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DIFFERENT KINDS OF SUFFICIENCY IN THE GENERAL GAUSS-MARKOV MODEL

Andrzej Kornacki

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ABSTRACT. Sufficiency is one of the fundamental notions in mathematical statistics. In connection with the general linear Gauss-Markov model **GM** $(y, X\beta, \sigma^2 V)$, there are some modifications of this notion such as *linear sufficiency* (Baksalary and Kala, Drygas) *invariant linearly sufficiency* (Oktaba, Kornacki, Wawrzosek) and *quadratic sufficiency* (Mueller). All these variants denote such transformations of the model **GM** that preserve properties essential in statistical inference. In the present paper we give mutual relations between above three classes of statistics.

Introduction

Sufficiency is one of the fundamental notions in mathematical statistics. In connection with the general linear Gauss-Markov model **GM** $(y, X\beta, \sigma^2 V)$, there are some modifications of this notion such as *linear sufficiency* (Baksalary and Kala [2], Drygas [3]), *invariant linearly sufficiency* (Oktaba, Kornacki, Wawrzosek [5]) and quadratic sufficiency (Mueller [4]). All these variants denote such transformations of the model **GM** that preserve properties essential in statistical inference. Baksalary and Kala considered transformations of the model **GM** which preserve information needed for linear estimation. Drygas [3] called them *linearly sufficient* statistics (LS). Mueller [4] and Baksalary and Drygas [1] have taken into consideration transformations of the model **GM** which do not lose information necessary for quadratic estimation of parameter. They were called *quadratic sufficient* statistics (QS). Oktaba, Kornacki and Wawrzosek [5] considered transformations of the model **GM** preserving all informations needed both for estimation and testing. They came into the literature under the name of *invariant linearly sufficient* statistics (ILS). In the present paper we give mutual relations between above three classes of statistics.The main result is formulated in Theorem 4.1.

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1. Notation

The transpose of matrix A, rank of matrix A and column space of matrix A are denoted by A', $r(A)$ and $R(A)$ respectively. The symbol A^- is reserved for the g-inverse of matrix A satisfying condition:

$$AA^-A = A. \tag{1}$$

A general, linear Gauss-Markov model is given by the triplet:

$$(y, X\beta, \sigma^2 V) \tag{2}$$

where a random vector y of n observations has the expected value $X\beta$ and dispersion matrix $\sigma^2 V$. The matrix X is known $n \times p$ matrix of arbitrary rank, β is vector of p unknown parameters, σ^2 is unknown skalar and V is known and nonnegative definite matrix. Then after linear transformation $u = Py$ model (2) obtained the form of the model:

$$(Py, PX\beta, \sigma^2 PVP') \tag{3}$$

where P is a matrix $k \times n$, $k > n$. The symbol BQUE denotes the best, quadratic, unbiased estimator of skalar σ^2 and BLUE means the best, linear, unbiased estimator of linear function of parameters.

2. Assumptions and lemmas

For the **GM** model (2) we introduce, following Rao [6], the matrix T of the form:

$$T = V + XUX' \tag{4}$$

where U is any symmetric and nonnegative matrix such that;

$$R(T) = R(X \vdots V) \tag{5}$$

Condition (5) is equivalent to the following two conditions:

$$R(X) \subset R(T) \quad \text{and} \quad R(V) \subset R(T) \tag{6}$$

The model **GM** (2) is called *consistent* (Rao, [6, p. 378]) when

$$y \in R(X \vdots V) \tag{7}$$

The following lemma on decomposition will be used in further part of this paper:

Lemma 2.1. (Rao [7, Lemma 2.1])

$$R(X \vdots V) = R(V) \oplus R(VZ) \tag{8}$$

where Z denotes any matrix of maximal rank such that $Z'X = 0$ and symbol $\otimes$ denotes direct sum of vector spaces.

3. Different kinds of sufficiency

We give now the different kinds of sufficiency that exist in literature in connection with general linear model **GM** (2).

Definition 3.1. (Drygas [3]) Statistic $u = Py$ is said to be *linearly sufficient* (LS) if there is a linear transformation $Gu = Gpy$ which is BLUE for X in model **GM** (2).

Lemma 3.1. (Bakasalary and Kala [1]) *Statistic $u = Py$ is linearly sufficient if and only if:*

$$R(X) \subseteq R(TP') \tag{9}$$

Definition 3.2. (Mueller [4]) Statistic $u = Py$ is said to be *quadratic sufficient* (QS) if there is a symmetric matrix Λ, $k \times k$, such that $(Py)'\Lambda(Py)$ is BQUE for σ^2 .

Lemma 3.2. (Baksalary and Drygas [1]) *Statistic $u = Py$ is quadratic sufficient if and only if;*

$$R(VZ) \subseteq R(TP') \tag{10}$$

or equivalently

$$R(VZ) = R(VP'(PX)) \tag{11}$$

Lemma 3.3. (Baksalary and Drygas [1]) *Statistic $u = Py$ is quadratic sufficient if and only if BQUE for σ^2 in **GM** model* (2) *coincide with BQUE for in **PGM** model* (3).

Definition 3.3. (Oktaba, Kornacki and Wawrzosek [5]) Statistic $u = Py$ is said to be *invariant linearly* sufficient (ILS) if

1. Py is linearly sufficient (LS);
2. Parametric function $\lambda\beta$ is estimable in model **GM** if and only if it is estimable in model **PGM**;
3. BQUE for σ^2 in models **GM** and **PGM** are the same;
4. Null hypothesis: $L\beta = \phi_0$ is consistent in model **GM** if and only if it is consistent in model **PGM**;
5. Under normality assumptions statistics for testing null hypothesis $L\beta = \phi_0$ are the same in both models **GM** and **PGM**.

Lemma 3.4. (Oktaba, Kornacki and Wawrzosek [5]) *If any one of the conditions*

$$R(T) = R(TP') \tag{12}$$

$$r(T) = r(TP') \tag{13}$$

$$R(T) \subseteq R(P') \tag{14}$$

is satisfied then the statistic $u = Py$ is ILS.

Let us note that conditions (12) and (13) are equivalent while (14) imply (12).

4. Main result

We give now mutual relations between considered classes of statistics: LS, QS and ILS.

Theorem 4.1. *Statistic $u = Py$ is ILS in model **GM** if and only if it is simultaneously LS and QS statistic.*

Proof.

$\Longrightarrow$ Let statistic $u = Py$ be ILS. Then from Def. 3.3 (conditions (1) and (3)) u is LS and BQUE for σ^2 in models **GM** and **PGM** are the same. In virtue of Lemma 3.3 it means that u Py is QS.

$\Longleftarrow$ Let us consider that $u = Py$ is simultaneously LS and QS statistic. Because the model **GM** is consistent so from (7) we get $y \in R(T) = R(X \vdots V)$. Using Lemma 2.1 on decomposition we have:

$$y \in R(X) \oplus R(VZ) \tag{15}$$

Because of assumption u is LS and QS simultaneously so from (9) and (10) we have $R(X) \subseteq R(TP')$ and $R(VZ) \subseteq R(TP')$ thus from (15) we get:

$$y \in R(TP') \tag{16}$$

Finally we have $R(T) \subseteq R(TP')$. Because is always $R(TP') \subseteq R(T)$ so we get $R(T) = R(TP')$ that is condition (12) from Lemma 3.4 which guaranties that statistics $u = Py$ is ILS. □

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Agricultural University in Lublin
Akademicka 13
PL–20-950 Lublin
POLAND
E-mail: andrzej.kornacki@ar.lublin.pl