1 - \nu^*(f_B) \rangle,$$
where
$$\mu^*(h_A) = \bigwedge \big\{ \mu(g_A) : \ h_A \leq g_A; \ \ g_A \in H_0 \big\},$$
$$\nu^*(h_B) = \bigvee \big\{ \nu(g_B) : \ h_B \geq g_B; \ \ g_B \in H_0 \big\}.$$
Let $F = (f_A, f_B) \in \mathscr{F}_0$, we need to show that
$$\mu^*(h_A) = \mu^*(h_A \wedge f_A) + \mu^*(h_A - h_A \wedge f_A)$$
and
$$\nu^*(h_B) = \nu^*(h_B \vee f_B) + \nu^*(h_B - h_B \vee f_B + 1) - 1$$
for each $f_A, f_B \in H_0$.

Since μ^* is subadditive, there holds for any $f_A \in H_0$:
$$\mu^*(h_A) \leq \mu^*(h_A \wedge f_A) + \mu^*(h_A - h_A \wedge f_A).$$
Let $f_A \in H_0$, $g_A \geq h_A$, $\varepsilon > 0$ and
$$\mu^*(h_A) + \varepsilon \geq \mu(g_A).$$
Measure μ is additive, i.e. for each $f_A, g_A \in H_0$ it holds that
$$\mu(g_A) = \mu(g_A \wedge f_A) + \mu(g_A - g_A \wedge f_A).$$
Then
$$\mu^*(h_A) + \varepsilon \geq \mu(g_A \wedge f_A) + \mu(g_A - g_A \wedge f_A).$$
Since $g_A \geq h_A$, $g_A \wedge f_A \geq h_A \wedge f_A$ and $\mu(g_A \wedge f_A) \geq \mu^*(h_A \wedge f_A)$.

We will use the following notation: $(g_A - f_A)^+ = 0 \vee (g_A - f_A)$.

Let $g_A \geq h_A$ and $(h_A - f_A)^+ = 0$, then $(g_A - f_A)^+ \geq (h_A - f_A)^+$.

Let $(h_A - f_A)^+ = h_A(x) - f_A(x) > 0$; then

$$(g_A - f_A)^+ = g_A(x) - f_A(x) \geq h_A(x) - f_A(x) = (h_A - f_A)^+.$$

Therefore

$$(g_A - f_A)^+ \geq (h_A - f_A)^+$$

and then

$$0 \vee (g_A - f_A) \geq 0 \vee (h_A - f_A),$$

$$(g_A - g_A) \vee (g_A - f_A) \geq (h_A - h_A) \vee (h_A - f_A),$$

$$g_A - (g_A \wedge f_A) \geq h_A - (h_A \wedge f_A).$$

Therefore

$$\mu(g_A - g_A \wedge f_A) \geq \mu^*(h_A - h_A \wedge f_A).$$

Hence

$$\mu^*(h_A) + \varepsilon \geq \mu^*(h_A \wedge f_A) + \mu^*(h_A - h_A \wedge f_A)$$

for each $\varepsilon > 0$ and

$$\mu^*(h_A) = \mu^*(h_A \wedge f_A) + \mu^*(h_A - h_A \wedge f_A)$$

for each $f_A \in H_0$.

The conditions of submeasure ν^* can be proved by the similar way. $\square$

Theorem 3.6. *Let λ be a measure on $\mathscr{F}_0$. T hen $\lambda^*\big|_{\mathscr{F}_0}$ is a measure and $\lambda^*\big|_{\mathscr{F}_0} = \lambda$.*

P r o o f. Let $f \in H_0$, $f \leq h$, then $\mu(f) \leq \mu(h)$ and

$$\mu(f) \leq \bigwedge\big\{\mu(h) :\ f \leq h,\ \ h \in H_0\big\} = \mu^*(f).$$

Put $h = f$; then $\mu^*(f) \leq \mu(f)$. Therefore if $f \in H_0$, then $\mu^*(f) = \mu(f)$. The equation $\nu^*(g) = \nu(g)$ can be prove by the similar way.

Therefore for each $(f, g) \in \mathscr{F}_0$ it holds that

$$\lambda^*((f, g)) = \langle \mu^*(f), 1 - \nu^*(g)\rangle = \langle \mu(f), 1 - \nu(g)\rangle = \lambda((f, g)).$$

$\square$

REFERENCES

[1] CIGNOLI, R.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Foundations of Many-Valued Reasoning*. Trends Log. Stud. Log. Libr. 7, Kluwer, Dordrecht, 2000.

[2] MICHALÍKOVÁ, A.: *Carathéodory outer measure on MV-algebras*. In: Proceeding of IWIFSGN'2005, September 16, Warsaw, Poland (To appear).

[3] RIEČAN, B: *On the Carathéodory method of the extension of measures and integrals*, Math. Slovaca **27** (1977), 365–374.

[4] RIEČAN, B.: *On the entropy on the Lukasiewicz square*. In: Joint EUSFLAT – LFA 2005, Barcelona, September 7–9, 2005, pp. 330–333.

[5] RIEČAN, B.—MUNDICI, D.: *Probability on MV-algebras*. In: Handbook of Measure Theory (E. Pap, ed.), Elsevier, Amsterdam, 2002, pp. 869–909.

[6] SRIVASTAVA, P.: *Carathéodory outer measure and measurability*, Indian J. Math. **45**, (2003), 89–104.

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