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REMARKS ON THE ORDER FOR QUANTUM OBSERVABLES

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ABSTRACT. Relations between generalized effect algebras and the sets of classical and quantum observables endowed with an ordering recently introduced in [GUDDER, S.: *An order for quantum observables*, Math. Slovaca **56** (2006), 573–589] are studied. In the classical case, a generalized OMP, while in the quantum case a weak generalized OMP is obtained. Existence of infima for arbitrary sets and suprema for above bounded sets in the quantum case is shown. Compatibility in the sense of Mackey is characterized.

1. Introduction

The set of bounded observables for a quantum system is usually represented by the set $\mathcal{S}(H)$ of bounded self-adjoint operators on a complex Hilbert space H. The traditional order for $A, B \in \mathcal{S}(H)$ is defined by $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for every $x \in H$. In [9], this order is called numerical. Under this ordering, $(\mathcal{S}(H), \le)$ becomes a partially ordered set (poset). A well-known theorem due to R. Kadison [14] shows that $(\mathcal{S}(H), \le)$ is not a lattice, it is even an anti-lattice in the sense that $A \wedge B$ exists if and only if $A \le B$ or $B \le A$, and $A \wedge B$ is the smaller of the two. If this ordering is applied to the self-adjoint elements $\mathcal{S}(\mathcal{A})$ of a von Neumann algebra $\mathcal{A}$, then $\mathcal{S}(\mathcal{A})$ is a lattice if and only if $\mathcal{A}$ is abelian.

Another ordering, so-called spectral order, was introduced in [17], [8] as follows: let $A, B \in \mathcal{S}(H)$, and let $(P^A_\lambda)_{\lambda\in\mathbb{R}}$, $(P^B_\lambda)_{\lambda\in\mathbb{R}}$ be spectral families of A, B,

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respectively. We define $A \leq_s B$ if $P_\lambda^B \leq P_\lambda^A$ for every $\lambda \in \mathbb{R}$. Then $\leq_s$ extends $\leq$ on projections, is coarser than the usual ordering, but agrees with it on abelian subalgebras, and turns $\mathcal{S}(H)$ into a boundedly complete lattice, that is, every bounded family in $\mathcal{S}(H)$ has a supremum and infimum.

In [9], a new order for quantum observables, represented by the set of bounded self-adjoint operators $\mathcal{S}(H)$ on a complex Hilbert space H, has been introduced. This new order is determined by assuming that $A \preceq B$ if the proposition that A has a value in Δ implies that B has a value in Δ for every Borel set Δ not containing 0. It is called the logical order. In the commutative case, we may represent observables by fuzzy random variables, and study the new ordering on them. There are several characterizations of the ordering $\preceq$, e.g., $A \preceq B$ if and only if $AB = A^2$. This shows that $\preceq$ is the restriction of D r a z i n 's order ([4]) from the set $B(H)$ of all bounded operators on H, to the self-adjoint part $\mathcal{S}(H)$. Indeed, the Drazin order $a \leq_d b$ is introduced by the binary relation $a^*a = a^*b = b^*a$ and $aa^* = ab^* = ba^*$. To the difference of both the traditional and the spectral order, the logical order is algebraic in the sense that a partial binary operation $\oplus$ can be introduced in $\mathcal{S}(H)$ such that $A \preceq B$ if there is C such that $A \oplus C = B$.

In the present paper, we will study the structure of classical and quantum observables with respect to the new ordering in more details. We will prove that in the classical case, the set $\mathcal{M}(\mathcal{A})$ of random variables on a probability space $(\Omega, \mathcal{A}, \mu)$ forms a generalized σ-orthocomplete orthomodular poset (GOMP), which satisfies the Riesz decomposition properties. Nevertheless, its unitization does not form a Boolean algebra. We also show that the set of functions with finite support form a Riesz ideal. In the quantum case, the set $\mathcal{S}(H)$ forms a weak generalized orthocomplete orthomodular poset (WGOMP). Moreover, the infimum of any two elements of $\mathcal{S}(H)$ exists, while the supremum exists if and only if the two elements have a common upper bound. This extends the results of [9], where the structure of a generalized σ-orthoalgebra has been shown for the classical, and a generalized orthoalgebra for the quantum case, and the existence of infima in $\mathcal{S}(H)$ has been proved only for the finite-dimensional H. More generally, we show that the infimum of an arbitrary family exists, and the supremum of an arbitrary above bounded family exists. We also find a characterization of the Mackey compatibility. Since Mackey compatibility is usually interpreted as simultaneous measurability, it turns out that two observables corresponding to self-adjoint operators A and B, are simultaneously measurable (with respect to $(\mathcal{S}(H), \oplus, 0)$) if and only if $AB = (A \wedge B)^2$.

2. Generalized effect algebras

Effect algebras were introduced in [6] (see also [7] and [15] for alternative definitions) as an abstract generalization of the Hilbert space effects, that is, self-adjoint operators between the zero and identity operators (in the usual ordering), which play an important role in the theory of quantum mechanical measurements ([1]).

Another important example is the unit interval $[0,1]$ of real numbers organized into an effect algebra by defining $a \perp b$ if $a+b \leq 1$, and then putting $a \oplus b = a+b$. We note that $[0,1]$ is also a prototypical example of an MV-algebra, a structure introduced by Chang [2] as an algebraic base for many-valued logic.

For the details about effect algebras and related structures see, e.g., [5].

Definition 2.1. An *effect algebra* (EA) is a system $(E, \oplus, 0, 1)$ consisting of a set E with two special elements $0, 1 \in E$ and with a partially defined binary operation $\oplus$ satisfying the following conditions for all $p, q, r \in E$:

(E1) if $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$ (commutative law),

(E2) if $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$ (associative law),

(E3) for every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = 1$ (orthosupplement law),

(E4) if $p \oplus 1$ is defined, then $p = 0$.

The element q in (E3) is denoted by p' and is called an *orthosupplement* of the element p.

We recall that an effect algebra E is an *orthoalgebra* (OA) if $a \perp a$ implies $a = 0$; E is an *orthomodular poset* (OMP) if whenever a, b, c are mutually orthogonal, then $a \perp b \oplus c$; E is an *orthomodular lattice* if it is a lattice ordered OMP.

Definition 2.2. ([11], [5]) A *generalized effect algebra* (GEA) is a system $(P, \oplus, 0)$ satisfying conditions (E1) and (E2) and

(E3') if $a \oplus b = a \oplus c$, then $b = c$ (cancelation law),

(E4') if $a \oplus b = 0$, then $a = 0 = b$ (positivity),

(E5) $a \oplus 0 = a$ for every $a \in P$.

On an effect algebra or a generalized effect algebra E, we further define:

• a binary relation $\perp$ by

$$a \perp b \iff a \oplus b \text{ exists},$$

• a dual partial binary operation $\ominus$ to the operation $\oplus$ by

$$c \ominus a = b \iff a \perp b \text{ and } a \oplus b = c,$$

• and a binary relation $\leq$ by

$$a \leq b \iff \exists c \in E: \ c \oplus a = b,$$

which is a partial order on E, where 0 is the least element and in the case of effect algebra, 1 is the greatest element. A generalized effect algebra becomes an effect algebra iff it contains a greatest element 1.

A GEA P is a *generalized orthoalgebra* (GOA) if $a \perp a$ implies $a = 0$; P is a *weak generalized orthomodular poset* (WGOMP) if $a \oplus b = a \vee b$ whenever $a \oplus b$ exists, and $a \perp (b \oplus c)$ whenever a, b, c are mutually orthogonal; P is a *generalized orthomodular poset* (GOMP) if $a \oplus b = a \vee b$ whenever $a \perp b$, and if $b \vee c$ exists and $a \perp b$, $a \perp c$ then $a \perp b \vee c$. A lattice ordered GOMP is a *generalized orthomodular lattice* (GOML).

It is well known that every generalized effect algebra P can be embedded into a uniquely defined effect algebra E such that for every $a \in E$, either $a \in P$ or its orthosupplement $a' \in P$. In analogy with the theory of rings, we call this effect algebra E the *unitization* of the GEA P (for more details see [11], [5], [19], [18]).

Moreover, a GEA P is a generalized orthoalgebra iff its unitization E is an orthoalgebra; P is a WGOMP iff its unitization E is an orthomodular poset ([16]); P is a GOMP iff E is an orthomodular poset and the embedding of P into E preserves existing suprema ([16]); and finally P is a generalized orthomodular lattice iff its unitization E is an orthomodular lattice ([12]).

3. The commutative case

Classical commuting observables are represented by random variables on a probability space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is usually thought of as the set of events for some statistical experiment. The set $\mathcal{A}$ can be organized into an effect algebra if we put $A \perp B$ if $A \cap B = \emptyset$ and define the orthosum $A \oplus B = A \cup B$ whenever $A \perp B$. Then $(\mathcal{A}, \oplus, \emptyset, \Omega)$ is an effect algebra with $A' = A^c (= \Omega \setminus A)$. In fact, $\mathcal{A}$ is a Boolean σ-algebra.

We identify an event $A \in \mathcal{A}$ with its characteristic function χ_A, which can be considered as a two-valued measurement with outcomes 0 and 1, or "no" and "yes". That is, for any $\omega \in \Omega$, $\chi_A(\omega)$ gives the values 1 or 0 depending on whether $\omega \in A$ or $\omega \notin A$. We have $\mu(A \oplus B) = \mu(A) + \mu(B)$ whenever $A \oplus B$ is defined, and $\mu(\Omega) = 1$. That is, μ is a state on $\mathcal{A}$ (i.e., an effect algebra morphism from $\mathcal{A}$ to $[0,1]$, considered as effect algebras). We note that here the effect algebra order $\preceq$ coincides with the usual order, i.e., $\chi_A \preceq \chi_B$ iff $\chi_A(\omega) \leq \chi_B(\omega)$ for all $\omega \in \Omega$.

In [9] it is suggested to extend the orthosum to all measurements associated with $\mathcal{A}$, which are represented by the set $\mathcal{M}(\mathcal{A})$ of all random variables on $(\Omega, \mathcal{A}, \mu)$. The extension is obtained by defining $f \perp g$ if $fg = 0$ for $f, g \in \mathcal{M}(\mathcal{A})$. Denote the support of f by $\operatorname{supp}(f) := \{\omega \in \Omega : \ f(\omega) \neq 0\}$, and the null space of f by $\operatorname{null}(f) := \{\omega \in \Omega : \ f(\omega) = 0\} = f^{-1}(0)$. Observe that $f \perp g$ iff $\operatorname{supp}(f) \subset \operatorname{null}(g)$ iff $\operatorname{supp}(f) \perp \operatorname{supp}(g)$. We define $f \oplus g = f + g$ iff $f \perp g$. Define a partial order $\preceq$ on $\mathcal{M}(\mathcal{A})$ by $f \preceq g$ if there is an $h \in \mathcal{M}(\mathcal{A})$ such that $f \perp h$ and $f \oplus h = g$. Then $(\mathcal{M}(\mathcal{A}), \preceq)$ is a poset and $0 \preceq f$ for all $f \in \mathcal{M}(\mathcal{A})$.

In fact, in [9], it was proved that $(\mathcal{M}(\mathcal{A}), \oplus, 0)$ admits a structure of a generalized σ-orthocomplete orthoalgebra. Moreover, the infimum $f \wedge g$ exists for all $f, g \in \mathcal{M}(\mathcal{A})$, while the supremum of f, g exists iff there is $h \in \mathcal{M}(\mathcal{A})$ such that $f, g \preceq h$. In what follows, we give a more precise description of the structure of $(\mathcal{M}(\mathcal{A}), \oplus, 0)$. We need the following characterization of the partial order $\preceq$, proved in [9, Th. 3.1].

Theorem 3.1. *The following statements are equivalent:*

(i) $f \preceq g$,

(ii) $f(\omega) = g(\omega)$ *for all* $\omega \in \operatorname{supp}(f)$,

(iii) $f = g \cdot \chi_{\operatorname{supp}(f)}$,

(iv) $fg = f^2$,

(v) $f^{-1}(\Delta) \subset g^{-1}(\Delta)$ *for every* $\Delta \in \mathcal{B}(\mathbb{R})$ *with* $0 \notin \Delta$.

Theorem 3.2. *The structure* $(\mathcal{M}(\mathcal{A}), \oplus, 0)$ *is a*

a) *WGOMP,*

b) *GOMP.*

Proof.

a) Recall that by [5, Th. 1.5.13] and [5, Remark 1.5.16], we have to prove:

(i) If $f, g \in \mathcal{M}(\mathcal{A})$ and $f \perp g$, then $f \oplus g = f \vee g$, i.e., $f \oplus g$ is the supremum of f and g.

(ii) If $f, g, h \in \mathcal{M}(\mathcal{A})$ are such that $f \perp h$, $g \perp h$, and $f \perp g$, then $h \perp f \oplus g$.

(i): Let $f, g \in \mathcal{M}(\mathcal{A})$, $f \perp g$. Clearly, $f, g \preceq f \oplus g$. Now let $h \in \mathcal{M}(\mathcal{A})$: $f, g \preceq h$. While for every $\omega \in \operatorname{supp}(f \oplus g)$ either $\omega \in \operatorname{supp}(f)$ or $\omega \in \operatorname{supp}(g)$, along with $f, g \preceq h$ we have $\forall\, \omega \in \operatorname{supp}(f \oplus g)$: $(f \oplus g)(\omega) = f(\omega) = h(\omega)$ or $(f \oplus g)(\omega) = g(\omega) = h(\omega)$ and therefore by Theorem 3.1, $f \oplus g \preceq h$.

ii): When f, g, h are mutually orthogonal, their supports are disjoint and so obviously $\operatorname{supp}(f \oplus g) \cap \operatorname{supp}(h) = [\operatorname{supp}(f) \cup \operatorname{supp}(g)] \cap \operatorname{supp}(h) = \emptyset \implies f \oplus g \perp h$.

b) A WGOMP is GOMP iff $\forall\, f, g, h \in \mathcal{M}(\mathcal{A})$: $f \perp h$, $g \perp h$ and the existence of $f \vee g$ imply $f \vee g \perp h$. So let us have $f, g, h \in \mathcal{M}(\mathcal{A})$ such that $f, g \perp h$ and $f \vee g$ exists. Observe that when the supremum of f, g exists, then the functions are equal on the set $\mathrm{supp}(f) \cap \mathrm{supp}(g)$. So we may define $p := f \cdot \chi_{\mathrm{supp}(f)\cap \mathrm{supp}(g)}$, $f_1 := f\bigl(1 - \chi_{\mathrm{supp}(g)}\bigr)$, $g_1 := g(1 - \chi_{\mathrm{supp}(f)})$. Therefore the function $f_1 \oplus p \oplus g_1$ is defined and it can be easily seen that it is in fact the supremum $f \vee g = f_1 \oplus p \oplus g_1$. Since $\mathrm{supp}(f_1 \oplus p \oplus g_1) = \mathrm{supp}(f) \cup \mathrm{supp}(g)$ and $f, g \perp h$, straightforwardly $\mathrm{supp}(f \vee\ g) \perp \mathrm{supp}(h)$ and thus $f \vee g \perp h$. □

Recall that a generalized effect algebra A satisfies the *Riesz decomposition property* (RDP in short) if for any $a, b, c \in A$, $a \le b \oplus c$ implies that there are $b_1 \le b$, $c_1 \le c$ such that $a = b_1 \oplus c_1$. More generally, A satisfies the *σ-Riesz decomposition property* if $a \le \bigoplus_{i\in\mathbb{N}} b_i$ implies $a = \bigoplus_{i\in\mathbb{N}} a_i$, where $a_i \le b_i$ for all $i \in \mathbb{N}$.

Theorem 3.3. *The structure $(\mathcal{M}(\mathcal{A}), \oplus, 0)$ satisfies the σ-RDP.*

P r o o f. Assume that $f \preceq \bigoplus_{i\in\mathbb{N}} g_i$. It follows that $\mathrm{supp}(f) \subseteq \bigcup_{i\in\mathbb{N}} \mathrm{supp}(g_i)$, and to every $\omega \in \mathrm{supp}(f)$ there is $i \in \mathbb{N}$ with $f(\omega) = g_i(\omega)$. Define $h_i := g_i \cdot \chi_{\mathrm{supp}(f)}$. Then $\mathrm{supp}(h_i) = \mathrm{supp}(g_i) \cap \mathrm{supp}(f)$, and clearly $h_i \preceq g_i$ for each $i \in \mathbb{N}$, and $h_i \perp h_j$ whenever $i \ne j$. Therefore $\bigoplus_{i\in\mathbb{N}} h_i$ exists, and since $\mathrm{supp}(f) = \bigcup_{i\in\mathbb{N}} \mathrm{supp}(h_i)$, we have $f = \bigoplus_{i\in\mathbb{N}} h_i$. □

Let us define

$$\mathcal{F} := \bigl\{ f \in \mathcal{M}(\mathcal{A}) :\ \mathrm{supp}(f) \text{ is finite} \bigr\}.$$

We recall that a subset I of a generalized effect algebra P is an *ideal* if for any $a, b \in P$,

(i) $a \in I$, $b \le a$ implies $b \in I$ (that is, I is an order ideal),

(ii) $a, b \in I$, $a \perp b$ implies $a \oplus b \in I$.

An ideal is a *Riesz ideal* if

(R1) $a \in I$, $b, c \in P$, $a \le b \oplus c$ implies $a \le b_1 \oplus c_1$ where $b_1, c_1 \in I$ and $b_1 \le b$, $c_1 \le c$;

(R2) $i \in I$, $i \le a$ and $a \ominus i \perp b$ implies $\exists\, j \in I$, $j \le b$ and $b \ominus j \perp a$.

The importance of Riesz ideals is given by the fact that the quotient P/I of a generalized effect algebra P with respect to a Riesz ideal is again a generalized effect algebra (for more details see [10], [5], [3], [18]).

Proposition 3.4. *The set $\mathcal{F}$ is an ideal in $\mathcal{M}(\mathcal{A})$.*

P r o o f. At first, it is obvious that $\mathcal{F}$ is an order ideal, because if $f \in \mathcal{F}$ and $g \preceq f$ then $\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$, so that $g \in \mathcal{F}$. As for the second property of an ideal, if $f, g \in \mathcal{F}$ and $f \perp g$ then since $\mathrm{supp}(f \oplus g) = \mathrm{supp}(f) \cup \mathrm{supp}(g)$, clearly $f \oplus g \in \mathcal{F}$. □

Theorem 3.5. *The set $\mathcal{F}$ is a Riesz ideal in $\mathcal{M}(\mathcal{A})$.*

Proof. We have to prove:

(R1) $f \in \mathcal{F}$, $g, h \in \mathcal{M}(\mathcal{A})$, $f \preceq g \oplus h$ implies that there are $g_1 \preceq g$, $h_1 \preceq h$ such that $g_1, h_1 \in \mathcal{F}$ and $f \preceq g_1 \oplus h_1$,

(R2a) $g, h \in \mathcal{M}(\mathcal{A})$, $f \in \mathcal{F}$, $h \preceq g$, $g \ominus h \perp f$ implies that there is $f_1 \in \mathcal{F}$ such that $f_1 \preceq h$, $(g \ominus f_1) \perp f$.

We notice that condition (R2a) is equivalent with (R2), which is in the original definition of a Riesz ideal (see [5]), what was proved in [13]. While (R1) follows by Theorem 3.3, we proceed immediately to (R2a):

Let $g, h \in \mathcal{M}(\mathcal{A})$, $f \in \mathcal{F}$, $h \preceq g$ and $g \ominus h \perp f$. We put $f_1 := g \cdot \chi_{\mathrm{supp}(f)}$ and show that it has required properties. At first, if $\omega \in \mathrm{supp}(f_1)$, then obviously $\omega \in \mathrm{supp}(f)$ and $\omega \in \mathrm{supp}(g)$. But since $g \ominus h \perp f$, this implies also $\omega \in \mathrm{supp}(h)$. Therefore we have $f_1 \preceq h$. In fact, we have $\mathrm{supp}(f_1) = \mathrm{supp}(f) \cap \mathrm{supp}(g)$, which already implies that $g \ominus f_1 \perp f$. Finally, it is clear from the definition of f_1 ($\mathrm{supp}(f_1) \subseteq \mathrm{supp}(f)$), that $f_1 \in \mathcal{F}$, which ends the proof. □

4. The quantum case

In this section, $\mathcal{S}(H)$ will denote the set of bounded self-adjoint operators on a complex Hilbert space H, and the set of orthogonal projections on H will be denoted by $\mathcal{P}(H)$. Usually, $\mathcal{P}(H)$ is interpreted as the set of events and $\mathcal{S}(H)$ as the set of bounded observables (measurable physical quantities) for a quantum system. If $A \in \mathcal{S}(H)$ and $P^A(\Delta)$, $\Delta \in \mathcal{B}(\mathbb{R})$, is the spectral measure for A, then $P^A(\Delta)$ is interpreted as the event that A admits a value in Δ. If ρ is a density operator on H (i.e., positive with trace 1), then ρ corresponds to a state of the system and $\mathrm{tr}(\rho P^A(\Delta))$ is interpreted as the probability that A has a value in Δ in the state ρ.

For $P, Q \in \mathcal{P}(H)$, we have $P \perp Q$ if $P + Q \leq I$, equivalently, if $PQ = 0$. In [9], the latter definition is extended to elements of $\mathcal{S}(H)$ by defining $A \perp B$ if $AB = 0$, in which case $A \oplus B := A + B$. In agreement with [9], we denote the closure of the range of A by $\overline{\mathrm{ran}}(A)$ and the projection on $\overline{\mathrm{ran}}(A)$ by P_A. By $\mathrm{null}(A)$ we denote the kernel of A. The proof of the next lemma ([9, Lemma 1]) is straightforward.

Lemma 4.1. *For $A, B \in \mathcal{S}(H)$, the following statements are equivalent.*

(i) $A \perp B$.

(ii) $\overline{\mathrm{ran}}(A) \subseteq \mathrm{null}(B)$.

(iii) $\overline{\mathrm{ran}}(B) \subseteq \mathrm{null}(A)$.

(iv) $P_A P_B = 0$.

(v) $\overline{\mathrm{ran}}(A) \perp \overline{\mathrm{ran}}(B)$.

By [9, Theorem 4.2], the structure $(\mathcal{S}(H), \oplus, 0)$ is a generalized orthoalgebra. The partial order is then defined by $A \preceq B$ if there is a $C \in \mathcal{S}(H)$ with $A \perp C$ and $A \oplus C = B$.

The following characterizations of the partial order were proved in [9, Lemma 4.3], [9, Theorem 4.6].

Proposition 4.2. *For $A, B \in \mathcal{S}(H)$, the following statements are equivalent.*

(i) $A \preceq B$.

(ii) $Ax = Bx$ *for all* $x \in \overline{\mathrm{ran}}(A)$.

(iii) $A = BP_A$.

(iv) $AB = A^2$.

(v) $P^A(\Delta) \le P^B(\Delta)$ *for all* $\Delta \in \mathcal{B}(\mathbb{R})$ *with* $0 \notin \Delta$.

By [9, Corollary 4.5], every $A \in \mathcal{S}(H)$ is principal. This implies the following statement, which in turn implies that the unitization of $(\mathcal{S}(H), \oplus, 0)$ is an orthomodular poset.

Theorem 4.3. *The structure $(\mathcal{S}(H), \oplus, 0)$ is a weak generalized orthomodular poset (WGOMP).*

Proof. We have to prove

(i) if $A \perp B$, then $A \oplus B = A \vee B$,

(ii) $A \perp B$, $B \perp C$, $C \perp A$ imply $A \oplus B \perp C$.

To prove (i), observe that $A, B \preceq A \oplus B$, and assume that for a $C \in \mathcal{S}(H)$, $A, B \preceq C$. Since C is principal, it follows that $A \oplus B \preceq C$, hence $A \oplus B$ is the least upper bound of A and B.

To prove (ii), observe that $AC = 0 = BC$ implies $(A+B)C = AC + BC = 0$, hence $A \oplus B \perp C$. □

Remark 1.

1. We do not know if $(\mathcal{S}(H), \oplus, 0)$ is a GOMP.

2. In analogy with the classical case, we may consider the set $\mathcal{F} := \{A \in \mathcal{S}(H) : \dim(\mathrm{ran}(A)) < \infty\}$. It can be easily seen that $\mathcal{F}$ is an ideal in $\mathcal{S}(H)$. We do not know if it is a Riesz ideal.

In what follows, we need the following lemma.

Lemma 4.4. *Let $(P_\nu)_\nu$ be an arbitrary set of projections on H, and let $P = \bigvee_\nu P_\nu$ in $\mathcal{P}(H)$. If $B \in \mathcal{S}(H)$ is such that $B \le I$ and $P_\nu \le B$ for all ν, then $P \le B$.*

Proof. Let us denote $M_\nu = \mathrm{ran}(P_\nu)$. We have $\|(I-B)^{1/2}x\|^2 \le \|P_\nu^\perp x\|^2$, for every $x \in H$ and all ν. Therefore, if $x \in \bigcup M_\nu$, $(I-B)^{1/2}x = 0$, hence $(I-B)x = 0$ holds for every $x \in \overline{\bigcup M_\nu}$. It follows that $I-B$ reduces $\overline{\bigcup M_\nu}$, hence also $(\overline{\bigcup M_\nu})^\perp$, and if $x \in (\overline{\bigcup M_\nu})^\perp = \bigcap (M_\nu)^\perp$, then $\langle (I-B)x, x\rangle \le \|x\|^2$. This entails that $\langle (I-B)x, x\rangle \le \|P_{\bigcap (M_\nu)^\perp} x\|^2 = \|\bigwedge P_\nu^\perp x\|^2 = \|P^\perp x\|^2$. Therefore $\langle Px, x\rangle \le \langle Bx, x\rangle$ for all $x \in H$. □

The following theorem extends [9, Theorem 4.8].

Theorem 4.5. *Let $(A_\alpha)_\alpha$ be a net of elements of $\mathcal{S}(H)$ (that is, for every α_1, α_2 there is β such that $A_{\alpha_1}, A_{\alpha_2} \preceq A_\beta$) such that for every α: $A_\alpha \preceq B$, $B \in \mathcal{S}(H)$. Then $A = \bigvee A_\alpha$ exists in $(\mathcal{S}(H), \preceq)$ and $A = \lim A_\alpha$ in the strong operator topology.*

P r o o f. Similarly as in the proof of [9, Theorem 4.8], we obtain that $A_\alpha \preceq A_\beta$ implies $P_{A_\alpha} \leq P_{A_\beta}$. Therefore $(P_{A_\alpha})_\alpha$ is a net of projections. According to Vigier's theorem (see e.g. [20, Lemma 1]), the supremum $P = \bigvee_\alpha P_{A_\alpha}$ (in $\mathcal{S}(H)$) exists, and $P = \lim P_{A_\alpha}$ in the strong operator topology. By Lemma 4.4, P coincides with the supremum of $(P_{A_\alpha})_\alpha$ in the complete lattice $\mathcal{P}(H)$, hence P is a projection. Since $A_\alpha \preceq B$ for every α, we have $A_\alpha = BP_{A_\alpha} = P_{A_\alpha}B$, and $P = \lim P_{A_\alpha}$ in strong operator topology, together with the one-sided continuity of product, implies that $BP = PB$. Proceeding similarly as in the proof of [9, Theorem 4.8], define the operator $A \in \mathcal{S}(H)$ by $A = BP$. Since

$$A_\alpha = BP_{A_\alpha} = BPP_{A_\alpha} = AP_{A_\alpha},$$

we conclude by Proposition 4.2 that $A_\alpha \preceq A$. Suppose $A_\alpha \preceq C$ for all α, where $C \in \mathcal{S}(H)$. Then $CP_{A_\alpha} = A_\alpha = BP_{A_\alpha}$ so that $CP = BP = A$. Since $C(CP) = C^2P^2 = (CP)^2$, by Proposition 4.2, $A = CP \preceq C$. Hence $A = \bigvee A_\alpha$. Since $A_\alpha = AP_{A_\alpha}$, we conclude that $\lim A_\alpha = A$ in the strong operator topology. □

In [9], it was proved that $(\mathcal{S}(H), \preceq)$ is a near lattice in the sense that for $A, B \in \mathcal{S}(H)$, $A \wedge B$ and $A \vee B$ exist in $\mathcal{S}(H)$ if there is $C \in \mathcal{S}(H)$ with $A, B \preceq C$. By [9, Theorem 4.17], if $\dim H < \infty$, then $A \wedge B$ exists for every $A, B \in \mathcal{S}(H)$. In the next theorem, we extend the latter result to every Hilbert space H and to arbitrary subsets of elements.

Corollary 4.6.

(i) *For every family $(A_\lambda)_\lambda \subseteq \mathcal{S}(H)$ and $B \in \mathcal{S}(H)$ such that $A_\lambda \preceq B$ for all λ, the supremum $\bigvee_\lambda A_\lambda$ exists.*

(ii) *For an arbitrary family $(A_\lambda)_\lambda \subseteq \mathcal{S}(H)$ the infimum $\bigwedge_\lambda A_\lambda$ exists.*

P r o o f.

(i) For any finite subfamily F of $(A_\lambda)_\lambda$ the supremum exists by [9, Corollary 4.13]. The finite suprema form a net in a natural way (with respect to $\preceq$) each element of which is above bounded by B. Applying Theorem 4.5 we obtain that the supremum of this net exists, and it is clearly also the supremum of $(A_\lambda)_\lambda$.

(ii) Consider the family of all lower bounds of $(A_\lambda)_\lambda$. By (i), supremum of this family exists, and it is the greatest lower bound of $(A_\lambda)_\lambda$. □

Notice that if $A \in \mathcal{S}(H)$ is invertible, then $P_A = I$, therefore $A \preceq B$ implies $A = BI = B$.

In conclusion we have the following statement.

Corollary 4.7. *The infimum $A \wedge B$ of any two elements $A, B \in \mathcal{S}(H)$ exists, while the supremum $A \vee B$ exists if and only if there is $C \in \mathcal{S}(H)$ with $A, B \preceq C$.*

5. Compatibility

We recall that two elements a, b in a (generalized) effect algebra P are called (Mackey) compatible (written $a \leftrightarrow b$) if there are elements $a_1, b_1, c \in P$ such that $a_1 \oplus b_1 \oplus c$ is defined, and $a = a_1 \oplus c$, $b = b_1 \oplus c$. If Q is a subset of P, we say that a, b are compatible in Q if a_1, b_1, c belong to Q. Notice that if E is the unitization of P, then $a, b \in P$ are compatible in E if and only if they are compatible in P. If P is a WGOMP, so that E is an OMP, then $c = a \wedge b$, and $a_1 \oplus b_1 \oplus c = a_1 \vee b_1 \vee c = a \vee b$. Consequently, any two compatible observables in a WGOMP have a supremum.

Proposition 5.1. *The unitization of the system $(\mathcal{M}(\mathcal{A}), \oplus, 0)$ is not a Boolean algebra.*

Proof. Let $f, g \in \mathcal{M}(\mathcal{A})$ be such that $f(\omega) \neq 0$ for all $\omega \in \Omega$, and $g \not\preceq f$. Then $f \vee g$ does not exist in $\mathcal{M}(\mathcal{A})$. Indeed, if $f \preceq h$, then $f(\omega) = h(\omega)$ for all $\omega \in \operatorname{supp}(f)$, hence $h = f$. Therefore f and g have no upper bound in $\mathcal{M}(\mathcal{A})$, consequently, $f \leftrightarrow g$ does hot hold. Since an OMP is a Boolean algebra iff all pairs of elements are compatible, we conclude that the unitization of $(\mathcal{M}(\mathcal{A}), \oplus, 0)$ is not a Boolean algebra. □

We remark that since an OMP with the Riesz decomposition properties is a Boolean algebra, the Riesz decomposition property is not satisfied in the unitization.

In the next theorem, we formulate a necessary and sufficient condition for the compatibility of two elements in $\mathcal{S}(H)$.

Theorem 5.2. *In $(\mathcal{S}(H), \oplus, 0)$ the following statements for $A, B \in \mathcal{S}(H)$ are equivalent.*

(i) $A \leftrightarrow B$.

(ii) $AB = (A \wedge B)^2$.

Proof.

(i) implies (ii): If $A \leftrightarrow B$, then A and B are compatible in the unitization of $\mathcal{S}(H)$, which is an OMP. Therefore we may write $A = (A \ominus (A \wedge B)) \oplus A \wedge B$, $B = (B \ominus (A \wedge B)) \oplus A \wedge B$, and $(A \ominus (A \wedge B)) \oplus (A \wedge B) \oplus (B \ominus (A \wedge B))$ is defined. It follows that $A \perp (B \ominus (A \wedge B))$, hence $AB - A(A \wedge B) = 0$. From $A \wedge B \preceq A$ we have $A(A \wedge B) = (A \wedge B)^2$.

(ii) implies (i) can be proved by reversing the order of the reasoning. □

Theorem 5.3. *If H is a finite dimensional Hilbert space, then $A, B \in \mathcal{S}(H)$ with $A \wedge B \neq 0$ are Mackey compatible if and only if their spectral resolutions have the following form:*

$$A = \sum_{i=1}^{t} \lambda_i P_i + \sum_{i=t+1}^{r-1} \nu_i P_i + 0 \cdot P_r, \tag{1}$$

$$B = \sum_{j=1}^{t} \lambda_j Q_j + \sum_{j=t+1}^{s-1} \mu_j Q_j + 0 \cdot Q_s, \tag{2}$$

where λ's, ν's and μ's are nonzero eigenvalues, and the following conditions are satisfied:

$$P_i \leq Q_s + Q_i, \quad i = 1, 2, \dots, t; \qquad P_i \leq Q_s, \quad i = t+1, \dots, r-1;$$
$$Q_j \leq P_r + P_j, \quad j = 1, 2, \dots, t; \qquad Q_j \leq P_r, \quad j = t+1, \dots, s-1.$$

Proof. According to [9, Theorem 4.17], operators A and B have a nonzero lower bound iff they are of the form (1) and (2), and $D = A \wedge B = \sum_{i=1}^{t} \lambda_i P_i \wedge Q_i$. Now the condition $AB = D^2$ yields $P_i Q_j = Q_j P_i$ for all i, j, and the rest follows by a routine computation. $\square$

Notice that if $A \wedge B = 0$, then A, B are Mackey compatible iff $AB = 0$, i.e., $A \perp B$. Moreover, if in the above theorem, 0 is not an eigenvalue of A, then $A \leftrightarrow B$ iff $B \preceq A$. Consequently, if none of A, B has zero eigenvalue, then they are compatible iff they are equal.

It is well known that $\mathcal{S}(H)$ can be covered by maximal families of mutually commuting elements. If H is separable, then every such family can be represented by the family of all real-valued bounded measurable functions $\mathcal{M}(\mathcal{A})$ on a measure space $(\Omega, \mathcal{A}, \mu)$. It is easily seen that every such $\mathcal{M}(\mathcal{A})$ is a subalgebra of $\mathcal{S}(H)$ in the sense that $0 \in \mathcal{M}(\mathcal{A})$, and if $A, B \in \mathcal{M}(\mathcal{A})$ with $A \perp B$, then $A \oplus B \in \mathcal{M}(\mathcal{A})$. Hence $\mathcal{S}(H)$ can be covered by subalgebras satisfying RDP, but these subalgebras do not belong to blocks of the unitization-OMP of $\mathcal{S}(H)$.

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