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ON MINIMAL RESIDUATED BOUNDED MAPPINGS IN ATOMISTIC LATTICES

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ABSTRACT. Relationship between automorphisms and residuated bounded mappings in atomistic lattices is studied.

Let P be an ordered set. For each $M \subseteq P$ we put

$$\mathrm{L}(M) := \{p \in P : \ (\forall m \in M)(p \leq m)\}$$
$$\uparrow M := \{p \in P : \ (\exists m \in M)(m \leq p)\}.$$

Let P, Q be ordered sets. The set of all isotone mappings of P to Q will be denoted by $Q^{\langle P \rangle}$.

Let $f \in Q^{\langle P \rangle}$ and $g \in P^{\langle Q \rangle}$. We write $f \dashv g$ if

$$(\forall l \in P)(\forall m \in Q)(f(l) \leq m \iff l \leq g(m))$$

and say that $[f, g]$ is an *adjoint pair*.

Clearly, $f \dashv g$ and $f' \dashv g'$ implies that $f' \circ f \dashv g \circ g'$ whenever $f \in Q^{\langle P \rangle}$, $g \in P^{\langle Q \rangle}$, $f' \in R^{\langle Q \rangle}$ and $g' \in Q^{\langle R \rangle}$.

An isotone mapping $f \in Q^{\langle P \rangle}$ is said to be *residuated* or a *left adjoint* if the following equivalent conditions hold:

(1) $(\exists g \in P^{\langle Q \rangle})\Big((\forall l \in P)\big(l \leq g \circ f(l)\big) \ \& \ (\forall m \in Q)\big(f \circ g(m) \leq m\big)\Big)$;

(2) $(\exists g \in P^{\langle Q \rangle})(f \dashv g)$;

(3) $(\forall m \in Q)\big(f^{-1}[\![\mathrm{L}(\{m\})]\!]$ is a principal ideal in $P\big)$.

If P and Q are complete lattices, the preceding conditions are also equivalent with

(4) f is a $\bigvee$-homomorphism.

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B l y t h ([1], [2]) may serve as a reference.

Let L be an atomistic complete lattice. We denote $\mathrm{R}(L)$ the pointwise ordered set of all residuated order endomorphisms f of L such that

$$f(1) = 1$$
$$f(y) = 0 \iff y = 0$$

where 0 is the bottom element and 1 is the top element of L. It is clear that $\mathrm{R}(L)$ is the set of all left adjoints from adjoint pairs of bounded order endomorphisms of L. The set of all minimal elements in $\mathrm{R}(L)$ is denoted by $\mathrm{MR}(L)$. It is obvious that each automorphism of L belongs to $\mathrm{MR}(L)$, that is $\mathrm{Aut}(L) \subseteq \mathrm{MR}(L)$. The conjecture formulated by K u c h m e i is as follows.

For each atomistic modular lattice L of finite length, $\mathrm{Aut}(L) = \mathrm{MR}(L)$.

We shall prove that the conjecture is false.

Theorem 1. *Let L be an atomistic lattice of finite length. Then the following conditions are equivalent:*

(i) *L is a finite lattice of length at most* 2*;*

(ii) $\mathrm{Aut}(L) = \mathrm{MR}(L)$.

The following lemmas and consequently the theorem require the axiom of choice. Recall that every lattice of finite length is complete.

Lemma 2. *Let L be an atomistic lattice of finite length. Then*

$$\mathrm{R}(L) = {\uparrow_{\mathrm{R}(L)}} \mathrm{MR}(L).$$

P r o o f. Let $h \in \mathrm{R}(L)$. There exists a maximal chain C in $\mathrm{R}(L)$ such that $h \in C$. Define $f \colon L \to L$ by

$$f(y) := \bigwedge \{g(y) : \ g \in C\}$$

and note that $f(y) \in \{g(y) : \ g \in C\}$ as $\{g(y) : \ g \in C\}$ is finite. Therefore $f(1) = 1$ and also $f(y) = 0$ implies $y = 0$. If $y < z$, then there exists $g \in C$ such that $f(y) = g(y)$ and $f(z) = g(z)$, and therefore $f(y) \le f(z)$. Hence f is isotone. Let $Y \subseteq L$. Then there exist $y_1, \ldots, y_n \in Y$ such that $\bigvee Y = y_1 \vee \cdots \vee y_n$. Consequently, there exists $g \in C$ such that $(\forall i) f(y_i) = g(y_i)$ and $f(\bigvee Y) = g(\bigvee Y)$. Therefore $f(\bigvee Y) = g(\bigvee Y) = g(y_1 \vee \cdots \vee y_n) = g(y_1) \vee \cdots \vee g(y_n) = f(y_1) \vee \cdots \vee f(y_n) \le \bigvee f[\![Y]\!]$. As f is isotone, the converse inequality is also true. Hence f is a $\bigvee$-homomorphism. To sum up, $f \in \mathrm{MR}(L)$ and $f \le h$. □

Lemma 3. *Let L be an infinite lattice of length* 2*. Then there exists a lattice endomorphism in* $\mathrm{MR}(L) \smallsetminus \mathrm{Aut}(L)$.

P r o o f. Consider an arbitrary countable subset in $\mathrm{At}(L)$, say $A = \{a_0, a_1, a_2, \ldots\}$. Define f by $f(a_i) := f(a_{i+1})$ and $f(a) := a$ if $a \in \mathrm{At}(L) \smallsetminus A$. □

Proposition 4. *Let L be an atomistic lattice. Then the following conditions are equivalent:*

(i) *L is a finite lattice of length at most 2;*

(ii) $\uparrow_{\mathrm{R}(L)} \mathrm{Aut}(L) = \mathrm{R}(L)$.

P r o o f.

(i) $\Longrightarrow$ (ii): Let L be a finite atomistic lattice of length at most 2, that is a singleton, a two-element chain or a finite diamond. If L is a singleton or a two-element chain, then $\mathrm{R}(L) = \{\mathrm{id}_L\}$ and the result is obvious. If L is a diamond with n atoms and $f \in \mathrm{MR}(L)$, then $f[\![\mathrm{At}(L)]\!] = \mathrm{At}(L)$. In order to prove this claim, denote $\mathrm{At}_0(L) := \{a \in \mathrm{At}(L) : \ f(a) \in \mathrm{At}(L)\}$ and $\mathrm{At}_1(L) := \{a \in \mathrm{At}(L) : \ f(a) = 1\}$. As $1 = f(1) = f(a_1 \vee a_2) = f(a_1) \vee f(a_2)$ for any distinct atoms a_1 and a_2, f is injective on $\mathrm{At}_0(L)$. Let g be an arbitrary bijection of $\mathrm{At}_1(L)$ onto $\mathrm{At}(L) \setminus f[\![\mathrm{At}_0(L)]\!]$. Then f' defined by $f'(a) := f(a)$ if $a \in \mathrm{At}_0(L)$ and $f'(a) := g(a)$ if $a \in \mathrm{At}_1(a)$ belongs to $\mathrm{R}(L)$. Clearly $f' \leq f$. As f was minimal, $f' = f$ and therefore $\mathrm{At}_0(L) = \mathrm{At}(L)$. Hence $f \in \mathrm{Aut}(L)$.

(ii) $\Longrightarrow$ (i): If L is not of length at most 2, that is $L \neq \mathrm{At}(L) \cup \{0, 1\}$, then take an arbitrary element $c \in L \setminus \mathrm{At}(L) \setminus \{0, 1\}$. Since L is atomistic, there exists an atom $a \in \mathrm{At}(L) \setminus \mathrm{L}(\{c\})$. Define f by

$$
\begin{aligned}
f(y) &:= a && \text{if} \quad y \in \mathrm{L}(\{c\}) \setminus \{0\} \\
f(a) &:= c && \\
f(0) &:= 0 && \\
f(y) &:= a \vee c && \text{if} \quad y \in \mathrm{L}(\{a \vee c\}) \setminus \mathrm{L}(\{a\}) \setminus \mathrm{L}(\{c\}) \\
f(y) &:= 1 && \text{otherwise.}
\end{aligned}
$$

It is obvious that $f \in \mathrm{R}(L)$. As a is the image of at least two different elements, namely c and atoms in $\mathrm{L}(\{c\})$, there exists no $g \in \mathrm{Aut}(L)$ such that $g \leq f$.

If L is an infinite lattice of length 2, then the result follows by Lemma 3. $\square$

The proof of the theorem follows from Proposition 4 and Lemma 2.

The set of all lattice endomorphisms of L will be denoted $\mathrm{End}(L)$.

Proposition 5. *Let L be a finite atomistic lattice. Then* $\mathrm{End}(L) \cap \mathrm{R}(L) = \mathrm{Aut}(L)$.

P r o o f. Let $f \in \mathrm{End}(L) \cap \mathrm{R}(L)$. Then for any two distinct atoms a_1, a_2 we have $f(a_1) \wedge f(a_2) = f(a_1 \wedge a_2) = f(0) = 0$, and hence $(\mathrm{L}(\{f(a_1)\}) \cap \mathrm{At}(L)) \cap (\mathrm{L}(\{f(a_2)\}) \cap \mathrm{At}(L)) = \emptyset$. It is immediate that $|\,\mathrm{L}(\{f(a)\}) \cap \mathrm{At}(L)| = 1$, and therefore $f(a) \in \mathrm{At}(L)$, for any atom a. Therefore f defines a bijection on $\mathrm{At}(L)$. Moreover, there exists a natural number n such that f^n is the identity on $\mathrm{At}(L)$. As f is a lattice homomorphism, so is f^n. If $b = a_1 \vee \cdots \vee a_k$ where $a_i \in \mathrm{At}(L)$, then $f^n(b) = f^n(a_1 \vee \cdots \vee a_k) = f^n(a_1) \vee \cdots \vee f^n(a_k) = a_1 \vee \cdots \vee a_k = b$. Hence $f^n = \mathrm{id}_L$. Finally, $f \in \mathrm{Aut}(L)$. The converse inclusion is obvious. $\square$

For infinite atomistic lattices of length 2 the statement is not true, see Lemma 3.

Proposition 6. *Let L be an atomistic lattice of finite length and $f \in \mathrm{End}(L) \cap \mathrm{R}(L)$, $g \in \mathrm{End}(L)$ be such that $f \dashv g$. Then $f \in \mathrm{Aut}(L)$ and $g = f^{-1}$.*

P r o o f. First we shall show that both $f \circ g$ and $g \circ f$ are the identity on $\mathrm{At}(L)$. It is clear that $g^{-1}[\![\{0\}]\!]$ is a principal ideal, say $\mathrm{L}(\{b\})$. We may assume without loss of generality that f, g are such that b is maximal. Note that $f \circ f$, $g \circ g$ also satisfy the assumptions. If $a_0, a_1 \in \mathrm{At}(L)$, then $f(a_0) \leq f(a_1)$ implies that $0 \neq f(a_0) = f(a_0) \wedge f(a_1) = f(a_0 \wedge a_1)$ which in turn yields $a_0 = a_1$. Hence f is injective on $\mathrm{At}(L)$. Let $a \in \mathrm{At}(L)$. If $a < g(f(a))$, then there would exist $c \in \mathrm{At}(L) \cap \mathrm{L}(\{g(f(a))\}) \setminus \mathrm{L}(\{a\})$. Then $f(a) \leq f(a) \vee f(c) \leq f(g(f(a))) = f(a)$, and hence $f(c) \leq f(a)$. Therefore $a = g(f(a))$. Suppose that $\{0\} \neq g^{-1}[\![\{0\}]\!]$. There exists $a \in \mathrm{At}(L) \cap g^{-1}[\![\{0\}]\!]$. Then $g(g(b)) = g(0) = 0$ and $g(g(f(a))) = g(a) = 0$. It is clear that $b < b \vee f(a) \in (g \circ g)^{-1}[\![0]\!]$. This is a contradiction with the maximality of b. Hence $0 \neq f(g(a)) \leq a$ and therefore $f(g(a)) = a$ for each $a \in \mathrm{At}(L)$. Let $y \in L$. There exist $a_0, \dots, a_k \in \mathrm{At}(L)$ such that $y = a_0 \vee \dots \vee a_k$. Then $f(g(y)) = f(g(a_0)) \vee \dots \vee f(g(a_k)) = a_0 \vee \dots \vee a_k = y$ and $g(f(y)) = g(f(a_0)) \vee \dots \vee g(f(a_k)) = a_0 \vee \dots \vee a_k = y$. $\square$

If we denote $\widetilde{\mathrm{R}(L)} := \{[f, g] : \ f \dashv g \ \& \ f \in \mathrm{R}(L)\}$ and $\widetilde{\mathrm{Aut}(L)} := \{[f, f^{-1}] : f \in \mathrm{Aut}(L)\}$, then Proposition 6 can be formulated as follows.

Let L be an atomistic lattice of finite length. Then

$$\widetilde{\mathrm{R}(L)} \cap (\mathrm{End}(L) \times \mathrm{End}(L)) = \widetilde{\mathrm{Aut}(L)}.$$

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