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SOME RESULTS ON $\mathbb{Q}$-GROUPS

M. R. Darafsheh — H. Sharifi

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ABSTRACT. A finite group G whose irreducible characters are rational valued is called a $\mathbb{Q}$-group. In this paper we will be concerned with the structure of a finite $\mathbb{Q}$-group that contains a strongly embedded subgroup and the structure of a finite $\mathbb{Q}$-group satisfying the property that none of its sections is isomorphic to $\mathbb{S}_4$.

1. Introduction

Let G be a finite group and χ be an irreducible complex character of G. The field generated by all $\chi(x)$, $x \in G$ is denoted by $\mathbb{Q}(\chi)$. By definition a complex character χ is called *rational* if $\mathbb{Q}(\chi) = \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers, and a finite group G is called a *rational group* or a $\mathbb{Q}$*-group* if every irreducible complex character of G is rational. Finite $\mathbb{Q}$-groups have been studied extensively, but classifying finite $\mathbb{Q}$-groups still remains an open research problem. It is easy to prove that a finite group G is a $\mathbb{Q}$-group if and only if for any $x \in G$ of order m the elements x and x^n are conjugate whenever n and m are relatively prime. Therefore the symmetric group $\mathbb{S}_n$ is an example of a $\mathbb{Q}$-group. Other examples of $\mathbb{Q}$-groups are the Weyl groups of the complex Lie algebras, see [2]. By [5] the prime divisors of the order of a finite solvable $\mathbb{Q}$-group can only be 2, 3 or 5. It is shown in [4] that the only non-abelian simple $\mathbb{Q}$-groups are the groups $SP_6(2)$ and $O_8^+(2)$. In [1, pp. 59-62] solvable $\mathbb{Q}$-groups with certain Sylow 2-subgroups are classified. It is shown that if a Sylow 2-subgroup of a $\mathbb{Q}$-group G is abelian, then G is a supersolvable $\{2, 3\}$-group. It is also shown that

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if G is a solvable non-nilpotent $\mathbb{Q}$-group with a Sylow 2-subgroup isomorphic to the quaternion group Q_8, then $G \cong E(3^n) : Q_8$ or $G \cong E(5^n) : Q_8$ where : denotes the semi-direct product of groups and $E(p^n)$ is an elementary abelian group of order p^n.

In [3] we described the structure of all Frobenius $\mathbb{Q}$-groups. In this note we first find all of $\mathbb{Q}$-groups that contain strongly embedded subgroups and using this, we classify $\mathbb{Q}$-groups that no section of them is isomorphic to the symmetric group on four letters, i.e. $\mathbb{S}_4$. All groups considered in this paper are finite and all characters are complex. The semi-direct product and central product of groups H and K is denoted by $H : K$ and $H \circ K$, respectively. Also, if p is a prime number, $E(p^n)$ denotes the elementary abelian p-group of order p^n. Finally, we write Q_8 for the quaternion group of order 8. Our main result is the following.

Main Theorem. *If G is a finite $\mathbb{Q}$-group in which no section of it is isomorphic to the symmetric group of degree four, then either $G \simeq P : \mathbb{Z}_2$ or $G \simeq E(p^n) : \mathbb{Q}_8$, where $n \in \mathbb{N}$, $p = 3$ or 5 and P is the Sylow 3-subgroup of G.*

2. $\mathbb{Q}$-groups with strongly embedded subgroups

First we will recall the following concept which is taken from ([8, Vol. 2, p. 391]).

Definition 1. A subgroup H of a finite group G is said to be *strongly embedded* in G if the following two conditions are satisfied:

(1) H is a proper subgroup of even order,

(2) For any element $x \in G - H$, the order of $H \cap H^x$ is odd.

In the following we will find the structure of a Sylow 2-subgroup of a $\mathbb{Q}$-group having a strongly embedded subgroup.

Lemma 1. *Let G be a finite $\mathbb{Q}$-group having a strongly embedded subgroup. Then a Sylow 2-subgroup of G is isomorphic to $\mathbb{Z}_2$ or Q_8.*

Proof. By [8, Vol. 2, p. 391] every Sylow 2-subgroup of G contains exactly one element of order 2. Thus a Sylow 2-subgroup P of G is either a cyclic group or a generalized quaternion group Q_{2^n}. If a Sylow 2-subgroup of G is the cyclic group $\mathbb{Z}_{2^n}$ of order 2^n, then $n = 1$, because by [7], $Z(P)$ is elementary abelian and hence $P \cong \mathbb{Z}_2$. Otherwise, $P = Q_{2^n}$ that is,

$$P = \langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ bab^{-1} = a^{-1} \rangle.$$

From what we mentioned in the introduction we deduce that G is a $\mathbb{Q}$-group if and only if $\frac{N_G(\langle x\rangle)}{C_G(\langle x\rangle)}$ is isomorphic to $\mathrm{Aut}(\langle x\rangle)$ for any arbitrary element x of G. Hence $[N_G(\langle a\rangle) : C_G(\langle a\rangle)] = \varphi(O(a)) = 2^{n-2}$ where $N_G(\langle a\rangle)$ and $C_G(\langle a\rangle)$ denote the normalizer and centralizer of a in G, respectively. Therefore $|\ N_G(\langle a\rangle)| = 2^{n-2}|C_G(\langle a\rangle)| \geq 2^{n-2}O(a) = 2^{2n-3}$. But Q_{2^n} is a Sylow 2-subgroup of G. Therefore $2n-3 \leq n$, it follows that $n = 3$ and $P = Q_8$. □

Now we can use [6] to determine the structure of finite $\mathbb{Q}$-groups having a strongly embedded subgroup.

Theorem 1. *Let G be a finite $\mathbb{Q}$-group with a strongly embedded subgroup. Then G is isomorphic to one of the following groups:*

(a) $G \cong G' : \mathbb{Z}_2$, *where G' is a 3-group.*

(b) $G \cong E(p^n) : Q_8$, *where $p = 3$ or 5.*

Proof. By Lemma 1 a Sylow 2-subgroup P of G is isomorphic to $\mathbb{Z}_2$ or Q_8. If $P \cong \mathbb{Z}_2$, then by [6] case (a) holds. If $P \cong Q_8$, then by [7, p. 35], case (b) holds. □

3. $\mathbb{Q}$-groups with strongly closed subgroups

In this section we will obtain the structure of $\mathbb{Q}$-groups with no section isomorphic to $\mathbb{S}_4$. First we recall the following concept.

Definition 2. Let P be a p-subgroup of a group G. A subgroup T of P is said to be *strongly closed subgroup* of G if for any element t of T and any element g of G, $t^g \in P$ implies that $t^g \in T$. Also T is said to be a *weakly closed subgroup*, if for any $g \in G$, $T^g \subset P$ implies that $T^g = T$.

It is easy to see that a strongly closed subgroup is weakly closed.

Before stating the next theorem, we will mention some well-known results. The proofs can be found in [8, Vol. 2, p. 601]. Glauberman proved the following theorem.

Theorem A. ([8, Vol. 2, p. 601]) *Let G be a group in which no section is isomorphic to the symmetric group of degree four. Then G contains a strongly closed abelian 2-subgroup.*

Definition 3. Let G be a p-group. For each natural number n we define $\Omega_n(G) = \langle x \mid x \in G,\ x^{p^n} = 1 \rangle$.

If G is an abelian p-group, then $\Omega_1(G)$ is the set of elements of order at most p. But, if G is not abelian, the statement is not necessarily true, it may happen that $\Omega_1(G)$ contains an element of order larger than p, for example consider $G = D_8$, the dihedral group of order 8.

Theorem B. ([8, Vol. 2, p. 590]) *Let S be a strongly closed 2-subgroup of G, and let H be the subgroup of G generated by all the conjugates of $\Omega_1(S)$. Let $I(S)$ denote the set of all involutions of S. Then we have one of the following two cases:*

(1) $H \subset \big\langle C_G(\langle s \rangle) \mid s \in I(S) \big\rangle$.

(2) *The group H contains a strongly embedded subgroup.*

Let us fix the notation of Theorem B and let T be a Sylow 2-subgroup of G such that $S \subset T$. Since $H \trianglelefteq G$, we have $T \cap H \in \mathrm{Syl}_2(H)$. So, for any $g \in G$, there is an element h of H such that $\Omega_1(S)^{gh} \subset T \cap H$. Since S is strongly closed, we get $\Omega_1(S)^{gh} \subset S$. This implies that $\Omega_1(S)^{gh} = \Omega_1(S)$. It follows that $H = \langle \Omega_1(S)^H \rangle$. Thus for the proof of Theorem B it is enough to assume $G = H$ as indicated in [8].

Goldschmidt proved the following theorem ([8, Vol. 2, p. 586]).

Theorem C. *Let S be a strongly closed abelian 2-subgroup of a group G such that $S \neq \{1\}$. Let K be the normal subgroup of G generated by all the conjugates of S, and set $\overline{K} = \frac{K}{O(K)}$. Then,*

(1) *$\overline{K}$ is the central product of a semi-simple group and an abelian 2-group, and each component of $\overline{K}$ is a quasi-simple group associated with one of the simple groups in the following list:*

$$\begin{aligned} &PSL(2, 2^n), \quad PSU(3, 2^n), \quad Sz(2^n), \\ &PSL(2, q)\ (q \equiv \pm 3 \pmod 8), \quad J_1, \quad \textit{or} \\ &\textit{The simple group of Ree type.} \end{aligned}$$

(2) $S = O_2(K)\Omega_1(T)$ *for some* $T \in \mathrm{Syl}_2(K)$.

Now we state and prove our main theorem.

Main Theorem. *Let G be a finite $\mathbb{Q}$-group such that no section of it is isomorphic to the symmetric group $\mathbb{S}_4$. Then we have one of the following cases:*

(1) $G \simeq G' : \mathbb{Z}_2$, *where G' is a 3-group.*

(2) $G \simeq E(p^n) : \mathbb{Q}_8$ *where n is a non-negative integer and $p = 3$ or 5.*

Proof. By Theorem A, G contains a strongly closed abelian 2-subgroup S. On the other hand by Theorem B, if H is a subgroup of G generated by all the conjugates of $\Omega_1(S)$, then either $H \subset \langle C_G(\langle s\rangle) \mid s \in I(S)\rangle$ or H contains a strongly embedded subgroup. But, we may suppose that $G = H$. To prove this theorem, it is enough to show that $G \neq \langle C_G(\langle s\rangle) \mid s \in I(S)\rangle$. Suppose $G = \langle C_G(\langle s\rangle) \mid s \in I(S)\rangle$, then since S is abelian, $\Omega_1(S) = I(S) \cup \{1\}$ implying $\Omega_1(S) \subset S$. Thus, by Theorem C, $G = \langle \Omega(S)^g \mid g \in G\rangle = \langle C_G(\langle s\rangle) \mid s \in I(S)\rangle$ implying that $G \subseteq K$, therefore $G = K = \langle S^g \mid g \in G\rangle$. Let $\overline{G} = \frac{G}{O(\overline{G})} \simeq A \circ B$, where A is semi-simple, B is an abelian 2-group and $\circ$ denotes the central product. By [8, Vol.1, p. 137], $A \cap B = Z(A) \cap B$ and $Z(A \circ B) = Z(A) \circ Z(B) = Z(A) \circ B$. Since the quotient group of a $\mathbb{Q}$-group is a $\mathbb{Q}$-group, $\frac{\overline{G}}{Z(\overline{G})}$ is a $\mathbb{Q}$-group. But

$$\frac{\overline{G}}{Z(\overline{G})} \simeq \frac{A \circ B}{Z(A \circ B)} \simeq \frac{\frac{A\circ B}{B}}{\frac{Z(A)\circ B}{B}} \simeq \frac{\frac{A}{A\cap B}}{\frac{Z(A)}{Z(A)\cap B}} \simeq \frac{A}{Z(A)}$$

and $\frac{A}{Z(A)}$ is a direct product of non-abelian simple groups listed in Theorem C. We see that none of them is a $\mathbb{Q}$-group. By [4] the only non-abelian simple rational $\mathbb{Q}$-groups are $SP_6(2)$ and $O_8^+(2)$. But $\frac{\overline{G}}{Z(\overline{G})}$ is a $\mathbb{Q}$-group and therefore we reached a contradiction. Therefore G contains a strongly embedded subgroup. Therefore by Theorem 1 the conclusion follows. □

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Department of Mathematics
Statistics and Computer Sciences
Faculty of Science
University of Tehran
Tehran
IRAN
E-mail: darafsheh@ut.ac.ir
hsharifi@khayam.ut.ac.ir