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A NON-ASSOCIATIVE GENERALIZATION OF MV-ALGEBRAS

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ABSTRACT. We consider a non-associative generalization of MV-algebras. The underlying posets of our non-associative MV-algebras are not lattices, but they are related to so-called λ-lattices.

1. Non-associative MV-algebras

As known, MV-algebras were introduced in the late-fifties by C. C. Chang as an algebraic semantics of the Łukasiewicz many-valued sentential logic (see [5], [6]). We recall the definition from [7] which is essentially due to P. Mangani [12]; Chang's original definition in [5] was a bit more complicated:

An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = \neg 0$ (the element $\neg 0$ is denoted by 1),

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The prototypical example of an MV-algebra is the algebra $\Gamma(G, u) = ([0, u], \oplus, \neg, 0)$, where $(G, +, -, 0, \vee, \wedge)$ is an Abelian lattice-ordered group, $0 < u \in G$ and $[0, u] = \{x \in G : 0 \leq x \leq u\}$, and the operations $\oplus$ and $\neg$ are defined via $x \oplus y := (x + y) \wedge u$ and $\neg x := u - x$, respectively. D. Mundici

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proved in [13] (see also [7]) that every MV-algebra A is isomorphic to (up to isomorphism) unique MV-algebra $\Gamma(G,u)$.

Another well-known fact is that for any MV-algebra A, the relation $\leq$ given by

$$x \leq y \ :\Longleftrightarrow \ \neg x \oplus y = 1 \tag{1}$$

is a lattice order on A with $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Obviously, if $A = \Gamma(G,u)$, then $\leq$ is the restriction of the group order to the interval $[0,u]$.

In the recent years, non-commutative generalizations of MV-algebras were considered by G. Georgescu and A. Iorgulescu [9] as *pseudo MV-algebras* and independetly by J. Rachůnek [14] as *GMV-algebras*. Although the respective definitions are slightly different, the resultant non-commutative MV-algebras are equivalent; they are algebras with a binary operation $\oplus$ and two unary operations $\neg$ and $\sim$, which coincide whenever $\oplus$ is commutative.

We have to remark that the name GMV-algebra appears e.g. in [2], [8] in a different sense. Here a *GMV-algebra* is a residuated lattice (in general non-commutative and unbounded) satisfying certain additional identities and bounded GMV-algebras correspond to pseudo MV-algebras.

In the paper we generalize MV-algebras omitting associativity of $\oplus$, but in such a way that the relation defined by (1) is still a partial order. However, without the identity (MV1) we would not be able to show that $\leq$ is transitive. Therefore we replace (MV1) by another two axioms which hold in all MV-algebras and which force $\leq$ to be transitive.

Definition 1. An algebra $(A, \oplus, \neg, 0)$ of type $(2,1,0)$ is called a *non-associative MV-algebra* or an *NMV-algebra* for short if it satisfies the identities (MV2)–(MV6) and

$$\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1, \tag{WA}$$

$$\neg x \oplus (x \oplus y) = 1. \tag{H}$$

If we put $y = 0$ in (H), we have $\neg x \oplus x = 1$, so $\leq$ is reflexive. It follows easily from (MV6) that it is antisymmetric. Finally, if $\neg x \oplus y = 1$ and $\neg y \oplus z = 1$, then (WA) entails $\neg x \oplus z = 1$, thus $\leq$ is also transitive. Altogether, $\leq$ is a partial order as desired. In addition, using (MV6) and (WA) with $z = 0$ it can be seen that $\neg(\neg x \oplus y) \oplus y$ is a common upper bound of x, y, but in contrast to MV-algebras, it need not be their supremum.

* * *

As usual, given a partially ordered set $(P, \leq)$, we write $L(x,y) = \{a \in P : a \leq x \text{ and } a \leq y\}$ and $U(x,y) = \{a \in P : a \geq x \text{ and } a \geq y\}$ for any $x, y \in P$. If

$U(x,y) \neq \varnothing$ for all $x, y \in P$, then $(P, \leq)$ is called an *upwards directed set*, and $(P, \leq)$ is called a *directed set* provided both $L(x,y)$ and $U(x,y)$ are non-empty.

V. Snášel in his unpublished thesis [15] (see also [16]) introduced the concept of a λ-*lattice* as a generalization of lattices:

An algebra $(L, \cup, \cap)$ of type $(2,2)$ is called a λ-*lattice* if it satisfies the identities

(L1) $x \cap x = x$, $x \cup x = x$,

(L2) $x \cap y = y \cap x$, $x \cup y = y \cup x$,

(L3) $x \cap ((x \cap y) \cap z) = (x \cap y) \cap z$, $x \cup ((x \cup y) \cup z) = (x \cup y) \cup z$,

(L4) $x \cap (x \cup y) = x$, $x \cup (x \cap y) = x$.

If we put $x \leq y$ iff $x \cap y = x$, or equivalently, $x \leq y$ iff $x \cup y = y$, then $(L, \leq)$ is a directed set and $x \cap y \in L(x,y)$ and $x \cup y \in U(x,y)$.

We can analogously introduce λ-semilattices (cf. [11]): An *upper λ-semilattice* is an algebra $(S, \cup)$ of type (2) satisfying the identities

(S1) $x \cup x = x$,

(S2) $x \cup y = y \cup x$,

(S3) $x \cup ((x \cup y) \cup z) = (x \cup y) \cup z$.

If we define $x \leq y$ iff $x \cup y = y$, then the relation $\leq$ is a partial order on S such that $x \cup y \in U(x,y)$, so $(S, \leq)$ is an upwards directed set.

The notion of a *lower λ-semilattice* can be defined dually, but we restrict ourselves to upper ones only, hence whenever we refer to a λ-*semilattice* we mean an upper λ-semilattice.

We notice that our λ-semilattices are equivalent to *commutative directoids* which were considered by J. Ježek and R. Quackenbush [10].

THEOREM 2. *Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. Then upon defining $x \cup y := \neg(\neg x \oplus y) \oplus y$ and $x \cap y := \neg(\neg x \cup \neg y)$, $(A, \cup, \cap)$ is a bounded λ-lattice with 0 at the bottom and 1 at the top.*

Proof. Putting $y = 0$ in (H) we obtain $\neg x \oplus x = 1$, so $x \cup x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = x$. Clearly, $x \cup y = y \cup x$ by (MV6). Further, by (WA) we have $\neg x \oplus ((x \cup y) \cup z) = 1$ whence $x \cup ((x \cup y) \cup z) = \neg(\neg x \oplus ((x \cup y) \cup z)) \oplus ((x \cup y) \cup z) = (x \cup y) \cup z$. It is plain that $x \cup 0 = x$ and $x \cup 1 = 1$ for every $x \in A$. Thus $(A, \cup)$ is a bounded λ-semilattice.

Further, observe that $x \oplus \neg(x \cap y) = x \oplus (\neg x \cup \neg y) = x \oplus (\neg(x \oplus \neg y) \oplus \neg y) = 1$ when we put $z = 0$ in (WA), whence it follows $x \cup (x \cap y) = \neg(\neg(x \cap y) \oplus x) \oplus x = x$. Using the definition of $\cap$ and just proved properties of $\cup$ it is straightforward to verity the remaining equations of (L1)–(L4). $\square$

2. λ-semilattices with involutions

A λ-*semilattice with involutions* is a λ-semilattice $(S, \cup)$ with the greatest element 1, where every interval $[a, 1] \subseteq S$ (so-called *section*) has an involution f_a with $f_a(1) = a$. We write simply x^a for $f_a(x)$. Clearly, a λ-semilattice with involutions can be considered as a structure $(S, \cup, ({}^a)_{a\in S}, 1)$.
A λ-*lattice with involutions* is defined analogously as a system $(L, \cup, \cap, ({}^a)_{a\in L}, 1)$.

Let $(S, \cup, ({}^a)_{a\in L}, 1)$ be a λ-semilattice with involutions. In order to overcome the difficulties concerning the number of partial unary operations ${}^a \colon [a, 1] \longrightarrow [a, 1]$, we define a new total binary operation $\to$ on S via

$$x \to y := (x \cup y)^y. \tag{2}$$

Lemma 3. *A λ-semilattice $(S, \cup)$ with the top element 1 is a λ-semilattice with involutions if and only if there exists a binary operation $\to$ on S that has the following properties, for all $x, y \in S$:*

(a) $1 \to x = x$,

(b) $x \cup y = (x \to y) \to y$,

(c) $((x \to y) \to y) \to y = x \to y$.

In this case, $x^a = x \to a$ for $x \in [a, 1]$, $a \in S$.

P r o o f. Let S be a λ-semilattice with involutions and let $\to$ be the operation given by (2). Then $1 \to x = (1 \cup x)^x = 1^x = x$, $(x \to y) \to y = ((x \cup y)^y \cup y)^y = (x \cup y)^{yy} = x \cup y$ and $((x \to y) \to y) \to y = (x \cup y) \to y = ((x \cup y) \cup y)^y = (x \cup y)^y = x \to y$. Obviously, $x^a = (x \cup a)^a = x \to a$ for every $x \in [a, 1]$.

Conversely, if $\to$ satisfies (a), (b) and (c), then we define $f_a(x) = x^a := x \to a$ for $x \in [a, 1]$, $a \in S$. By (b) and (c), $(x \to a) \cup a = ((x \to a) \to a) \to a = x \to a$, i.e. $a \leq x \to a$ and $x^a \in [a, 1]$. Further, we have $x^{aa} = (x \to a) \to a = x \cup a = x$, so f_a is an involution on $[a, 1]$, and $1^a = 1 \to a = a$. Thus S is a λ-semilattice with involutions. Moreover, due to (c) and (b) we obtain $x \to y = ((x \to y) \to y) \to y = (x \cup y) \to y = (x \cup y)^y$. $\square$

Consequently, λ-(semi)lattices can be treated as algebras $(S, \cup, \to, 1)$ of type $(2, 2, 0)$ or $(L, \cup, \cap, \to, 1)$ of type $(2, 2, 2, 0)$, respectively.

Remark 4. Note that the partial order $\leq$ can be retrieved via $x \leq y$ iff $x \to y = 1$, however, the operation $\to$ does not determine $\cup$. To be more precise, if $\to$ is a total binary operation satisfying all the equations in the language $\{\to, 1\}$ which are derivable in λ-semilattices with involutions, in particular, $1 \to x = x$ and $(x \to y) \to y = (y \to x) \to x$, then $(x \to y) \to y$ need not be equal to $x \cup y$.

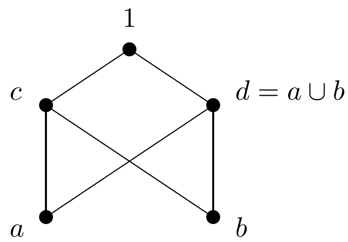

Figure 1

Example 5. Let $(S, \cup)$ be a λ-semilattice as shown in Fig. 1. Let the involutions f_a and f_b in the non-trivial sections $[a, 1]$ and $[b, 1]$, respectively, be defined as follows: $f_a(c) = c$, $f_a(d) = d$ and $f_b(c) = d$, $f_b(d) = c$. The operation $\to$ is then given by Table 1. However, the operation $\rightsquigarrow$ given by Table 2 also fulfils the equations $1 \rightsquigarrow x = x$ and $(x \rightsquigarrow y) \rightsquigarrow y = (y \rightsquigarrow x) \rightsquigarrow x$, but $(a \rightsquigarrow b) \rightsquigarrow b = c \neq d = a \cup b$. Observe that $\rightsquigarrow$ is obtained by (2) when $a \cup b$ is defined as c.

$\to$	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	c	d	1	d	1
d	d	c	c	1	1
1	a	b	c	d	1

Table 1

$\rightsquigarrow$	a	b	c	d	1
a	1	d	1	1	1
b	c	1	1	1	1
c	c	d	1	d	1
d	d	c	c	1	1
1	a	b	c	d	1

Table 2

Lemma 6. *Let $(S, \cup, \to, 1)$ be a λ-semilattice with involutions. Then for all $x, y \in S$,*

(i) $x \to 1 = 1$, $x \to x = 1$,

(ii) $y \leq x \to y$.

Proof.

(i) We have $x \to 1 = (x \cup 1)^1 = 1^1 = 1$ and $x \to x = (x \cup x)^x = x^x = 1$.

(ii) This is obvious since $x \to y = (x \cup y)^y \geq y$. □

Theorem 7. *The variety of all λ-lattices with involutions is regular and arithmetical.*

Proof. Let $\mathscr{V}$ be the variety of λ-lattices with involutions.

$\mathscr{V}$ is regular: Let

$$t_1(x,y,z) = ((x \to y) \cap (y \to x)) \cap z,$$
$$t_2(x,y,z) = ((x \to y) \to z) \cup ((y \to x) \to z).$$

We show that $t_1(x,y,z) = t_2(x,y,z) = z$ iff $x = y$.

Obviously, $t_1(x,x,z) = z$ and $t_2(x,x,z) = z$. Conversely, let $t_1(x,y,z) = t_2(x,y,z) = z$. Then $z \leq x \to y, y \to x$ and $z \geq (x \to y) \to z, (y \to x) \to z$. But by Lemma 6(ii) we have $(x \to y) \to z, (y \to x) \to z \geq z$, so that $(x \to y) \to z = z = (y \to x) \to z$, whence $x \to y = (x \to y) \cup z = ((x \to y) \to z) \to z = z \to z = 1$, so $x \leq y$. Similarly $y \leq x$, and hence $x = y$.

$\mathscr{V}$ is arithmetical: Let

$$m(x,y,z) = (((x \to y) \to z) \cap ((z \to y) \to x)) \cap (x \cup z).$$

We prove that $m(x,y,y) = m(x,y,x) = m(y,y,x) = x$.

We have $m(x,y,y) = (((x \to y) \to y) \cap ((y \to y) \to x)) \cap (x \cup y) = ((x \cup y) \cap x) \cap (x \cup y) = x$, $m(x,y,x) = (((x \to y) \to x) \cap ((x \to y) \to x)) \cap (x \cup x) = ((x \to y) \to x) \cap x = x$ since $(x \to y) \to x \geq x$ by Lemma 6, and $m(y,y,x) = (((y \to y) \to x) \cap ((x \to y) \to y)) \cap (y \cup x) = (x \cap (x \cup y)) \cap (y \cup x) = x$. □

* * *

There is a one-to-one correspondence between NMV-algebras and bounded λ-(semi)lattices with involutions that satisfy a simple additional identity:

Theorem 8.

(i) *Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. Define $x \cup y := \neg(\neg x \oplus y) \oplus y$ and $x \to y := \neg x \oplus y$. Then $\phi(A) = (A, \cup, \to, 0, 1)$ is a bounded λ-semilattice with involutions that satisfies the identity*

$$x \to (y \to 0) = y \to (x \to 0). \tag{WE}$$

(ii) *Let $(S, \cup, \to, 0, 1)$ be a bounded λ-semilattice with involutions satisfying* (WE). *If we define $x \oplus y := (x \to 0) \to y$ and $\neg x := x \to 0$, then $\psi(S) = (S, \oplus, \neg, 0)$ is an NMV-algebra.*

(iii) *For any NMV-algebra A and any bounded λ-semilattice with involutions S satisfying* (WE), *$\psi(\phi(A)) = A$ and $\phi(\psi(S)) = S$.*

P r o o f.

(i) We already know from Theorem 2 that $(A, \cup)$ is a bounded λ-semilattice. We show that the conditions (a), (b) and (c) of Lemma 3 are satisfied. It is obvious that $1 \to x = \neg 1 \oplus x = x$ and $x \cup y = \neg(\neg x \oplus y) \oplus y = (x \to y) \to y$. Now, due to the axiom (H), we have $y \leq y \oplus \neg x = \neg x \oplus y$ whence

$$((x \to y) \to y) \to y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = (\neg x \oplus y) \cup y = \neg x \oplus y = x \to y$$

verifying (c). So by Lemma 3, $\phi(A) = (A, \cup, \to, 0, 1)$ is a bounded λ-semilattice with involutions. Finally, $\phi(A)$ fulfils (WE) since

$$\begin{aligned} x \to (y \to 0) &= \neg x \oplus (\neg y \oplus 0) = \neg x \oplus \neg y = \neg y \oplus \neg x \\ &= \neg y \oplus (\neg x \oplus 0) = y \to (x \to 0). \end{aligned}$$

(ii) Let $(S, \cup, \to, 0, 1)$ be a bounded λ-semilattice with involutions that satisfies (WE). It is worth noticing that $\neg x \oplus y = ((x \to 0) \to 0) \to y = (x \cup 0) \to y$ $= x \to y$.

(MV2): $x \oplus y = (x \to 0) \to y = (x \to 0) \to ((y \to 0) \to 0) = (y \to 0) \to ((x \to 0) \to 0) = (y \to 0) \to x = y \oplus x$ by (WE).

(MV3): $x \oplus 0 = (x \to 0) \to 0 = x$.

(MV4): $\neg\neg x = (x \to 0) \to 0 = x$.

(MV5): $x \oplus 1 = (x \to 0) \to 1 = 1$.

(MV6): $\neg(\neg x \oplus y) \oplus y = (x \to y) \to y = x \cup y = (y \to x) \to x = \neg(\neg y \oplus x) \oplus x$.

(WA): $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) =$
$x \to ((((x \to y) \to y) \to z) \to z) = x \to ((x \cup y) \cup z) = 1$
since $x \leq (x \cup y) \cup z$ by (S3).

(H): $\neg x \oplus (x \oplus y) = x \to ((x \to 0) \to y) = 1$ since $x \leq (y \to 0) \to x = (x \to 0) \to y$ by Lemma 6 (ii).

(iii) Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. Define $\phi(A) = (A, \cup, \to, 0, 1)$ and $\psi(\phi(A)) = (A, \oplus', \neg', 0)$. We have $x \oplus' y = (x \to 0) \to y = \neg(\neg x \oplus 0) \oplus y = x \oplus y$ and $\neg' x = x \to 0 = \neg x \oplus 0 = \neg x$. Thus $\psi(\phi(A)) = A$.

Conversely, let $(S, \cup, \to, 0, 1)$ be a bounded λ-semilattice with involutions that fulfils (WE). Define $\psi(S) = (S, \oplus, \neg, 0)$ and $\phi(\psi(S)) = (S, \cup', \to', 0, 1')$. We have $x \cup' y = \neg(\neg x \oplus y) \oplus y = (x \to y) \to y = x \cup y$, $x \to' y = \neg x \oplus y = x \to y$ and $1' = \neg 0 = 0 \to 0 = 1$, so that $\phi(\psi(S)) = S$. □

Corollary 9. *Let* $(S, \cup, \to, 0, 1)$ *be a bounded* λ*-semilattice with involutions satisfying* (WE)*. Then* $(S, \cup, \cap, \to, 0, 1)$*, where* $x \cap y = ((x \to y) \to (x \to 0)) \to 0$*, is a bounded* λ*-lattice with involutions.*

P r o o f. By Theorem 8(ii), $(S, \oplus, \neg, 0)$ is an NMV-algebra and by Theorem 2 we know that $(S, \cup, \cap)$ is a bounded λ-lattice in which

$$\begin{aligned} x \cap y &= \neg(\neg x \cup \neg y) \\ &= (((y \to 0) \to (x \to 0)) \to (x \to 0)) \to 0 \\ &= ((x \to ((y \to 0) \to 0)) \to (x \to 0)) \to 0 \\ &= ((x \to y) \to (x \to 0)) \to 0. \end{aligned}$$

$\square$

Remark 10. Though every NMV-algebra, as well as every bounded λ-semilattice with involutions satisfying (WE), is a λ-lattice, *Theorem 8 does not hold for λ-lattices.* The reason is that $x \cap y$ need not be the greatest lower bound of $\{x, y\}$, and consequently, the operation $\cap$ defined in Corollary 9 is not the only possible one which makes $(S, \cup, \to, 0, 1)$ into a λ-lattice:

Example 11. Consider the λ-lattice $(S, \cup, \cap_1)$ from Figure 2. Let the involutions f_0, f_a and f_b in the non-trivial sections be given as follows:

- $f_0(a) = d$, $f_0(b) = c$, $f_0(c) = b$ and $f_0(d) = a$,
- $f_a(c) = c$ and $f_a(d) = d$,
- $f_b(c) = d$ and $f_b(d) = c$.

The operation $\to$ is given by Table 3, so that $(S, \cup, \cap_1, \to, 0, 1)$ is a bounded λ-lattice with involutions. A straightforward verification yields that $\to$ obeys (WE), and hence $(S, \oplus, \neg, 0)$ is an NMV-algebra, where the operations $\oplus$ and $\neg$ are given by Table 4. Now, upon setting $x \cap y := \neg(\neg x \cup \neg y)$, $(S, \cup, \cap)$ is a λ-lattice, but $\cap$ does not agree with the initial $\cap_1$. Indeed, we have $c \cap d = \neg(\neg c \cup \neg d) = \neg c = b \neq a = c \cap_1 d$. Therefore, the part (iii) of Theorem 8 does not work in the case of λ-lattices with involutions.

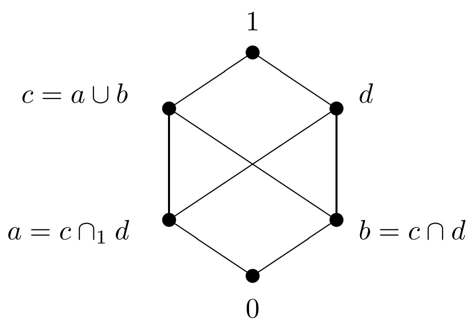

FIGURE 2

By Theorem 7 and Theorem 8 (i) we get

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	d	c	c	1	1
1	0	a	b	c	d	1

Table 3

$\oplus$	0	a	b	c	d	1	$\neg$
0	0	a	b	c	d	1	1
a	a	d	c	c	1	1	d
b	b	c	d	1	d	1	c
c	c	c	1	1	1	1	b
d	d	1	d	1	1	1	a
1	1	1	1	1	1	1	0

Table 4

Corollary 12. *The variety of all NMV-algebras is regular and arithmetical.*

3. Implication reducts

There exist several equivalent counterparts of MV-algebras; for instance, MV-algebras are term equivalent to bounded weak implication algebras which were introduced in [4] as a generalization of J. C. Abbott's implication algebras (see [1]). We recall that an *implication algebra* is an algebra $(A, \to)$ satisfying the equations

(I1) $(x \to y) \to x = x$,

(I2) $(x \to y) \to y = (y \to x) \to x$,

(I3) $x \to (y \to z) = y \to (x \to z)$.

These axioms capture the basic properties of the implication in the classical propositional calculus. Starting from the implication in the Łukasiewicz logic, we obtain weak implication algebras: An algebra $(A, \to, 1)$ with a binary operation $\to$ and a constant 1 is called a *weak implication algebra* if it fulfils (I2), (I3) and

(I0) $x \to 1 = 1$, $1 \to x = x$.

It is not hard to show that if $(A, \oplus, \neg, 0)$ is an MV-algebra then $(A, \to, 1)$ is a weak implication algebra, where $x \to y$ is defined as $\neg x \oplus y$.

Every weak implication algebra is a join-semilattice with 1 at the top with respect to the partial order given by $x \leq y$ iff $x \to y = 1$; $x \vee y = (x \to y) \to y$ is the supremum of any pair x, y.

A *bounded weak implication algebra* is a structure $(A, \to, 0, 1)$ such that $(A, \to, 1)$ is a weak implication algebra with the least element 0. Clearly, this is equivalent to the identity $0 \to x = 1$. Bounded weak implication algebras are known in the literature under the name *bounded commutative BCK-algebras* (see e.g. [7]).

This motivates us to describe the generalization of weak implication algebras which corresponds to our NMV-algebras.

Definition 13. An *NMV-implication algebra* is an algebra $(A, \to, 0, 1)$ of type $(2, 0, 0)$ that satisfies the following identities:

(NI1) $x \to 1 = 1$, $1 \to x = x$ and $0 \to x = 1$,

(NI2) $(x \to y) \to y = (y \to x) \to x$,

(NI3) $x \to (y \to 0) = y \to (x \to 0)$,

(NI4) $x \to ((((x \to y) \to y) \to z) \to z) = 1$,

(NI5) $((x \to y) \to y) \to y = x \to y$.

Comparing the above axioms with those of (weak) implication algebras, (NI1) includes (I0), (NI2) is precisely (I2) and (NI3) is another name for (WE) and rises as a weakening of (I3) by replacing z by 0. Furthermore, (NI4) captures (WA) and (NI5) is just (c) of Lemma 3.

Weak implication algebras are a particular case of NMV-implication ones. Indeed, any weak implication algebra fulfils (NI4) and (NI5) since in weak implication algebras we have $x \to ((((x \to y) \to y) \to z) \to z) = x \to (x \vee y \vee z) = 1$ and $((x \to y) \to y) \to y = (x \to y) \vee y = x \to y$.

Let us note that from (NI1) we can easily infer $x \to x = 1$.

Theorem 14. *Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. If we define $x \to y := \neg x \oplus y$, then $(A, \to, 0, 1)$ is an NMV-implication algebra.*

Conversely, if $(A, \to, 0, 1)$ is an NMV-implication algebra and if we put $x \oplus y := (x \to 0) \to y$ and $\neg x := x \to 0$, then $(A, \oplus, \neg, 0)$ is an NMV-algebra.

P r o o f. It is obvious at once that for each NMV-algebra $(A, \oplus, \neg, 0)$, the operation $\to$ satisfies all the identities (NI1)–(NI5), so $(A, \to, 0, 1)$ is an NMV-implication algebra.

Conversely, assume that $(A, \to, 0, 1)$ is an NMV-implication algebra. First, we note that for any $x \in A$ we have $(x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$ by (NI2) and (NI1), and hence $\neg x \oplus y = ((x \to 0) \to 0) \to y = x \to y$.

(MV2): $x \oplus y = (x \to 0) \to y = (x \to 0) \to ((y \to 0) \to 0) = (y \to 0) \to ((x \to 0) \to 0) = (y \to 0) \to x = y \oplus x$.

(MV3): $x \oplus 0 = (x \to 0) \to 0 = x$.

(MV4): $\neg\neg x = (x \to 0) \to 0 = x$.

(MV5): $x \oplus 1 = (x \to 0) \to 1 = 1$.

(MV6): Using $\neg x \oplus y = x \to y$ we obtain $\neg(\neg x \oplus y) \oplus y = (x \to y) \to y = (y \to x) \to x = \neg(\neg y \oplus x) \oplus x$ by (NI2). □

(WA): $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = x \to ((((x \to y) \to y) \to z) \to z) = 1$ by (NI4).

(H): We have $\neg x \oplus (x \oplus y) = x \to ((x \to 0) \to y) = x \to ((y \to 0) \to x)$, hence it is enough to show that $x \to (y \to x) = 1$ for all $x, y \in A$. This follows from (NI5), (NI2) and (NI1): $x \to (y \to x) = ((x \to (y \to x)) \to (y \to x)) \to (y \to x) = (((y \to x) \to x) \to x) \to (y \to x) = (y \to x) \to (y \to x) = 1$.

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