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HOW TO PLAY METASQUARES

J. W. Sander

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ABSTRACT. The rules of the game MetaSquares as well as computational results suggest to follow a "lattice" strategy. This strategy is presented, and by counting lattice points it is shown to be essentially best possible.

1. Introduction

MetaSquares, a game invented by Scott Kim, was originally implemented by AOL and soon found a large community of players worldwide. The object of the game, played by two people on a chess-board, is to score 150 points and to defeat the opponent by a margin of at least 15 points. The two players (white and black) cover empty positions on the board by alternately putting down stones of their colour and score points by making squares. More precisely, each player tries to occupy the four corners of a square (for example: C2-F3-E6-B5, using chess labeling). The number of points for such a uni-coloured square σ is determined by the size of the so-called "bounding box" of σ, i.e. the smallest orthogonally-aligned square that σ fits into (the corners of the bounding box in our example are B2-F2-F6-B6; hence the bounding box is a (5×5)-square of value 25).

A lot of information on MetaSquares can be found on the homepage of Long [7]. In particular, some strategies for playing MetaSquares are described, however only from a vague heuristical point of view. In the language of game theory, MetaSquares is a finite two-person zero-sum game and standard results (cf. [6]) can be applied. They show for example that one player has a strategy which guarantees him at least a draw against all possible strategies of the other player (this is true for chess as well). Our object, however, is to illustrate an explicit strategy and provide mathematical evidence for it.

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The more systematical approach by Makoui (cf. [8] and [9]) reports on recent computational investigations concerning the optimal positioning of a fixed number m of one-coloured stones for particular values of m. Our paper provides a mathematical explanation for the phenomena which became evident there. For this reason we consider the following slightly more general problem:
Given an $(n \times n)$-board and m stones, which position of these stones maximizes the number of points scored according to the MetaSquares rules?

We present a strategy that will be proved to be essentially best possible and suggest to call it the "lattice strategy".

Given $X \subseteq \mathbb{Z}^2$, we denote by $S(X)$ the set of all squares whose four corners lie in X. Let $\mu\colon S(X) \to \mathbb{R}_{\geq 0}$ be a function invariant with respect to translations, i.e. $\mu(\sigma + \vec{v}) = \mu(\sigma)$ for all $\vec{v} \in \mathbb{Z}^2$ with $\sigma, \sigma + \vec{v} \in S(X)$. As a natural example, we could define $\mu(\sigma)$ to be the area of the square σ. MetaSquares uses instead the area of the bounding box, i.e. the smallest square containing σ which has edges parallel with the coordinate axes. We call this measure μ_{MSq}, having the following formal definition: Each $\sigma \in S(X)$ has the four vertices $\vec{x}, \vec{x}+\vec{u}, \vec{x}+\vec{u}_\perp, \vec{x}+\vec{u}+\vec{u}_\perp$ for a unique pair of vectors $\vec{x}, \vec{u} \in \mathbb{Z}^2$ with $\vec{x} := (x_1, x_2)$, $\vec{u} := (u_1, u_2)$, $\vec{u}_\perp := (-u_2, u_1)$ and $u_1 > 0$, $u_2 \geq 0$. Then

$$\mu_{MSq}(\sigma) := (u_1 + u_2 + 1)^2. \tag{1}$$

For a positive integer n, let $Q_n := \{0, 1, 2, \ldots, n-1\}^2 \subseteq \mathbb{Z}^2$ (this represents the $(n \times n)$-board). Then $S_n(X)$ shall be the set of all squares in $S(X)$ whose four corners lie in Q_n. We define

$$G(X, \mu, n) := \sum_{\sigma \in S_n(X)} \mu(\sigma)$$

and observe that $G(X, \mu_{MSq}, n)$ counts exactly the points a MetaSquares player gets for putting down his stones at the positions $\vec{x} \in X \cap Q_n$ of an $(n \times n)$-board.

We shall prove that, given a fixed number of stones, an optimal MetaSquares strategy is to choose X as the points of a suitable lattice. Let $\Gamma := \mathbb{Z}\vec{w} + \mathbb{Z}\vec{w}_\perp$ with $\vec{w} = (w_1, w_2) \in \mathbb{Z}^2 \setminus \{\vec{0}\}$ and $\vec{w}_\perp := (-w_2, w_1)$; hence Γ is a quadratic sublattice of $\mathbb{Z}^2$ with determinant $d(\Gamma) = |\vec{w}|^2 = w_1^2 + w_2^2$. We may clearly assume without loss of generality that $w_1 > 0$ and $w_2 \geq 0$, because each quadratic sublattice of $\mathbb{Z}^2$ can be generated by a vector $\vec{w}$ satisfying these conditions. A function $\mu\colon S(\Gamma) \to \mathbb{R}_{\geq 0}$ is invariant with respect to translations if and only if $\mu(\sigma + \vec{v}) = \mu(\sigma)$ for all $\sigma \in S$ and all $\vec{v} \in \Gamma$.

Our main tool will be an asymptotic formula for $G(\Gamma, \mu, n)$. We define

$$\Gamma^+ := \{\vec{v} \in \Gamma :\ v_1 > 0,\ \ v_2 \geq 0\},$$

and for a given vector $\vec{u} \in \Gamma^+$ we denote by $\sigma_{\vec{0}}(\vec{u})$ the square with vertices $\vec{0}, \vec{u}, \vec{u}_\perp, \vec{u} + \vec{u}_\perp$.

Theorem 1. *Let* $\Gamma := \mathbb{Z}\vec{w} + \mathbb{Z}\vec{w}_\perp$ *for some integers* $w_1 > 0$, $w_2 \geq 0$, *and define* $W := d(\Gamma) = w_1^2 + w_2^2$. *For each positive integer* n *and each function* $\mu\colon S(\Gamma) \to \mathbb{R}_{\geq 0}$ *invariant with respect to translations, we have*

$$G(\Gamma, \mu, n) = \sum_{\substack{\vec{u} \in \Gamma^+ \cap Q_n \\ u_1 + u_2 \leq n-1}} \mu(\sigma_{\vec{0}}(\vec{u})) \left(\frac{1}{W}(n - u_1 - u_2)^2 + R_n(\vec{w}, \vec{u}) \right)$$

for an error term $R_n(\vec{w}, \vec{u})$ *satisfying*

$$|R_n(\vec{w}, \vec{u})| \leq \frac{4\sqrt{2} + 2}{\sqrt{W}}(n - u_1 - u_2) + 4. \tag{2}$$

The application of Theorem 1 for $\mu = \mu_{MSq}$ leads to an asymptotic formula for the score obtained by playing MetaSquares according to the lattice strategy.

Theorem 2. *Let* $\Gamma := \mathbb{Z}\vec{w} + \mathbb{Z}\vec{w}_\perp$ *for some integers* $w_1 > 0$, $w_2 \geq 0$. *Define* $W := d(\Gamma) = w_1^2 + w_2^2$ *and* $g := \gcd(w_1 + w_2, w_1 - w_2)$. *Let* n *be a positive integer.*

(i) *If* $n \leq w_1 + w_2$, *then* $G(\Gamma, \mu_{MSq}, n) = 0$.

(ii) *For* $n > w_1 + w_2$, *we have*

$$G(\Gamma, \mu_{MSq}, n) = \frac{1}{60W^2}n^6 + \frac{C(n)}{gW}n^5,$$

where $C(n)$ *satisfies* $|C(n)| \leq 41$ *for all* n *and* $|C(n)| \leq \frac{1}{2}$ *for* $n > 325^2$.

Given an $(n \times n)$-board and $m \leq n^2$ stones, the next result and its proof show how to play MetaSquares.

Theorem 3. *Let* n *and* $m \leq n^2$ *be positive integers. Then there is a quadratic lattice* $\Gamma_{MSq} \subseteq \mathbb{Z}^2$ *with* $|\Gamma_{MSq} \cap Q_n| \leq m$, *and there are absolutely bounded numbers* $C_1(n,m)$, $C_2(n,m)$ *such that*

$$G(\Gamma_{MSq}, \mu_{MSq}, n) = \frac{1}{60}m^2n^2 - C_1(n,m)m^{11/4}n^{1/2} + C_2(n,m)mn^3.$$

For $m \geq 100$ *the numbers* C_1 *and* C_2 *satisfy* $0 < C_1(n,m) < \frac{2}{5}$ *and* $|C_2(n,m)| < |C(n)| + \frac{3}{10}$ *with* $C(n)$ *as in* Theorem 2.

Theorem 3 gives an asymptotic formula only for $n = o(m)$. It is clear from the proof of Theorem 3 that we could also get explicit bounds for C_1 and C_2 in case $m < 100$. We like to point out that the use of the computer algebra package MAPLE in the proofs of Theorems 2 and 3 could easily be avoided if we did not care for sharp explicit bounds for the numbers $C(n)$, $C_1(n,m)$, $C_2(n,m)$.

The strategy suggested by Theorem 3 for placing $m \leq n^2$ stones on an $(n \times n)$-board is to look for the smallest integer $W = w_1^2 + w_2^2$ with $w_1 > 0$, $w_2 \geq 0$ such that $\Gamma_{MSq} := \mathbb{Z}\vec{w} + \mathbb{Z}\vec{w}_\perp$ satisfies $|\Gamma_{MSq} \cap Q_n| \leq m$ and cover

$\Gamma_{MSq} \cap Q_n$. Observe that $W \sim n^2/m$, which means $W/(n^2/m)$ tends to 1 for large n and m.

The following upper bound shows that the order of the main term in Theorem 3 is best possible.

Theorem 4. *Let n be a positive integer and let $X \subseteq Q_n$ with $|X| = m$. Then*

$$G(X, \mu_{MSq}, n) \leq \frac{1}{2}(m^2 - m)n^2.$$

From Theorem 3 and Theorem 4 we immediately obtain the following consequence, which shows that the lattice strategy is essentially best possible.

Corollary. *Let n and $n < m \leq n^2$ be positive integers. Then*

$$\max_{\substack{X \subseteq Q_n \\ |X| = m}} G(X, \mu_{MSq}, n) \asymp m^2 n^2,$$

where $f(x) \asymp g(x)$ means that $|f(x)/g(x)|$ and $|g(x)/f(x)|$ are both bounded by an absolute constant.

2. Counting lattice points

For a given set $X \subseteq \mathbb{Z}^2$ we extend $S(X)$ to $S^*(X) := S(X) \cup X$, i.e., we consider the points of X as squares with side length 0. For a given function $\mu\colon S(X) \to \mathbb{R}_{\geq 0}$ invariant with respect to translations, we define $\mu^*\colon S^*(X) \to \mathbb{R}_{\geq 0}$ by setting

$$\mu^*(\sigma) = \begin{cases} \mu(\sigma) & \text{for } \sigma \in S(X), \\ 0 & \text{for } \sigma \in X. \end{cases}$$

For each $\sigma \in S^*(\Gamma)$ there is a unique

$$\vec{u} = (u_1, u_2) \in \Gamma^* := \{\vec{v} \in \Gamma : \ v_1 > 0, v_2 \geq 0 \text{ or } v_1 = v_2 = 0\}$$

such that the square $\sigma_{\vec{0}}(\vec{u})$ with vertices $\vec{0}$, $\vec{u}$, $\vec{u}_\perp$, $\vec{u}+\vec{u}_\perp$ is a translation of σ. We define $T\colon S^*(\Gamma) \to S^*(\Gamma)$ by $T(\sigma) := \sigma_{\vec{0}}(\vec{u})$. Since μ^* is invariant with respect to translations, we clearly have $\mu^*(T(\sigma)) = \mu^*(\sigma)$. Consequently

$$G(\Gamma, \mu, n) = \sum_{\vec{u} \in \Gamma^+ \cap Q_n} \mu(\sigma_{\vec{0}}(\vec{u})) N_n(\vec{u}) = \sum_{\vec{u} \in \Gamma^* \cap Q_n} \mu^*(\sigma_{\vec{0}}(\vec{u})) N_n(\vec{u}), \tag{3}$$

where

$$N_n(\vec{u}) := |T^{-1}(\sigma_{\vec{0}}(\vec{u})) \cap S_n^*(\Gamma)|$$

and $S_n^*(X) := S^*(X) \cap Q_n$. It is obvious that

$$N_n(\vec{u}) = 0 \qquad (u_1 + u_2 \geq n). \tag{4}$$

Proposition. *Let* $\Gamma := \mathbb{Z}\vec{w} + \mathbb{Z}\vec{w}_\perp$ *for some integers* $w_1 > 0$, $w_2 \geq 0$, *and define* $W := d(\Gamma) = w_1^2 + w_2^2$. *Let* n *be a positive integer, and let* $\vec{u} = (u_1, u_2) \in \Gamma^* \cap Q_n$ *satisfy* $u_1 + u_2 \leq n - 1$. *Then*

$$\left| N_n(\vec{u}) - \frac{1}{W}(n - u_1 - u_2)^2 \right| \leq \frac{4\sqrt{2} + 2}{\sqrt{W}}(n - u_1 - u_2) + 4. \tag{5}$$

Proof. For a given $\vec{x} \in \Gamma$ let $\sigma_{\vec{x}}(\vec{u})$ be the square with corners $\vec{x}$, $\vec{x} + \vec{u}$, $\vec{x} + \vec{u}_\perp$, $\vec{x} + \vec{u} + \vec{u}_\perp$. Then

$$N_n(\vec{u}) = \left| \left\{ \vec{x} \in \Gamma : \ \sigma_{\vec{x}}(\vec{u}) \in S_n^*(\Gamma) \right\} \right|.$$

By definition, the square $\sigma_{\vec{x}}(\vec{u})$ lies in $S_n^*(\Gamma)$ if and only if it has all four corners in Q_n, i.e. if the following conditions are satisfied:

$$\begin{array}{lll}
\text{(i)} & 0 \leq x_1 \leq n-1, & 0 \leq x_2 \leq n-1; \\
\text{(ii)} & 0 \leq x_1 + u_1 \leq n-1, & 0 \leq x_2 + u_2 \leq n-1; \\
\text{(iii)} & 0 \leq x_1 - u_2 \leq n-1, & 0 \leq x_2 + u_1 \leq n-1; \\
\text{(iv)} & 0 \leq x_1 - u_2 + u_1 \leq n-1, & 0 \leq x_2 + u_1 + u_2 \leq n-1.
\end{array}$$

Since $\vec{u} \in \Gamma^*$, this system of inequalities is equivalent with the following:

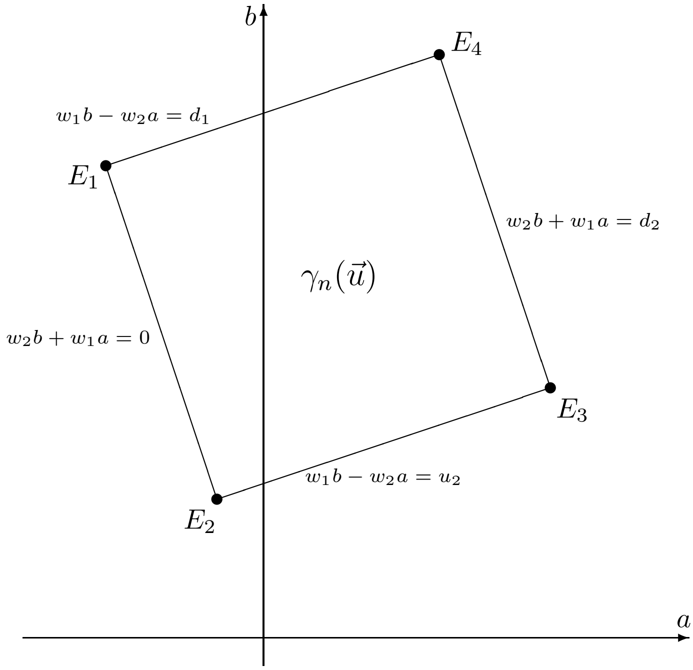
$$\left.\begin{array}{l}
x_2 \geq 0, \\
x_1 + u_1 \leq n - 1, \\
x_1 - u_2 \geq 0, \\
x_2 + u_1 + u_2 \leq n - 1.
\end{array}\right\} \tag{6}$$

We therefore have

$$\begin{aligned}
N_n(\vec{u}) &= \left| \left\{ \vec{x} \in \Gamma : \ \vec{x} \text{ satisfies (6)} \right\} \right| \\
&= \left| \left\{ (a,b) \in \mathbb{Z}^2 : \ \vec{x} := b\vec{w} + a\vec{w}_\perp \text{ satisfies (6)} \right\} \right| \qquad (7) \\
&= \left| \left\{ (a,b) \in \mathbb{Z}^2 : \ 0 \leq w_2 b + w_1 a \leq d_2, \ \ u_2 \leq w_1 b - w_2 a \leq d_1 \right\} \right|,
\end{aligned}$$

where $d_1 := n - 1 - u_1$ and $d_2 := n - 1 - u_1 - u_2$. By what we have just seen, $N_n(\vec{u})$ counts all the points with integer coordinates (a, b) in the quadrilateral $\gamma_n(\vec{u})$, say, with vertices

$$E_1 := \frac{d_1}{W}\vec{w}_\perp, \ \ E_2 := \frac{u_2}{W}\vec{w}_\perp, \ \ E_3 := \frac{1}{W}(d_2\vec{w} + u_2\vec{w}_\perp), \ \ E_4 := \frac{1}{W}(d_2\vec{w} + d_1\vec{w}_\perp).$$



It is easy to verify that $\gamma_n(\vec{u})$ is in fact a square with side length $d_2/\sqrt{W}$. Hence the area of $\gamma_n(\vec{u})$ equals d_2^2/W. By a standard method in lattice point theory (cf. [2], [3], [4]), the number of integer points in $\gamma_n(\vec{u})$ is counted as follows: We assign to each integer point $\vec{x} \in \mathbb{Z}^2$ the one-by-one square with edges parallel to the coordinate axes and centered at $\vec{x}$. This dissects the plane into unit squares, and we have a 1-1 mapping between these squares and the integer points. Therefore the area of $\gamma_n(\vec{u})$ is approximately equal to the number of integer points in $\gamma_n(\vec{u})$. The error is apparently bounded by the length of the boundary of $\gamma_n(\vec{u})$; more precisely, by the area of the strip along the edges of $\gamma_n(\vec{u})$ of width $\sqrt{2}$, where $\sqrt{2}$ is the length of the diagonal of a unit square. This means

$$\left| N_n(\vec{u}) - \frac{d_2^2}{W} \right| \le \left(\frac{d_2}{\sqrt{W}} + \sqrt{2} \right)^2 - \left(\frac{d_2}{\sqrt{W}} - \sqrt{2} \right)^2 = 4\sqrt{2}\frac{d_2}{\sqrt{W}}, \tag{8}$$

but we have to assume that $d_2/\sqrt{W} \ge \sqrt{2}$ (otherwise the corresponding square ring is in fact a complete square). In this case it follows from (8) that

$$\begin{aligned} \left| N_n(\vec{u}) - \frac{(d_2+1)^2}{W} \right| &= \left| N_n(\vec{u}) - \frac{d_2^2}{W} - \frac{2d_2+1}{W} \right| \\ &\le 4\sqrt{2}\frac{d_2}{\sqrt{W}} + \frac{2d_2+1}{W} \le \frac{4\sqrt{2}+2}{\sqrt{W}}(d_2+1), \end{aligned}$$

which proves (5) in this situation. Now we consider the case $d_2/\sqrt{W} < \sqrt{2}$. A little thought reveals that a square of side length $d_2/\sqrt{W}$ then contains at most four integer points, i.e. $N_n(\vec{u}) \le 4$. This implies

$$\begin{aligned}\left|N_n(\vec{u}) - \frac{(d_2+1)^2}{W}\right| &\le \max\left\{N_n(\vec{u}),\, \frac{(d_2+1)^2}{W}\right\}\\ &\le \max\left\{4,\, \frac{(d_2+1)^2}{W}\right\} \le \frac{4\sqrt{2}+2}{\sqrt{W}}(d_2+1)+4.\end{aligned}$$

Again this is what (5) asserts, and this completes the proof of the proposition. □

P r o o f o f T h e o r e m 1. By use of (3) and (4), Theorem 1 is an immediate consequence of the proposition. □

3. The special case: MetaSquares

We now consider the special measure μ_{MSq} used in MetaSquares as defined in (1). Hence

$$\mu_{MSq}(\sigma_{\vec{0}}(\vec{u})) = (u_1+u_2+1)^2.$$

Before we are able to prove the asymptotic formula for $G(\Gamma, \mu_{MSq}, n)$, we need the following technical result.

Lemma. *Let $\Gamma := \mathbb{Z}\vec{w} + \mathbb{Z}\vec{w}_\perp$ for some integers $w_1 > 0$, $w_2 \ge 0$, and define $W := d(\Gamma) = w_1^2 + w_2^2$. Let n and $u \le n-1$ be positive integers. Then*

$$A(\Gamma, n, u) = |\{\vec{u} \in \Gamma^+ \cap Q_n :\ u_1+u_2 = u\}|$$

satisfies

$$\left|A(\Gamma, n, u) - \frac{gu}{W}\right| < 1$$

if $g := \gcd(w_1+w_2, w_1-w_2) \mid u$, and otherwise $A(\Gamma, n, u) = 0$.

P r o o f. We have $\vec{u} \in \Gamma$ if and only if $\vec{u} = x\vec{w} + y\vec{w}_\perp$ for some integers x, y. Hence

$$\begin{aligned}B(\Gamma, u) &:= \{\vec{u} \in \Gamma :\ u_1+u_2 = u\}\\ &= \big\{(x,y) \in \mathbb{Z}^2 :\ (w_1x - w_2y) + (w_2x + w_1y) = u\big\}.\end{aligned}$$

The Diophantine equation

$$(w_1+w_2)x + (w_1-w_2)y = u \tag{9}$$

has a solution x_0, y_0 if and only if $g \mid u$, and the set of all solutions is then given by

$$x = x_0 + \frac{w_1-w_2}{g}t, \quad y = y_0 - \frac{w_1+w_2}{g}t \qquad (t \in \mathbb{Z}) \tag{10}$$

(cf. [5], or any other introductory book to number theory). Consequently, $A(\Gamma, n, u) = 0$ for $g \nmid u$. If $g \mid u$ we have

$$\begin{aligned}
A(\Gamma, n, u) &= |\{\vec{u} \in \Gamma : \ u_1 + u_2 = u, \ 0 < u_1 \le n-1, \ 0 \le u_2 \le n-1\}| \\
&= |\{\vec{u} \in B(\Gamma, u) : \ 0 < u_1 \le n-1, \ 0 \le u_2 \le n-1\}| \\
&= |\{(x, y) \in \mathbb{Z}^2 : \ (x, y) \text{ satisfies (9)}, \ 0 < w_1 x - w_2 y \le n-1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le w_2 x + w_1 y \le n-1\}| \\
&= |\{t \in \mathbb{Z} : \ 0 < w_1 x - w_2 y \le n-1, \ 0 \le w_2 x + w_1 y \le n-1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } x, \, y \text{ as in (10)}\}| \\
&= \left|\left\{t \in \mathbb{Z} : \ c_1 < \tfrac{W}{g} t \le n-1+c_1, \ c_2 - (n-1) \le \tfrac{W}{g} t \le c_2\right\}\right|,
\end{aligned}$$

where $c_1 := w_2 y_0 - w_1 x_0$ and $c_2 := w_2 x_0 + w_1 y_0$. Since x_0, y_0 satisfy (9), we have by hypothesis of the lemma

$$c_2 - c_1 = (w_1 + w_2) x_0 + (w_1 - w_2) y_0 = u \le n - 1.$$

Thus $c_1 \ge c_2 - (n-1)$ and $c_2 \le n - 1 + c_1$. It follows that

$$\begin{aligned}
A(\Gamma, n, u) &= \left|\left\{t \in \mathbb{Z} : \ c_1 < \tfrac{W}{g} t \le c_2\right\}\right| \\
&= \left\lfloor \frac{g}{W} c_2 \right\rfloor - \left\lfloor \frac{g}{W} c_1 \right\rfloor \\
&= \frac{g}{W}(c_2 - c_1) + \theta = \frac{g}{W} u + \theta
\end{aligned}$$

for some $|\theta| < 1$. □

P r o o f o f T h e o r e m 2. By Theorem 1, we have

$$G(\Gamma, \mu_{MSq}, n) = \frac{1}{W} G_2 + \frac{4\sqrt{2} + 2}{\sqrt{W}} G_1 + 4 G_0 \tag{11}$$

with

$$G_j := \sum_{\substack{\vec{u} \in \Gamma^+ \cap Q_n \\ u_1 + u_2 \le n-1}} (u_1 + u_2 + 1)^2 (n - u_1 - u_2)^j.$$

Substituting $u := u_1 + u_2$, the lemma implies the existence of a function $\theta(u)$ with $|\theta(u)| < 1$ for all u and numbers θ_j with $|\theta_j| < 1$ such that for

$g := \gcd(w_1 + w_2, w_1 - w_2)$

$$
\begin{aligned}
G_j &= \sum_{u=1}^{n-1} (u+1)^2 (n-u)^j A(\Gamma, n, u) \\
&= \sum_{\substack{u=1\\ g|u}}^{n-1} (u+1)^2 (n-u)^j \left(\frac{gu}{W} + \theta(u) \right) \qquad (12) \\
&= \frac{g}{W} \sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2 (n-gv)^j gv + \theta_j \sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2 (n-gv)^j .
\end{aligned}
$$

It is well known that for positive integers k and m

$$
S_k(m) := \sum_{u=1}^{m} u^k = \frac{1}{k+1} m^{k+1} + \frac{1}{2} m^k + \sum_{j=2}^{k} \frac{B_j}{j} \binom{k}{j-1} m^{k-j+1},
$$

where the B_j are the Bernoulli numbers, in particular $B_2 = 1/6$, $B_4 = -1/30$ and $B_3 = B_5 = 0$. It follows that

$$
\begin{aligned}
S_1(m) &= \frac{1}{2}\left(m^2 + m\right), \\
S_2(m) &= \frac{1}{6}\left(2m^3 + 3m^2 + m\right), \\
S_3(m) &= \frac{1}{4}\left(m^4 + 2m^3 + m^2\right), \\
S_4(m) &= \frac{1}{30}\left(6m^5 + 15m^4 + 10m^3 - m\right), \\
S_5(m) &= \frac{1}{12}\left(2m^6 + 6m^5 + 5m^4 - m^2\right).
\end{aligned}
$$

Part (i) of the theorem follows trivially from (4). We may therefore assume

$$
n > w_1 + w_2. \qquad (13)
$$

We shall present a detailed computation of the G_j only for the main term G_2, while the other terms can be treated in a completely analogous fashion. We have with $m := \lfloor \frac{n-1}{g} \rfloor$

$$
\begin{aligned}
\sum_{v=1}^{m} (gv+1)^2 (n-gv)^2 gv &= g^5 S_5(m) - 2(n-1) g^4 S_4(m) + (n^2 - 4n + 1) g^3 S_3(m) \\
&\quad + 2(n^2 - n) g^2 S_2(m) + n^2 g S_1(m), \\
\sum_{v=1}^{m} (gv+1)^2 (n-gv)^2 &= g^4 S_4(m) - 2(n-1) g^3 S_3(m) + (n^2 - 4n + 1) g^2 S_2(m) \\
&\quad + 2(n^2 - n) g S_1(m) + n^2 m.
\end{aligned}
$$

Putting $m = \frac{n-1}{g} - \delta$ for a suitable $0 \le \delta < 1$, we obtain (admittedly by virtue of MAPLE)

$$\begin{aligned}
\sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2(n-gv)^2 gv &= \frac{1}{60g}n^6 + \frac{1}{15g}n^5 + \frac{1}{12g}n^4 + \delta_1 g^2 n^3, \\
\sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2(n-gv)^2 &= \frac{1}{30g}n^5 + \frac{1}{6g}n^4 + \frac{1}{3g}n^3 + \delta_2 g^2 n^2
\end{aligned}$$

for some numbers δ_1, δ_2 (depending on n), where we have used $g \le w_1 + w_2 < n$, which follows from (13). By simple, though tedious calculations, explicit bounds for δ_1, δ_2 can be found, namely $|\delta_1| \le 1$ and $|\delta_2| \le 5/3$.

Similarly we obtain

$$\begin{aligned}
\sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2(n-gv)gv &= \frac{1}{20g}n^5 + \frac{1}{6g}n^4 + \delta_3 g n^3, \\
\sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2(n-gv) &= \frac{1}{12g}n^4 + \frac{1}{3g}n^3 + \delta_4 g n^2, \\
\sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2 gv &= \frac{1}{4g}n^4 + \delta_5 n^3, \\
\sum_{v=1}^{\lfloor \frac{n-1}{g} \rfloor} (gv+1)^2 &= \frac{1}{3g}n^3 + \delta_6 n^2
\end{aligned}$$

with numbers δ_3, δ_4, δ_5, δ_6 satisfying $|\delta_3| \le 1$, $|\delta_4| \le 3/2$, $|\delta_5| \le 1$ and $|\delta_6| \le 4/3$. By (11) and (12), this implies

$$\begin{aligned}
& G(\Gamma, \mu_{MSq}, n) \\
= \;& \frac{1}{W}G_2 + \frac{4\sqrt{2}+2}{\sqrt{W}}G_1 + 4G_0 \\
= \;& \frac{1}{60W^2}n^6 + \left(\frac{1}{15W^2} + \frac{\theta_1}{30gW} + \frac{4\sqrt{2}+2}{20W^{3/2}} \right) n^5 \\
& + \left(\frac{1}{12W^2} + \frac{\theta_1}{6gW} + \frac{4\sqrt{2}+2}{6W^{3/2}} + \frac{(4\sqrt{2}+2)\theta_2}{12gW^{1/2}} + \frac{1}{W} \right) n^4
\end{aligned}$$

$$+\left(\frac{\delta_1 g^3}{W^2}+\frac{\theta_1}{3gW}+\frac{(4\sqrt{2}+2)\delta_3 g^2}{W^{3/2}}+\frac{(4\sqrt{2}+2)\theta_2}{3gW^{1/2}}+\frac{4\theta_5 g}{W}+\frac{4\theta_3}{3g}\right)n^3$$
$$+\left(\frac{\theta_1\delta_2 g^2}{W}+\frac{(4\sqrt{2}+2)\theta_2\delta_4 g}{W^{1/2}}+4\theta_3\delta_6\right)n^2.$$

By use of the bounds given for the θ_j and the δ_j, we get

$$\begin{aligned}
&\left|G(\Gamma,\mu_{MSq},n)-\frac{1}{60W^2}n^6\right| \\
&\qquad\le\left(\frac{1}{15W^2}+\frac{1}{30gW}+\frac{2\sqrt{2}+1}{10W^{3/2}}\right)n^5 \\
&\qquad\quad+\left(\frac{1}{12W^2}+\frac{1}{6gW}+\frac{4}{3W^{3/2}}+\frac{2}{3gW^{1/2}}+\frac{1}{W}\right)n^4 \\
&\qquad\quad+\left(\frac{g^3}{W^2}+\frac{1}{3gW}+\frac{8g^2}{W^{3/2}}+\frac{8}{3gW^{1/2}}+\frac{4g}{W}+\frac{4}{3g}\right)n^3 \\
&\qquad\quad+\left(\frac{5g^2}{3W}+\frac{12g}{W^{1/2}}+\frac{16}{3}\right)n^2.
\end{aligned}$$

Since $g=\gcd(w_1+w_2,w_1-w_2)\le|w_1-w_2|$, it follows that

$$g^2\le(w_1-w_2)^2=w_1^2-2w_1w_2+w_2^2\le W.$$

By (13), we also have

$$n^2>(w_1+w_2)^2=w_1^2+2w_1w_2+w_2^2\ge W.$$

Therefore

$$\frac{1}{15W^2}+\frac{1}{30gW}+\frac{2\sqrt{2}+1}{10W^{3/2}}\le\left(\frac{1}{15}+\frac{1}{30}+\frac{2\sqrt{2}+1}{10}\right)\frac{1}{gW}\le\left(\frac{1}{2}-\frac{1}{100}\right)\frac{1}{gW},$$

$$\begin{aligned}
\frac{1}{12W^2}+\frac{1}{6gW}+\frac{4}{3W^{3/2}}+\frac{2}{3gW^{1/2}}+\frac{1}{W}&\le\left(\frac{1}{12}+\frac{1}{6}+\frac{4}{3}+\frac{2}{3}+1\right)\frac{1}{gW^{1/2}} \\
&=\frac{39}{12gW^{1/2}},
\end{aligned}$$

$$\frac{g^3}{W^2}+\frac{1}{3gW}+\frac{8g^2}{W^{3/2}}+\frac{8}{3gW^{1/2}}+\frac{4g}{W}+\frac{4}{3g}\le\left(1+\frac{1}{3}+8+\frac{8}{3}+4+\frac{4}{3}\right)\frac{1}{g}=\frac{52}{3g},$$

$$\frac{5g^2}{3W}+\frac{12g}{W^{1/2}}+\frac{16}{3}\le\frac{5}{3}+12+\frac{16}{3}=19.$$

This yields

$$\begin{aligned}\left|G(\Gamma,\mu_{MSq},n)-\frac{1}{60W^2}n^6\right| &\le \left(\frac{1}{2}-\frac{1}{100}+\frac{39W^{1/2}}{12n}+\frac{52W}{3n^2}+\frac{19gW}{n^3}\right)\frac{1}{gW}n^5\\ &\le \left(\frac{1}{2}-\frac{1}{100}+\frac{39}{12n^{1/2}}+\frac{52}{3n}+\frac{19}{n^{3/2}}\right)\frac{1}{gW}n^5.\end{aligned}$$

Consequently,

$$\left|G(\Gamma,\mu_{MSq},n)-\frac{1}{60W^2}n^6\right| \le \begin{cases} \frac{41}{gW}n^5 & \text{for all } n,\\ \frac{1}{2gW}n^5 & \text{for all } n>(325)^2.\end{cases}$$

□

P r o o f o f T h e o r e m 3. For $m<100$ the formula is trivial, because then the last term on the right hand side is of greater order than the first and the second. We may therefore assume that $m\ge 100$. Let $m_1 := m-9\sqrt{m}$, hence $0<m_1<n^2$. By a result essentially due to B a m b a h and C h o w l a [1], which S h i u [10] recently improved with a two-sentence argument, there is an integer W in the interval

$$\frac{n^2}{m_1}\le W\le \frac{n^2}{m_1}+2\sqrt{2}\left(\frac{n^2}{m_1}\right)^{1/4}+1 \tag{14}$$

such that $W=w_1^2+w_2^2$ for suitable integers $w_1>0$ and $w_2\ge 0$. Define $\Gamma_{MSq}:=\mathbb{Z}\vec{w}+\mathbb{Z}\vec{w}_\perp$. The proposition used for $\vec{u}=\vec{0}$ implies

$$\left||\Gamma_{MSq}\cap Q_n|-\frac{n^2}{W}\right|\le (4\sqrt{2}+2)\frac{n}{\sqrt{W}}+4.$$

Since $m\ge 100$, it follows by (14) that

$$\begin{aligned}|\Gamma_{MSq}\cap Q_n| &\le \frac{n^2}{W}+(4\sqrt{2}+2)\frac{n}{\sqrt{W}}+4\le m_1+(4\sqrt{2}+2)\sqrt{m_1}+4\\ &\le m-9\sqrt{m}+(4\sqrt{2}+2)\sqrt{m}+4\le m.\end{aligned}$$

It remains to show the asymptotic formula. Inequality (14) implies that $(w_1+w_2)^2\le 2W<n^2$, because $m_1\ge 10$ for $m\ge 100$. Thus we have by

Theorem 2 and (14)

$$\begin{aligned}
G(\Gamma_{MSq}, \mu_{MSq}, n) &= \frac{1}{60W^2}n^6 + \frac{c_1}{W}n^5 \\
&= \frac{1}{60}n^6 \left(\frac{n^2}{m_1} + \theta_1 \left(\frac{n^2}{m_1} \right)^{1/4} \right)^{-2} + c_2 m_1 n^3 \\
&= \frac{1}{60}m_1^2 n^2 \left(1 + \theta_1 \frac{m_1^{3/4}}{n^{3/2}} \right)^{-2} + c_2 m_1 n^3 \\
&= \frac{1}{60}m_1^2 n^2 \left(1 + \theta_2 \frac{m_1^{3/4}}{n^{3/2}} \right)^{-1} + c_2 m_1 n^3,
\end{aligned}$$

where the numbers involved satisfy $0 \leq \theta_1 \leq 2\sqrt{2} + 1$ and $0 \leq \theta_2 \leq 2\theta_1 + \theta_1^2 \leq 8\sqrt{2} + 11$, since $m_1/n^2 \leq 1$, and $|c_2| \leq |c_1| \leq |C|$ with C as in Theorem 2. Since for any $\varepsilon > 0$

$$(1 + \theta_2\varepsilon)^{-1} = (1 - \theta_2\varepsilon) + \frac{\theta_2^2\varepsilon^2}{1 + \theta_2\varepsilon},$$

we obtain

$$\begin{aligned}
G(\Gamma_{MSq}, \mu_{MSq}, n) &= \frac{1}{60}m_1^2 n^2 \left(1 - \theta_3 \frac{m_1^{3/4}}{n^{3/2}} \right) + c_2 m_1 n^3 \\
&= \frac{1}{60}m_1^2 n^2 - \theta_4 m_1^{11/4} n^{1/2} + c_2 m_1 n^3
\end{aligned}$$

with $0 \leq \theta_3 \leq \theta_2$ and $0 \leq \theta_4 \leq \theta_2/60 \leq (8\sqrt{2} + 11)/60 < 2/5$. Since $m \geq 100$, we have $m_1 = \delta m$ for some positive $\delta < 1$. Consequently

$$\begin{aligned}
G(\Gamma_{MSq}, \mu_{MSq}, n) &= \frac{1}{60}\left(m - 9\sqrt{m}\right)^2 n^2 - \theta_4 (\delta m)^{11/4} n^{1/2} + c_2 \delta m n^3 \\
&= \frac{1}{60}m^2 n^2 - \lambda m^{3/2} n^2 - \theta_5 m^{11/4} n^{1/2} + c_3 m n^3 \\
&= \frac{1}{60}m^2 n^2 - \theta_5 m^{11/4} n^{1/2} + c_4 m n^3
\end{aligned}$$

with $|c_3| \leq |c_2|$, $0 \leq \lambda \leq 3/10$ and $|c_4| \leq |c_3| + \frac{3}{10}$, since $m \leq n^2$ and $m \geq 100$. This completes the proof of Theorem 3. $\square$

4. The upper bound

We shall prove Theorem 4. We define a mapping $\rho\colon S_n(X) \to X \times X$. For each $\sigma \in S_n(X)$ there is a unique pair of vectors $\vec{x}, \vec{u} \in \mathbb{Z}^2$ with $u_1 > 0$, $u_2 \geq 0$, such that $\vec{x}, \vec{x}+\vec{u}, \vec{x}+\vec{u}_\perp, \vec{x}+\vec{u}+\vec{u}_\perp$ are the vertices of σ. Then let $\rho(\sigma) := (\vec{x}, \vec{x}+\vec{u})$. It is completely obvious that ρ is an embedding. Consequently

$$|S_n(X)| \leq |X \times X| = |X|^2 = m^2.$$

A slight improvement can be obtained by simply observing that $(\vec{x}, \vec{x}) \notin \rho(S_n(X))$ and at most one of the two pairs $(\vec{x}, \vec{y})$ and $(\vec{y}, \vec{x})$ lies in $\rho(S_n(X))$ for any $\vec{x}, \vec{y} \in X$. Hence

$$|S_n(X)| \leq \frac{1}{2}(m^2 - m).$$

This implies

$$G(X, \mu_{MSq}, n) = \sum_{\sigma \in S_n(X)} \mu_{MSq}(\sigma) \leq n^2 \sum_{\sigma \in S_n(X)} 1 = n^2 |S_n(X)| \leq \frac{1}{2}(m^2 - m)n^2.$$

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Institut für Mathematik
Technische Universität Clausthal
Erzstr. 1
D–38678 Clausthal-Zellerfeld
GERMANY
E-mail: majsa@math.tu-clausthal.de