

MODAL OPERATORS ON BOUNDED
COMMUTATIVE RESIDUATED ℓ -MONOIDS

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ABSTRACT. Bounded commutative residuated lattice ordered monoids ($R\ell$ -monoids) are a common generalization of, e.g., Heyting algebras and BL -algebras, i.e., algebras of intuitionistic logic and basic fuzzy logic, respectively. Modal operators (special cases of closure operators) on Heyting algebras were studied in [MacNAB, D. S.: *Modal operators on Heyting algebras*, Algebra Universalis **12** (1981), 5–29] and on MV -algebras in [HARLENEROVÁ, M.—RACHŮNEK, J.: *Modal operators on MV-algebras*, Math. Bohem. **131** (2006), 39–48]. In the paper we generalize the notion of a modal operator for general bounded commutative $R\ell$ -monoids and investigate their properties also for certain derived algebras.

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Commutative residuated lattice ordered monoids ($R\ell$ -monoids) are duals to commutative DRL -monoids which were introduced by Swamy [16] as a common generalization of Abelian lattice ordered groups and Brouwerian algebras. By [11], [12], [13], also algebras of logics behind fuzzy reasoning can be considered as particular cases of bounded commutative $R\ell$ -monoids. Namely from this point of view, MV -algebras, an algebraic counterpart of the Łukasiewicz infinite-valued propositional logic, are precisely bounded commutative $R\ell$ -monoids satisfying the double negation law. Further, BL -algebras, an algebraic semantics of the Hájek basic fuzzy logic, are just bounded commutative $R\ell$ -monoids isomorphic to subdirect products of linearly ordered commutative $R\ell$ -monoids. Heyting algebras (duals to Brouwerian algebras), i.e. algebras of intuitionistic logic, are characterized as bounded commutative $R\ell$ -monoids with idempotent multiplication.

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Modal operators (special cases of closure operators) on Heyting algebras were introduced and studied by Macnab in [10]. Analogously, modal operators on *MV*-algebras were introduced in [7] recently.

In this paper we define modal operators for arbitrary bounded commutative *Rℓ*-monoids and we study their properties in the class of normal *Rℓ*-monoids in particular.

For concepts and results relating to *MV*-algebras, *BL*-algebras and Heyting algebras see for instance [3], [6], [1].

DEFINITION 1. A *bounded commutative Rℓ-monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions.

- (i) $(M; \odot, 1)$ is a commutative monoid.
- (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.
- (iii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, for any $x, y, z \in M$.
- (iv) $x \odot (x \rightarrow y) = x \wedge y$, for any $x, y \in M$.

Bounded commutative *Rℓ*-monoids are special cases of residuated lattices, more precisely (see for instance [4]), they are exactly commutative integral generalized *BL*-algebras in the sense of [2] and [8].

In what follows, by an *Rℓ-monoid* we will mean a bounded commutative *Rℓ-monoid*.

Let us define on any *Rℓ-monoid* M the unary operation of negation “ $-$ ” by $x^- := x \rightarrow 0$ for any $x \in M$. Further, we put $x \oplus y := (x^- \odot y^-)^-$ for any $x, y \in M$.

Algebras of the above mentioned propositional logics can be characterized in the class of all *Rℓ-monoids* as follows: An *Rℓ-monoid* M is

- a) a *BL*-algebra ([13]) if and only if M satisfies the identity of pre-linearity $(x \rightarrow y) \vee (y \rightarrow x) = 1$;
- b) an *MV*-algebra ([11], [12]) if and only if M fulfills the double negation law $x^{--} = x$;
- c) a Heyting algebra ([16]) if and only if the operations “ $\odot$ ” and “ $\wedge$ ” coincide on M .

LEMMA 1. ([16], [15]) *In any bounded commutative Rℓ-monoid M we have for any $x, y \in M$:*

- (1) $x \leq y \iff x \rightarrow y = 1$.
- (2) $x \odot y \leq x \wedge y \leq x, y$.
- (3) $x \leq y \implies x \odot z \leq y \odot z$.
- (4) $x \leq y \implies z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$.

- (5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
- (6) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.
- (7) $1^{--} = 1, 0^{--} = 0$.
- (8) $x \leq x^{--}, x^- = x^{---}$.
- (9) $x \leq y \implies y^- \leq x^-$.
- (10) $(x \vee y)^- = x^- \wedge y^-$.
- (11) $(x \wedge y)^{--} = x^{--} \wedge y^{--}$.
- (12) $(x \odot y)^- = y \rightarrow x^- = y^{--} \rightarrow x^- = x \rightarrow y^- = x^{--} \rightarrow y^-$.
- (13) $(x \odot y)^{--} \geq x^{--} \odot y^{--}$.
- (14) $(x \rightarrow y)^{--} = x^{--} \rightarrow y^{--}$.

Remark 2. It is obvious that $x \oplus z \leq y \oplus z$ holds for any $x, y, z \in M$ such that $x \leq y$. Further by [14, Lemma 2.11], $x^{--} \oplus y^{--} = x \oplus y$ for any $x, y \in M$, hence also $x \oplus y = x^{--} \oplus y = x \oplus y^{--} = x^{--} \oplus y^{--}$.

DEFINITION 2. Let M be an $R\ell$ -monoid. A mapping $f: M \longrightarrow M$ is called a *modal operator* on M if, for any $x, y \in M$,

- 1. $x \leq f(x)$;
- 2. $f(f(x)) = f(x)$;
- 3. $f(x \odot y) = f(x) \odot f(y)$.

If, moreover, for any $x, y \in M$,

- 4. $f(x \oplus y) = f(x \oplus f(y))$,

then f is called a *strong modal operator* on M .

PROPOSITION 3. If f is a modal operator on an $R\ell$ -monoid M and $x, y \in M$, then

- (i) $x \leq y \implies f(x) \leq f(y)$;
- (ii) $f(x \rightarrow y) \leq f(x) \rightarrow f(y) = f(f(x) \rightarrow f(y)) = x \rightarrow f(y) = f(x \rightarrow f(y))$;
- (iii) $f(x) \leq (x \rightarrow f(0)) \rightarrow f(0)$;
- (iv) $f(x) \odot x^- \leq f(0)$;
- (v) $x \oplus f(0) \geq f(x^{--}) \geq f(x)$;
- (vi) $f(x \vee y) = f(x \vee f(y)) = f(f(x) \vee f(y))$.

Proof.

$$(i) \quad x \leq y \implies f(x \wedge y) = f(x) \implies f(y \odot (y \rightarrow x)) = f(x) \implies f(y) \odot f(y \rightarrow x) = f(x) \implies f(x) \leq f(y).$$

$$(ii) \quad \text{By (i), } f(x) \odot f(x \rightarrow y) = f(x \odot (x \rightarrow y)) = f(x \wedge y) \leq f(y).$$

This implies

$$f(x \rightarrow y) \leq f(x) \rightarrow f(y).$$

From this we get

$$\begin{aligned} f(f(x) \rightarrow f(y)) &\leq f(f(x)) \rightarrow f(f(y)) = f(x) \rightarrow f(y) \leq x \rightarrow f(y) \\ &\leq f(x \rightarrow f(y)) \leq f(x) \rightarrow f(f(y)) \\ &= f(x) \rightarrow f(y) \leq f(f(x) \rightarrow f(y)), \end{aligned}$$

therefore

$$f(x) \rightarrow f(y) = f(f(x) \rightarrow f(y)) = x \rightarrow f(y) = f(x \rightarrow f(y)).$$

(iii) By use of (ii) and (i), we have

$$\begin{aligned} f(x) \odot (f(x) \rightarrow f(0)) &= f(x) \wedge f(0) = f(0) \implies f(x) \leq (f(x) \rightarrow f(0)) \rightarrow f(0) \\ &\implies f(x) \leq (x \rightarrow f(0)) \rightarrow f(0). \end{aligned}$$

(iv) By (ii), we obtain

$$0 \leq f(0) \implies x^- = x \rightarrow 0 \leq x \rightarrow f(0) = f(x) \rightarrow f(0),$$

thus

$$f(x) \odot x^- \leq f(x) \odot (f(x) \rightarrow f(0)) = f(x) \wedge f(0) = f(0).$$

(v) According to Remark 2, Lemma 1(12), (8) and the part (ii) consecutively,

$$\begin{aligned} x \oplus f(0) &= x^{--} \oplus f(0) = (x^{---} \odot f(0)^-)^- = x^{---} \rightarrow f(0)^{-} \\ &= x^- \rightarrow f(0)^{-} = f(x^- \rightarrow f(0)^{-}) \geq f(x^- \rightarrow f(0)) \geq f(x^- \rightarrow 0) \\ &= f(x^{--}) \geq f(x). \end{aligned}$$

Hence

$$x \oplus f(0) \geq f(x^{--}) \geq f(x).$$

(vi) $f(x \vee y) \leq f(x \vee f(y)) \leq f(f(x) \vee f(y)) = f(f(x \vee y)) = f(x \vee y)$. $\square$

Remark 4. By the definition of a modal operator and Proposition 3(i) every modal operator on an $R\ell$ -monoid M is a closure operator on the lattice $(M; \vee, \wedge)$.

Remark 5. M. Galatos and C. Tsinakis introduced in [5] the notion of a *nucleus* of a residuated lattice L as a closure operator γ on L satisfying $\gamma(a)\gamma(b) \leq \gamma(ab)$, to represent generalizations of MV -algebras (dropping integrality, commutativity and the existence of bounds) by means of ℓ -groups and nuclei of negative cones of ℓ -groups. From this point of view, a modal operator f on an $R\ell$ -monoid M is a nucleus of M satisfying $f(x) \odot f(y) \geq f(x \odot y)$.

PROPOSITION 6. *If f is a strong modal operator on an $R\ell$ -monoid M and $x, y \in M$, then*

- (vii) $f(x \oplus y) = f(f(x) \oplus f(y))$;
- (viii) $x \oplus f(0) = f(x^{--})$.

Proof. Let us suppose that f is a strong modal operator. Then

- (vii) $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus f(y))$;
- (viii) By Proposition 3(v), $f(x \oplus f(0)) = f(x \oplus 0) = f(x^{--})$ implies $f(x^{--}) = f(x \oplus f(0)) \geq x \oplus f(0) \geq f(x^{--})$. $\square$

THEOREM 7. *Let M be an $R\ell$ -monoid and $f: M \longrightarrow M$ be a mapping. Then f is a modal operator on M if and only if for any $x, y \in M$ it is satisfied:*

- (a) $x \rightarrow f(y) = f(x) \rightarrow f(y)$;
- (b) $f(x) \odot f(y) \geq f(x \odot y)$.

Proof. Let a mapping f fulfil conditions (a) and (b).

1. For any $x \in M$ we have $x \rightarrow f(x) = f(x) \rightarrow f(x) = 1$. Therefore $x \leq f(x)$.
2. For all $x \in M$ it holds $1 = f(x) \rightarrow f(x) = f(f(x)) \rightarrow f(x)$. This implies $f(f(x)) \leq f(x)$. Therefore, by 1, $f(f(x)) = f(x)$.
3. For any $x, y \in M$ it is true

$$\begin{aligned} x \odot y \leq f(x \odot y) &\implies y \leq x \rightarrow f(x \odot y) = f(x) \rightarrow f(x \odot y) \implies y \odot f(x) \leq \\ &f(x \odot y) \implies f(x) \leq y \rightarrow f(x \odot y) = f(y) \rightarrow f(x \odot y) \implies f(x) \odot f(y) \leq \\ &f(x \odot y) \implies f(x) \odot f(y) = f(x \odot y). \end{aligned}$$

The converse implication is obvious. $\square$

COROLLARY 8. *If M is an $R\ell$ -monoid and $f: M \longrightarrow M$ is a mapping, then f is a nucleus of M if and only if f satisfies (a) of Theorem 7 and it is isotone.*

Remark 9. If M is a Heyting algebra and $x, y \in M$, then $f(x) \odot f(y) = f(x) \wedge f(y) \geq f(x \wedge y) = f(x \odot y)$. Therefore, by Theorem 7, f is a modal operator on M iff it satisfies condition (a) (see also [10]).

We say that an $R\ell$ -monoid M is *normal* if M satisfies the identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

Remark 10. By [15, Proposition 5], every BL -algebra and every Heyting algebra is normal, hence the variety of normal $R\ell$ -monoids is considerably wide.

Let M be an $R\ell$ -monoid. For arbitrary element $a \in M$ we denote by $\varphi_a: M \longrightarrow M$ the mapping such that $\varphi_a(x) = a \oplus x$ for every $x \in M$.

Denote by

$$I(M) = \{a \in M : a \odot a = a\}$$

the set of all multiplicative idempotents in an $R\ell$ -monoid M . It is obvious that $0, 1 \in I(M)$. By [9, Lemma 2.8.3], $a \odot x = a \wedge x$ holds for any $a \in I(M)$, $x \in M$. Further, if M is a normal $R\ell$ -monoid and $a \in I(M)$, then also $a^{--} \in I(M)$.

THEOREM 11. *If M is a normal $R\ell$ -monoid and $a \in M$, then φ_a is a strong modal operator on M if and only if $a^-, a^{--} \in I(M)$.*

Proof.

a) Let $a, x, y \in M$, $a^-, a^{--} \in I(M)$.

1. $\varphi_a(x) = a \oplus x = (a^- \odot x^-)^- \geq x^{--} \geq x$.
2. $\varphi_a(\varphi_a(x)) = a \oplus (a \oplus x) = a \oplus (a^- \odot x^-)^- = (a^- \odot (a^- \odot x^-)^{-})^- = (a^- \odot (a^- \odot x^-))^- = ((a^- \odot a^-) \odot x^-)^- = (a^- \odot x^-)^- = a \oplus x = \varphi_a(x)$.

3. We first prove that $a \oplus x = (a \vee x)^{--}$.

By Lemma 1(10), we obtain $a \oplus x = (a^- \odot x^-)^- = (a^- \wedge x^-)^- = ((a \vee x)^-)^- = (a \vee x)^{--}$.

We will now prove condition 3 from the definition of a modal operator.

We have

$$\begin{aligned} \varphi_a(x) \odot \varphi_a(y) &= (a \oplus x) \odot (a \oplus y) = (a^{--} \oplus x) \odot (a^{--} \oplus y) \\ &= (a^{--} \vee x)^{--} \odot (a^{--} \vee y)^{--} = ((a^{--} \vee x) \odot (a^{--} \vee y))^{--} \\ &= ((a^{--} \odot a^{--}) \vee (x \odot a^{--}) \vee (a^{--} \odot y) \vee (x \odot y))^{--} \\ &= (a^{--} \vee (x \odot y))^{--} = a^{--} \oplus (x \odot y) = a \oplus (x \odot y) \\ &= \varphi_a(x \odot y). \end{aligned}$$

4. According to [14, Proposition 2.10], $(M; \oplus)$ is a commutative semigroup.

For this reason

$$\begin{aligned} \varphi_a(x \oplus y) &= a \oplus (x \oplus y) = a^{--} \oplus (x \oplus y) = (a \oplus a) \oplus (x \oplus y) \\ &= a \oplus (x \oplus (a \oplus y)) = \varphi_a(x \oplus \varphi_a(y)). \end{aligned}$$

b) Let φ_a be a strong modal operator on M . Then on account of condition 3, we have $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$. Then for $x = y = 0$ we obtain $a \oplus (0 \odot 0) = (a \oplus 0) \odot (a \oplus 0)$, hence $a \oplus 0 = (a \oplus 0) \odot (a \oplus 0)$. Since $a \oplus 0 = a^{--}$ (see [14, Lemma 2.11]), we conclude that $a^{--} = a^{--} \odot a^{--}$, which yields $a^{--} \in I(M)$.

From condition 4 we have $a \oplus (x \oplus y) = a \oplus (x \oplus (a \oplus y))$. Then for $x = y = 0$ it follows that $a^{--} = a \oplus 0 = a \oplus (0 \oplus 0) = a \oplus (0 \oplus (a \oplus 0)) = (a \oplus 0) \oplus a^{--} = a^{--} \oplus a^{--}$, thus $a^{--} = a^{--} \oplus a^{--}$. From this $a^{--} = (a^- \odot a^-)^-$, hence $a^- = (a^- \odot a^-)^{-}$. Since M is normal, we obtain $a^- = a^- \odot a^-$ and so $a^- \in I(M)$. $\square$

Remark 12. If M is an MV -algebra and $a \in M$, then $a \in I(M)$ if and only if $a^-, a^{--} \in I(M)$. Concurrently, by [7], in any MV -algebra it is true that φ_a is a modal operator on M if and only if φ_a is a strong modal operator on M (namely if and only if $a \in I(M)$). The question, whether φ_a is a modal operator on M if and only if it is a strong modal operator also for any normal $R\ell$ -monoid M , remains open.

COROLLARY 13. *Let M be a normal $R\ell$ -monoid and f be a modal operator on M such that $f(x) = f(x^{--})$ for all $x \in M$. Then f is strong if and only if $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$.*

Proof. Suppose that a modal operator f on M satisfies the condition $f(x) = f(x^{--})$ for every $x \in M$. Then by Proposition 6 and Theorem 11, f is strong if and only if $f = \varphi_a$ for some $a \in M$ such that $a^-, a^{--} \in I(M)$.

If f is strong and $x \in M$, then $f(x) = f(x^{--}) = f(0) \oplus x$. Hence $f = \varphi_{f(0)}$ and we have $f(0), f(0)^- \in I(M)$.

For any modal operator f we have $f(0)^{--} \in I(M)$. In fact, $f(0)^{--} = f(0 \odot 0)^{--} = (f(0) \odot f(0))^{--} = f(0)^{--} \odot f(0)^{--}$. Hence, if $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$, then by Theorem 11, f is strong. $\square$

COROLLARY 14. *Specially for MV -algebras, we obtain (see [7]): If M is an MV -algebra and f is a modal operator on M , then f is strong if and only if $f = \varphi_{f(0)}$.*

Let M be an $R\ell$ -monoid and $a \in M$. Consider mappings $\psi_a: M \longrightarrow M$ and $\chi_a: M \longrightarrow M$ such that $\psi_a(x) := a \rightarrow x$ and $\chi_a(x) := (x \rightarrow a) \rightarrow a$ for every $x \in M$. These mappings are significant modal operators in Heyting algebras (see [10]). We will now deal with the mappings ψ_a and χ_a in arbitrary $R\ell$ -monoids.

PROPOSITION 15. *If M is an $R\ell$ -monoid and $a \in I(M)$, then for any $x, y \in M$*

$$x \rightarrow \psi_a(y) = \psi_a(x) \rightarrow \psi_a(y).$$

Proof. By the definition of ψ_a , $x \rightarrow \psi_a(y) = x \rightarrow (a \rightarrow y)$ and $\psi_a(x) \rightarrow \psi_a(y) = (a \rightarrow x) \rightarrow (a \rightarrow y)$. At the same time, by Lemma 1(5), $(a \rightarrow x) \rightarrow (a \rightarrow y) = ((a \rightarrow x) \odot a) \rightarrow y = (a \wedge x) \rightarrow y = (a \odot x) \rightarrow y = x \rightarrow (a \rightarrow y)$, whence the assertion follows. $\square$

From Theorem 7 and Proposition 15 we obtain as an immediate consequence the following claim.

COROLLARY 16. *Let M be an $R\ell$ -monoid and $a \in I(M)$. Then ψ_a is a modal operator on M if and only if for any $x, y \in M$*

$$\psi_a(x) \odot \psi_a(y) \geq \psi_a(x \odot y).$$

LEMMA 17. *If M is an $R\ell$ -monoid and $a \in M$, then for any $x, y \in M$*

$$x \rightarrow \chi_a(y) \leq \chi_a(x) \rightarrow \chi_a(y).$$

Proof. By the definition of χ_a and by Lemma 1(5), $x \rightarrow \chi_a(y) = x \rightarrow ((y \rightarrow a) \rightarrow a) = (x \rightarrow (y \rightarrow a)) \rightarrow a$, $\chi_a(x) \rightarrow \chi_a(y) = ((x \rightarrow a) \rightarrow a) \rightarrow ((y \rightarrow a) \rightarrow a)$. Since by [14, Lemma 2.3], $(y \rightarrow a) \rightarrow (x \rightarrow a) \leq ((x \rightarrow a) \rightarrow a) \rightarrow ((y \rightarrow a) \rightarrow a)$, we have $x \rightarrow \chi_a(y) \leq \chi_a(x) \rightarrow \chi_a(y)$. $\square$

For any $R\ell$ -monoid M , let us denote by $B(M)$ the set of all elements from M having the complement in the lattice $(M; \vee, \wedge, 0, 1)$. Note that $0, 1 \in B(M)$. If $a \in B(M)$ then its complement a' is equal to the element a^- . By [9, Lemma 2.8.8], $B(M) \subseteq I(M)$.

PROPOSITION 18. *Let M be an $R\ell$ -monoid and $a \in B(M)$. Then for any $x, y \in M$*

$$x \rightarrow \chi_a(y) = \chi_a(x) \rightarrow \chi_a(y).$$

Proof. Let $a \in B(M)$, $x, y \in M$. Then

$$x \rightarrow \chi_a(y) = x \rightarrow ((y \rightarrow a) \rightarrow a) = (y \rightarrow a) \rightarrow (x \rightarrow a),$$

$$\begin{aligned} \chi_a(x) \rightarrow \chi_a(y) &= ((x \rightarrow a) \rightarrow a) \rightarrow ((y \rightarrow a) \rightarrow a) \\ &= (y \rightarrow a) \rightarrow (((x \rightarrow a) \rightarrow a) \rightarrow a), \end{aligned}$$

$$x \rightarrow a = x \rightarrow a^{--} = (x \odot a^-)^-,$$

$$\begin{aligned} (x \rightarrow a) \rightarrow a &= ((x \rightarrow a) \odot a^-)^- = ((x \odot a^-)^- \odot a^-)^- = ((x \wedge a^-)^- \wedge a^-)^- \\ &= ((x \wedge a^-) \vee a)^{--} = ((x \vee a) \wedge (a^- \vee a))^{--} = (x \vee a)^{--} = x \oplus a, \end{aligned}$$

$$\begin{aligned} ((x \rightarrow a) \rightarrow a) \rightarrow a &= (((x \rightarrow a) \rightarrow a) \odot a^-)^- = ((x \oplus a) \odot a^-)^- = (x \oplus a) \rightarrow a^{--} \\ &= (x \vee a)^{--} \rightarrow a^{--} = ((x \vee a) \rightarrow a)^{--} = ((x \vee a) \odot a^-)^- \\ &= ((x \odot a^-) \vee (a \odot a^-))^+ = (x \odot a^-)^+ = x \rightarrow a. \end{aligned}$$

Hence

$$\begin{aligned} \chi_a(x) \rightarrow \chi_a(y) &= (y \rightarrow a) \rightarrow (((x \rightarrow a) \rightarrow a) \rightarrow a) \\ &= (y \rightarrow a) \rightarrow (x \rightarrow a) = x \rightarrow \chi_a(y). \end{aligned}$$

□

COROLLARY 19. *Let M be an $R\ell$ -monoid and $a \in B(M)$. Then χ_a is a modal operator on M if and only if for any $x, y \in M$*

$$\chi_a(x) \odot \chi_a(y) \geq \chi_a(x \odot y).$$

Let M be an $R\ell$ -monoid and f be a modal operator on M . Then $\text{Fix}(f) = \{x \in M : f(x) = x\}$ will denote the set of all fixed elements of the operator f . By the definition of a modal operator it is obvious that $\text{Fix}(f) = \text{Im}(f)$.

Since f is a closure operator on the lattice $(M; \vee, \wedge)$, we infer that $(\text{Fix}(f); \vee_F, \wedge)$, where $y \vee_F z = f(y \vee z)$ and “ $\wedge$ ” is the restriction of the corresponding operation from M on $\text{Fix}(f)$, is a lattice.

THEOREM 20. *If f is a modal operator on an $R\ell$ -monoid M , then $\text{Fix}(f)$ is closed under the operations “ $\odot$ ” and “ $\rightarrow$ ” and $\text{Fix}(f) = (\text{Fix}(f); \odot, \vee_F, \wedge, \rightarrow, f(0), 1)$ is an $R\ell$ -monoid.*

Proof.

- (i) Let $x, y \in \text{Fix}(f)$. Then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, thus $x \odot y \in \text{Fix}(f)$.
 - (ii) $(\text{Fix}(f); \vee_F, \wedge, f(0), 1)$ is a bounded lattice.
 - (iii) If $y, z \in \text{Fix}(f)$, then by Proposition 3 we have $y \rightarrow z = f(y) \rightarrow f(z) = f(f(y) \rightarrow f(z)) = f(y \rightarrow z)$, hence $y \rightarrow z \in \text{Fix}(f)$.
- Therefore, if $x, y, z \in \text{Fix}(f)$, then $x \odot y, y \rightarrow z \in \text{Fix}(f)$ and for this reason $x \odot y \leq z$ holds in $\text{Fix}(f)$ if and only if $x \leq y \rightarrow z$.
- (iv) By foregoing, $\text{Fix}(f)$ also satisfies the identity $x \odot (x \rightarrow y) = x \wedge y$. $\square$

Remark 21. The above theorem strengthens general Lemma 3.3 of [5] proved for any residuated lattices in our special case of bounded (commutative) $R\ell$ -monoids.

THEOREM 22. *Let M be an $R\ell$ -monoid, $a \in I(M)$ and*

$$I(a) := [0, a] = \{x \in M : 0 \leq x \leq a\}.$$

For any $x, y \in I(a)$ we set $x \odot_a y = x \odot y$ and $x \rightarrow_a y := (x \rightarrow y) \wedge a$. Then $I(a) = (I(a); \odot_a, \vee, \wedge, \rightarrow_a, 0, a)$ is an $R\ell$ -monoid.

Proof.

- (i) If $x, y \in I(a)$, then $x \odot y \in I(a)$ and $x \odot a = x \wedge a = x$, hence $(I(a); \odot_a, a)$ is a commutative monoid.
- (ii) Obviously, $(I(a); \vee, \wedge, 0, a)$ is a bounded lattice.
- (iii) Let $x, y \in I(a)$. It holds that $x \rightarrow y$ is the greatest element $z \in M$ such that $x \odot z \leq y$. Therefore $(x \rightarrow y) \wedge a$ is the greatest element in $I(a)$ with this property. That means, $x \odot_a z \leq y$ if and only if $z \leq (x \rightarrow y) \wedge a = x \rightarrow_a y$ for every $z \in I(a)$.
- (iv) For any $x, y \in I(a)$ we have $x \odot_a (x \rightarrow_a y) = x \odot ((x \rightarrow y) \wedge a) = x \odot (x \rightarrow y) \odot a = (x \wedge y) \wedge a = x \wedge y$. $\square$

Remark 23. If for any $x, y \in I(a)$ we denote by x^{-a} the negation of an element x and by $x \oplus_a y$ the sum of elements x and y in the $R\ell$ -monoid $I(a)$, then it holds

$$x^{-a} = x^{-} \wedge a, \quad x \oplus_a y = (x \oplus y) \wedge a.$$

Indeed

$$x^{-a} = x \rightarrow_a 0 = (x \rightarrow 0) \wedge a = x^{-} \wedge a,$$

$$\begin{aligned} x \oplus_a y &= (x^{-a} \odot y^{-a})^{-} \wedge a = (x^{-} \odot a \odot y^{-} \odot a)^{-} \wedge a = (x^{-} \odot y^{-} \odot a)^{-} \wedge a \\ &= (a \rightarrow (x^{-} \odot y^{-}))^{-} \odot a = a \wedge (x^{-} \odot y^{-})^{-} = a \wedge (x \oplus y). \end{aligned}$$

Now, let M be arbitrary $R\ell$ -monoid (still bounded and commutative), $a \in I(M)$ and let f be a modal operator on M . Let us consider a mapping $f^a: I(a) \longrightarrow I(a)$ such that $f^a(x) = f(x) \wedge a (= f(x) \odot a)$, for every $x \in I(a)$.

THEOREM 24. *Let M be an $R\ell$ -monoid, $a \in I(M)$ and f be a modal and strong modal, respectively, operator on M . Then f^a is a modal and strong modal, respectively, operator on the $R\ell$ -monoid $I(a)$.*

Proof. Assume $x, y \in I(a)$.

1. $x \leq a$ and $x \leq f(x)$, hence $x \leq a \wedge f(x) = f^a(x)$.
2. $f^a(f^a(x)) = f(f(x) \wedge a) \wedge a = f(f(x) \odot a) \wedge a = (f(f(x)) \odot f(a)) \wedge a$
 $= f(x) \wedge f(a) \wedge a = f(x) \wedge a = f^a(x)$.
3. $f^a(x \odot y) = f(x \odot y) \wedge a = f(x) \odot f(y) \odot a \odot a = (f(x) \wedge a) \odot (f(y) \wedge a)$
 $= f^a(x) \odot f^a(y)$.
4. Let f be strong. Then

$$\begin{aligned}
 f^a(x \oplus_a f^a(y)) &= f^a((x \oplus (f(y) \wedge a)) \wedge a) = f((x \oplus (f(y) \wedge a)) \wedge a) \wedge a \\
 &= f(x \oplus (f(y) \wedge a)) \wedge f(a) \wedge a = f(x \oplus f(f(y) \wedge a)) \wedge a \\
 &= f(x \oplus ((f(f(y)) \wedge f(a))) \wedge a = f(x \oplus (f(y) \wedge f(a))) \wedge a \\
 &= f(x \oplus f(y \wedge a)) \wedge a = f(x \oplus f(y)) \wedge a = f(x \oplus y) \wedge a \\
 &= f^a(x \oplus y).
 \end{aligned}$$

□

THEOREM 25.

a) *Let M be an $R\ell$ -monoid, let f be a modal operator on M and $\hat{f} = f|_{I(M)}$. Then $I(M)$ is a subalgebra of the reduct $(M; \odot, \vee, \wedge, 0, 1)$ and $\hat{f}$ is a mapping of $I(M)$ into $I(M)$ satisfying conditions 1, 2, 3 from the definition of a modal operator.*

b) *Let M be a normal $R\ell$ -monoid and let $x^- \in I(M)$ for each $x \in I(M)$. Then $I(M)$ is closed also under the operation “ $\oplus$ ”. Moreover, if f is a strong modal operator on M , then $\hat{f}$ satisfies condition 4 from the definition of a strong modal operator.*

c) *Let M be a BL -algebra. Then $I(M)$ is a subalgebra of the algebra M which is a Heyting algebra. If f is a modal operator on M , then $\hat{f}$ is a modal operator on the Heyting algebra $I(M)$. If $x^- \in I(M)$ holds for each $x \in I(M)$ and f is a strong modal operator on M , then $\hat{f}$ is a strong modal operator on $I(M)$.*

Proof.

a) Let M be an $R\ell$ -monoid and $x, y \in I(M)$. Then

$$(x \odot y) \odot (x \odot y) = (x \odot x) \odot (y \odot y) = x \odot y,$$

thus $x \odot y = x \wedge y \in I(M)$. Further,

$$(x \vee y) \odot (x \vee y) = (x \odot x) \vee (y \odot x) \vee (x \odot y) \vee (y \odot y) = x \vee y \vee (x \odot y) = x \vee y,$$

therefore also $x \vee y \in I(M)$.

Obviously, $0, 1 \in I(M)$.

Finally, if f is a modal operator on M , then for each $x \in I(M)$ we have

$$f(x) = f(x \odot x) = f(x) \odot f(x).$$

It follows that $f(x) \in I(M)$. Therefore $\hat{f}$ is a mapping of $I(M)$ into $I(M)$ satisfying conditions 1–3.

b) If $x^- \in I(M)$ holds for every $x \in I(M)$, then (similarly to the third part of the proof of Theorem 11) for any $x, y \in I(M)$ we obtain $x \oplus y = (x \vee y)^{--}$, and hence provided M is normal we have

$$\begin{aligned} (x \oplus y) \odot (x \oplus y) &= (x \vee y)^{--} \odot (x \vee y)^{--} = ((x \vee y) \odot (x \vee y))^{--} = (x \vee y)^{--} \\ &= x \oplus y, \end{aligned}$$

therefore $x \oplus y \in I(M)$.

At the same time it is obvious that if f is a strong modal operator on M , then $\hat{f}$ fulfills condition 4 as well.

c) By [13], an $R\ell$ -monoid M is a BL -algebra if and only if M is isomorphic to a subdirect product of $R\ell$ -chains ($=BL$ -chains). Let now a BL -algebra M be a subdirect product of BL -chains M_α , $\alpha \in \Gamma$. If $a \in M$, then $a = (a_\alpha; \alpha \in \Gamma) \in I(M)$ if and only if $a_\alpha \in I(M_\alpha)$ for each $\alpha \in \Gamma$. Let $x = (x_\alpha; \alpha \in \Gamma)$, $y = (y_\alpha; \alpha \in \Gamma) \in I(M)$. Then $x_\alpha \rightarrow y_\alpha = 1$ for $y_\alpha \geq x_\alpha$ and $x_\alpha \rightarrow y_\alpha = y_\alpha$ for $x_\alpha > y_\alpha$. Whence $(x_\alpha \rightarrow y_\alpha; \alpha \in \Gamma) \in I(M)$ and it is equal to the element $x \rightarrow y$. By [13], furthermore, $I(M)$ is a Heyting algebra.

Then it is clear that $\hat{f}$ is a modal operator on $I(M)$ for any modal operator f on M . Moreover, by [15, Proposition 5], every BL -algebra is a normal $R\ell$ -monoid. Therefore, if $x^- \in I(M)$ for each $x \in I(M)$, then $\hat{f}$ is a strong modal operator on the Heyting algebra $I(M)$ for every strong modal operator f on M . $\square$

Remark 26. For any $a \in M$, also mappings $\pi_a: M \rightarrow M$ (in our notation) defined by $\pi_a(x) = a \vee x$ for each $x \in M$ were introduced and studied for Heyting algebras in [10]. Evidently, if M is an arbitrary $R\ell$ -monoid, then π_a satisfies conditions 1 and 2 from the definition of a modal operator on M . This begs the question if π_a fulfills condition 3 from this definition as well and in which cases $\pi_a = \varphi_a$ holds, respectively.

a) If M is a Heyting algebra then $x \odot y = x \wedge y$ for any $x, y \in M$. From the distributivity of the lattice $(M; \vee, \wedge)$ it follows that condition 3 is satisfied for any $a \in M$. At the same time, $a \oplus x = (a \vee x)^{--}$, hence π_a need not generally be equal to φ_a . For example, $\pi_0(x) = x$, $\varphi_0(x) = x^{--}$.

b) If M is an MV -algebra, then $a \vee x = a \oplus x$ holds for any $a \in I(M)$ and $x \in M$, and $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$. Therefore, we have $\varphi_a = \pi_a$ for each $a \in I(M)$ and hence, for each $a \in I(M)$, moreover π_a is a strong modal operator on M .

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