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# QUASI-UNIFORM COMPLETIONS OF PARTIALLY ORDERED SPACES

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ABSTRACT. In this paper we define partially ordered quasi-uniform spaces $(X, \mathfrak{U}, \leq)$ (PO-quasi-uniform spaces) as those spaces with a biconvex quasi-uniformity $\mathfrak{U}$ on the poset $(X, \leq)$ and give a construction of a (transitive) biconvex compatible quasi-uniformity on a partially ordered topological space when its topology satisfies certain natural conditions. We also show that under certain conditions on the topology $\tau_{\mathfrak{U}^*}$ of a PO-quasi-uniform space $(X, \mathfrak{U}, \leq)$, the bicompletion $(\widetilde{X}, \widetilde{\mathfrak{U}})$ of $(X, \mathfrak{U})$ is also a PO-quasi-uniform space $(\widetilde{X}, \widetilde{\mathfrak{U}}, \preceq)$ with a partial order $\preceq$ on $\widetilde{X}$ that extends $\leq$ in a natural way.



## 1. Introduction

Throughout the paper by a quasi-uniformity on a set $X$ we understand a quasi-uniformity defined by entourages of $X$ ([8]). Recall that if $X$ is a set and $A$, $B$ are relations on $X$, then the *inverse relation of* $A$ is the set $A^{-1} = \{(x, y) : (y, x) \in A\}$, and the *composition of* $A$ *and* $B$ is the set $A \circ B = \{(x, y) :$ there exists $z \in X$ such that $(x, z) \in A$ and $(z, y) \in B\}$. The diagonal of the Cartesian product $X \times X$ is denoted by $\Delta$. Every relation on $X$ that contains $\Delta$ is called an *entourage*. If $x$ is a point in $X$ and $U$ is an entourage on $X$, we denote by $U(x)$ the set $\{y \in X : (x, y) \in U\}$.

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**Definition 1.1.** A *quasi-uniformity* on a set $X$ is a non-empty collection $\mathfrak{U}$ of entourages on $X$ which satisfies the following conditions:

(QU1) If $V \in \mathfrak{U}$ and $U$ is an entourage on $X$ such that $V \subset U$, then $U \in \mathfrak{U}$.

(QU2) If $U, V \in \mathfrak{U}$, then $U \cap V \in \mathfrak{U}$.

(QU3) For every $U \in \mathfrak{U}$ there exists a $V \in \mathfrak{U}$ such that $V \circ V \subset U$.

The pair $(X, \mathfrak{U})$ is called a quasi-uniform space.

Although other ways of defining quasi-uniformities are known ([10]), the above is the most widely used.

**Definition 1.2.** A subfamily $\mathfrak{B} \subset \mathfrak{U}$ is called a *base for the quasi-uniformity* $\mathfrak{U}$ if for each $U \in \mathfrak{U}$ there exists a $V \in \mathfrak{B}$ such that $V \subset U$. A subfamily $\mathfrak{S} \subset \mathfrak{U}$ is said to be a subbase for the quasi-uniformity $\mathfrak{U}$ if the family of finite intersections of members of $\mathfrak{S}$ is a base for $\mathfrak{U}$.

If $\mathfrak{U}$ is a quasi-uniformity on a set $X$, then the *conjugate* of $\mathfrak{U}$, that is the collection $\mathfrak{U}^{-1} = \{U^{-1} : U \in \mathfrak{U}\}$ is also a quasi-uniformity on $X$. A quasi-uniformity $\mathfrak{U}$ on $X$ that is equal to its conjugate is called a *uniformity*. In this case the pair $(X, \mathfrak{U})$ is called a *uniform space*. For further reading on the topic of uniformities see [2], [11], [14].

If $X$ is a set and $\mathfrak{U}$ a quasi-uniformity on $X$, then $\{U \cap U^{-1} : U \in \mathfrak{U}\}$ is a base for a uniformity on $X$ denoted by $\mathfrak{U}^*$ and $\mathfrak{U}^*$ is the coarsest uniformity containing $\mathfrak{U}$.

A (sub)base $\mathfrak{B}$ for a quasi-uniformity is said to be transitive if each $B \in \mathfrak{B}$ is a transitive relation. A quasi-uniformity with a transitive (sub)base is called a transitive quasi-uniformity.

For a quasi-uniformity $\mathfrak{U}$, by $\tau_{\mathfrak{U}}$ we understand the topology on $X$ generated by this quasi-uniformity. Precisely, $\tau_{\mathfrak{U}}$ is the collection $\{A \subset X :$ for each $x \in A$ there is $U \in \mathfrak{U}$ such that $U(x) \subset A\}$. If $\tau_{\mathfrak{U}} = \tau$, for some topology $\tau$, then $\mathfrak{U}$ is said to be *compatible* with $\tau$ and $(X, \tau)$ is said to *admit* $\mathfrak{U}$.

For further reading and recent advancements on quasi-uniform spaces one can consult [12], [13].

Let $X$ be a partially ordered set (poset) with order relation $\leq$. To exclude trivial cases assume that all posets are nonempty. For any subset $Y$ of a poset $X$, we define $Y^{\rightarrow} = \{x \in X :$ there is a $y \in Y$ such that $y \leq x\}$ and $Y^{\leftarrow} = \{x \in X :$ there is a $y \in Y$ such that $x \leq y\}$. The set of all *lower* (*upper*) *bounds* of $Y$ is denoted by $Y^{-}$ ($Y^{+}$).

There are many different intrinsic topologies that one can consider on a poset (see for example [3], [4], [5]). Two particular topologies that have been investigated are the following:

**Definition 1.3.** Let $(X, \leq)$ be a poset. The *open interval topology* on $X$, denoted by $\theta_{\mathrm{oi}}(X)$ is the topology obtained from the subbase $\mathcal{S}_{\mathrm{oi}}(X) = \bigl\{]x, \rightarrow[, ]\leftarrow, x[ :\ x \in X\bigr\}$. The interval topology on $X$, denoted by $\theta_{\mathrm{i}}(X)$ is the topology obtained from the subbase $\mathcal{S}_{\mathrm{i}}(X) = \bigl\{X \setminus ]\leftarrow, x],\ X \setminus [x, \rightarrow[ :\ x \in X\bigr\}$.

Clearly, all open intervals are open in the open interval topology ("oi-open"), and the open interval topology is the coarsest topology possessing this property. Also, all closed intervals are closed in the interval topology ("i-closed"), and the interval topology is the coarsest such topology. The open interval topology and the interval topology are in general incomparable. Indeed, let $L(\mathbb{R}^2)$ be the lattice of all linear subspaces of $\mathbb{R}^2$, and let $p$ be any one dimensional subspace of $\mathbb{R}^2$. The set $L(\mathbb{R}^2) \setminus [p, \rightarrow[ \cap L(\mathbb{R}^2) \setminus ]\leftarrow, \{0\}]$ is open in $(L(\mathbb{R}^2), \theta_{\mathrm{i}})$, but it is not open in $(L(\mathbb{R}^2), \theta_{\mathrm{oi}})$. This is because for every one dimensional subspace $q$ of $\mathbb{R}^2$, $q \neq p$, every oi-open set in $L(\mathbb{R}^2)$ containing $q$ will also contain $p$. On the other hand, for every one dimensional subspace $p$ of $\mathbb{R}^2$, $\{\{0\}\} = ]\leftarrow, p[$. Hence $\{\{0\}\}$ is open in $(L(\mathbb{R}), \theta_{\mathrm{oi}})$, but it is not open in $(L(\mathbb{R}), \theta_{\mathrm{i}})$.

The following proposition will be needed later.

**Proposition 1.1.** *Let $(X, \theta)$ be a topological space such that $\theta_{\mathrm{oi}} \subset \theta$, then the following holds: If $G \in \theta$, then $G^{\leftarrow}, G^{\rightarrow} \in \theta$.*

Proof. Let $G \in \theta$, then $G^{\leftarrow} = \bigcup\bigl\{]\leftarrow, x[ :\ x \in G\bigr\} \cup G$ and $G^{\rightarrow} = \bigcup\bigl\{]x, \rightarrow[ :\ x \in G\bigr\} \cup G$. Thus $G^{\leftarrow}, G^{\rightarrow} \in \theta$. □

Below we would need the following topology on a poset $(X, \leq)$ which we denote as $\theta_{\mathrm{r}}(X)$.

**Definition 1.4.** The *real topology* on $X$, denoted by $\theta_{\mathrm{r}}(X)$ is the topology obtained from the subbase $\mathcal{S}_{\mathrm{r}}(X) = \bigl\{]x, \rightarrow[, ]\leftarrow, x[, X \setminus [x, \rightarrow[,\ X \setminus ]\leftarrow, x] :\ x \in X\bigr\}$.

Thus the real topology is the coarsest topology with the property that all open intervals are open ("r-open") and all closed intervals are closed ("r-closed"). Since the open interval topology and the interval topology are in general incomparable, $\theta_{\mathrm{r}}$ is in general finer than both $\theta_{\mathrm{oi}}$ and $\theta_{\mathrm{i}}$. It is not difficult to see that for linearly ordered sets the three topologies $\theta_{\mathrm{oi}}, \theta_{\mathrm{i}}$ and $\theta_{\mathrm{r}}$ coincide.

## 2. Partially ordered quasi-uniform spaces

**Definition 2.1.** An entourage $U$ on a poset $(X, \leq)$ is said to be *biconvex* if for every $x \in X$, $U(x)$ and $U^{-1}(x)$ are convex subsets of $X$. A quasi-uniformity on $(X, \leq)$ is called biconvex if it has a base consisting of biconvex entourages.

**Proposition 2.1.** *If $\mathfrak{U}$ is a biconvex quasi-uniformity on a poset $X$, then $\mathfrak{U}^*$ is a convex uniformity on $X$.*

Proof. Assume that $\mathfrak{U}$ is a biconvex quasi-uniformity on a poset $X$. Then there exists a base $\mathfrak{B}$ for $\mathfrak{U}$ such that for each $U \in \mathfrak{B}$, $U(x)$ and $U^{-1}(x)$ are convex for all $x \in X$, and consequently $U(x) \cap U^{-1}(x)$ is convex for all $x \in X$. Since $\{U \cap U^{-1} : \ U \in \mathfrak{B}\}$ is a symmetric base for the uniformity $\mathfrak{U}^*$ on $X$, $\mathfrak{U}^*$ is a convex uniformity on $X$. □

Observe that the converse does not hold. For let $\mathfrak{V}$ be the quasi-uniformity on a poset $X$ generated by the base $\{\Delta\}$, and let $a$, $b$ be distinct points in $X$ such that $]a, b[ \neq \emptyset$. Then any base for the quasi-uniformity $\mathfrak{U} = \big\{V \cup \{(a, b)\} : V \in \mathfrak{V}\big\}$ must include $\mathfrak{B} = \big\{\Delta \cup \{(a, b)\}\big\}$, which is itself a base for $\mathfrak{U}$. Let $U_0 \in \mathfrak{B}$, then $(U_0 \cap U_0^{-1})(x) = \Delta(x)$ is convex for all $x \in X$, but neither $U_0(a) = \{a, b\}$ nor $U_0^{-1}(b) = \big(\Delta \cup \{(b, a)\}\big)(b) = \{b, a\}$ is convex. Thus although $\mathfrak{U}^*$ is a convex uniformity on $X$, $\mathfrak{U}$ is not a biconvex quasi-uniformity on $X$.

**Definition 2.2.** A triple $(X, \mathfrak{U}, \leq)$ is called a *partially ordered quasi-uniform space* (PO-quasi-uniform space) if $\mathfrak{U}$ is a biconvex quasi-uniformity on the poset $(X, \leq)$.

The following proposition gives a construction of a (transitive) biconvex compatible quasi-uniformity on a partially ordered topological space when its topology satisfies certain natural conditions. Below by a partially ordered topological space $(X, \theta, \leq)$ we understand a partially ordered set $(X, \leq)$ with a topology $\theta$ on $X$.

**Proposition 2.2.** *Let $(X, \theta, \leq)$ be a partially ordered topological space which satisfies the following conditions:*

(a) *$\theta$ has a convex base $\mathcal{B}_\theta$,*

(b) *for every $G \in \mathcal{B}_\theta$: $G^{\leftarrow}, G^{\rightarrow} \in \theta$.*

*Then $\mathfrak{S} = \{S(G) : \ G \in \mathcal{B}_\theta\}$, where*

$$S(G) = [G \times G] \cup [(G^{\leftarrow} \setminus G) \times G^{\leftarrow}] \cup [(G^{\rightarrow} \setminus G) \times G^{\rightarrow}] \cup [(X \setminus (G^{\leftarrow} \cup G^{\rightarrow})) \times X]\ ,$$

*for a subset $G$ of $X$, is a subbase for a quasi-uniformity $\mathfrak{U}$ on $X$ which is:*

(1) *transitive,*

(2) *biconvex,*

(3) *compatible with $\theta$, i.e., $\tau_{\mathfrak{U}} = \theta$.*

One can note that by Proposition 1.1, (b) is satisfied if $\theta_{\mathrm{oi}} \subset \theta$. Also,

$$S^{-1}(G) = [G \times X] \cup [(G^{\leftarrow} \setminus G) \times (X \setminus G^{\rightarrow})] \cup [(G^{\rightarrow} \setminus G) \times (X \setminus G^{\leftarrow})] \cup [(X \setminus (G^{\leftarrow} \cup G^{\rightarrow})) \times (X \setminus (G^{\leftarrow} \cup G^{\rightarrow}))]$$

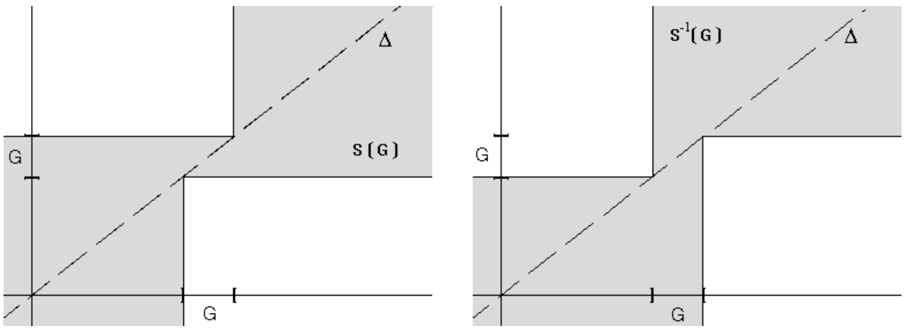


FIGURE 1. $S(G)$ and $S^{-1}(G)$ in a linearly ordered space

Proof. For every $G \in \theta$ we have that $\Delta \subset S(G)$ and it can be easily seen that $S(G) \circ S(G) = S(G)$. Therefore $\mathfrak{S}$ is a subbase for a transitive quasi-uniformity, which we denote by $\mathfrak{U}$.

Let $B \in \mathfrak{B}$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$ consisting of finite intersections of elements of $\mathfrak{S}$. There exists $\{S(G_i) : G_i \in \mathcal{B}_\theta,\ 1 \le i \le n\}$, for some $n \in \mathbb{N}$, such that $B = \bigcap_{i=1}^{n} S(G_i)$. Let $x \in X$, then $B(x) = \Big[\bigcap_{i=1}^{n} S(G_i)\Big](x) = \bigcap_{i=1}^{n} S(G_i)(x)$ and since each $S(G_i)(x)$ is convex, $B(x)$ is convex. Similarly, $B^{-1}(x) = \Big[\bigcap_{i=1}^{n} S^{-1}(G_i)\Big](x) = \bigcap_{i=1}^{n} S^{-1}(G_i)(x)$ and since each $S^{-1}(G_i)(x)$ is convex, $B^{-1}(x)$ is convex.

Let $A \in \theta$ and let $x \in A$. There exists convex $G \in \mathcal{B}_\theta$ such that $x \in G \subset A$. Since $S(G)(x) = G$, $S(G)(x) \subset A$. Hence $A \in \tau_{\mathfrak{U}}$, and consequently $\theta \subset \tau_{\mathfrak{U}}$. Conversely, let $A \in \tau_{\mathfrak{U}}$ and let $x \in A$. There exists $U \in \mathfrak{U}$ such that $U(x) \subset A$ and there exist $n \in \mathbb{N}$ and $G_i \in \mathcal{B}_\theta$, $1 \le i \le n$, such that $\bigcap_{i=1}^{n} S(G_i) \subset U$. Since $S(G_i)(x) \in \theta$ for $i = 1, \dots, n$, we have $\bigcap_{i=1}^{n} S(G_i)(x) \in \theta$ and $U(x)$ is a neighborhood of $x$ in $\theta$. Consequently, $A \in \theta$ and $\tau_{\mathfrak{U}} \subset \theta$. □

**Proposition 2.3.** *Let $(X,\theta,\leq)$ be a partially ordered topological space, and let $\{\mathfrak{U}_i : i \in \mathcal{I}\}$ be a nonempty collection of biconvex quasi-uniformities compatible with $\theta$. Then $\mathfrak{U} = \sup\{\mathfrak{U}_i : i \in \mathcal{I}\}$ is a biconvex quasi-uniformity compatible with $\theta$. If for each $i \in \mathcal{I}$, $\mathfrak{U}_i$ is transitive, then $\mathfrak{U}$ is also transitive.*

Proof. Let $(X,\theta,\leq)$ be a partially ordered topological space, and let $\{\mathfrak{U}_i : i \in \mathcal{I}\}$ be an arbitrary collection of biconvex quasi-uniformities compatible with $\theta$. Then a subbase for the quasi-uniformity $\mathfrak{U} = \sup\{\mathfrak{U}_i : i \in \mathcal{I}\}$ is $\bigcup\{\mathfrak{U}_i : i \in \mathcal{I}\}$. Let $U \in \mathfrak{U}$. Then there exists $B \in \mathfrak{U}$ such that $B \subset U$ and $B = \bigcap_{j=1}^{n} U_j$, where $n \in \mathbb{N}$ and $U_j \in \mathfrak{U}_{i_j}$, for $i_j \in \mathcal{I}$. Since each $\mathfrak{U}_{i_j}$ is biconvex, we may assume that for $1 \leq j \leq n$, $U_j(x)$ and $U_j^{-1}(x)$ are convex for all $x \in X$. It follows that $B(x)$ and $B^{-1}(x)$ are convex for all $x \in X$. Hence the quasi-uniformity $\mathfrak{U}$ is biconvex.

The proof that $\mathfrak{U}$ is compatible with $\theta$ is standard.

If each $\mathfrak{U}_i$ is transitive, then each $\mathfrak{U}_i$ has a transitive base, say $\mathfrak{B}_i$. For each $U \in \mathfrak{U}$ there exist $\{U_j : U_j \in \mathfrak{U}_{i_j},\ 1 \leq j \leq n,\ i_j \in \mathcal{I}\}$ for some $n \in \mathbb{N}$ such that $\bigcap_{j=1}^{n} U_j \subset U$. Then there exist $\{B_j : B_j \in \mathfrak{B}_{i_j},\ 1 \leq j \leq n,\ i_j \in \mathcal{I}\}$ such that $\bigcap_{j=1}^{n} B_j \subset \bigcap_{j=1}^{n} U_j \subset U$ and $\bigcap_{j=1}^{n} B_j$ is a transitive relation. Hence $\bigcup\{\mathfrak{B}_i : i \in \mathcal{I}\}$ is a transitive subbase of $\mathfrak{U}$ and so $\mathfrak{U}$ is transitive. □

**Corollary 2.4.** *If $(X,\theta,\leq)$ is a partially ordered topological space such that $\theta$ has a base $\mathcal{B}_\theta$ consisting of convex sets and for every $G \in \mathcal{B}_\theta$ both $G^{\leftarrow}, G^{\rightarrow} \in \theta$, then there exists a finest biconvex (transitive) quasi-uniformity compatible with $\theta$.*

## 3. (Bi)Completions of a PO-quasi-uniform space

Recall that a quasi-uniform space $(X,\mathfrak{U})$ is said to be bicomplete if every $\mathfrak{U}^*$-Cauchy filter on $X$ has a $\tau_{\mathfrak{U}^*}$-cluster point. This is equivalent to saying that every $\mathfrak{U}^*$-Cauchy filter on $X$ has a $\tau_{\mathfrak{U}^*}$-limit point. Thus a quasi-uniform space $(X,\mathfrak{U})$ is bicomplete if, and only if, the uniform space $(X,\mathfrak{U}^*)$ is complete. Uniform completions of linearly ordered topological spaces and generalized ordered spaces have been studied in [1], [8], [9]. One can also add that ideals have also been used to construct completions in [6], [7].

Below, given a point $x$ of a quasi-uniform space $(X,\mathfrak{U})$, by $\mathcal{N}_x$ we denote the $\tau_{\mathfrak{U}}$-neighborhood filter of $x$, by $\mathcal{N}_x^{-1}$ we denote the $\tau_{\mathfrak{U}^{-1}}$-neighborhood filter of $x$, and by $\mathcal{N}_x^*$ we denote the $\tau_{\mathfrak{U}^*}$-neighborhood filter of $x$.

Let $(X, \mathfrak{U})$ be a quasi-uniform space. A $\mathfrak{U}^*$-Cauchy filter on $X$ is said to be *minimal* if it contains no $\mathfrak{U}^*$-Cauchy filter other than itself. It is known that for each point $x$ in a quasi-uniform space $(X, \mathfrak{U})$, $\mathcal{N}_x^*$ is a minimal $\mathfrak{U}^*$-Cauchy filter on $X$.

A bicompletion of a quasi-uniform space $(X, \mathfrak{U})$ is a bicomplete quasi-uniform space $(Y, \mathfrak{V})$ that has a $\tau_{\mathfrak{V}^*}$-dense subspace quasi-unimorphic to $(X, \mathfrak{U})$. Every $T_0$ quasi-uniform space $(X, \mathfrak{U})$ has a $T_0$ bicompletion $(\widetilde{X}, \widetilde{\mathfrak{U}})$. The construction is as follows. Let $\widetilde{X}$ be the set of all minimal $\mathfrak{U}^*$-Cauchy filters on $X$ and map each point $x \in X$ to $\mathcal{N}_x^* \in \widetilde{X}$. For every $U \in \mathfrak{U}$, one defines $\widetilde{U} = \{(\mathcal{F}, \mathcal{G}) \in \widetilde{X} \times \widetilde{X} :$ there exists $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subset U\}$. Then $\{\widetilde{U} : U \in \mathfrak{U}\}$ is a base for a bicomplete quasi-uniformity $\widetilde{\mathfrak{U}}$ on $\widetilde{X}$.

**Theorem 3.1.** *Let $(X, \mathfrak{U}, \leq)$ be a PO-quasi-uniform space such that $\theta_{\mathrm{r}} \subset \tau_{\mathfrak{U}^*}$ and let $(\widetilde{X}, \widetilde{\mathfrak{U}})$ be the bicompletion of $(X, \mathfrak{U})$. Then there exists a partial order $\preceq$ on $\widetilde{X}$ which extends $\leq$ such that $(\widetilde{X}, \widetilde{\mathfrak{U}}, \preceq)$ is a PO-quasi-uniform space.*

**Remark 3.1.** If the quasi-uniformity $\mathfrak{U}$ is a uniformity and $\leq$ is a linear order, then the above extension $\preceq$ is the unique linear order on $(\widetilde{X}, \widetilde{\mathfrak{U}})$ that extends $\leq$ and makes $\widetilde{\mathfrak{U}}$ convex.

P r o o f. As explained above, $\widetilde{X}$ is the set of all minimal $\mathfrak{U}^*$-Cauchy filters on $X$. For $\mathcal{F}, \mathcal{G} \in \widetilde{X}$ we say that $\mathcal{F} \preceq \mathcal{G}$ if, and only if, $\mathcal{F} = \mathcal{G}$ or for any two sets $F \in \mathcal{F}$, $G \in \mathcal{G}$, there exist convex sets $F_1 \in \mathcal{F}$ and $G_1 \in \mathcal{G}$ such that $F_1 \subset F$, $G_1 \subset G$, and $F \cap G_1^- \neq \emptyset \neq F_1^+ \cap G$.

Evidently $\preceq$ is reflexive. To check that $\preceq$ is antisymmetric let $\mathcal{F} \preceq \mathcal{G}$ and $\mathcal{G} \preceq \mathcal{F}$ for some $\mathcal{F}, \mathcal{G} \in \widetilde{X}$. Assume that $\mathcal{F} \neq \mathcal{G}$, otherwise we are done. Choose any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then there exist convex $F_1, F_2, F_3 \in \mathcal{F}$ and convex $G_1, G_2, G_3 \in \mathcal{G}$ such that $F_3 \subset F_2 \subset F_1 \subset F$, $G_3 \subset G_2 \subset G_1 \subset G$, $F \cap G_1^- \neq \emptyset \neq F_1^+ \cap G$, $G_1 \cap F_2^- \neq \emptyset \neq G_2^+ \cap F_1$, and $F_2 \cap G_3^- \neq \emptyset \neq F_3^+ \cap G_2$. Let $a \in G_1 \cap F_2^-$, $b \in G_2^+ \cap F_1$, $c \in F_2 \cap G_3^-$ and $d \in F_3^+ \cap G_2$. Then $G_3$ is bounded below by $c \in F_2 \subset F_1$ and $F_3$ is bounded above by $d \in G_2 \subset G_1$. Since $G_2$ is bounded above by $b \in F_1$ and $G_3 \subset G_2$, $G_3$ is bounded above by $b$. Also since $F_2$ is bounded below by $a \in G_1$ and $F_3 \subset F_2$, $F_3$ is bounded below by $a$. Since both $F_1$ and $G_1$ are convex, $G_3 \subset [c, b] \subset F_1 \subset F$ and $F_3 \subset [a, d] \subset G_1 \subset G$. It follows that $F \in \mathcal{G}$ and $G \in \mathcal{F}$, and by the arbitrariness of $F$ and $G$, $\mathcal{F} = \mathcal{G}$. To check that $\preceq$ is transitive, let $\mathcal{F} \preceq \mathcal{G}$ and $\mathcal{G} \preceq \mathcal{H}$ for some $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \widetilde{X}$. Assume that $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are distinct, otherwise we are done. Choose any $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $H \in \mathcal{H}$, then there exist convex $F_1, F_2 \in \mathcal{F}$, $G_1, G_2, G_3, G_4 \in \mathcal{G}$ and $H_1, H_2 \in \mathcal{H}$ such that $F_2 \subset F_1 \subset F$, $G_4 \subset G_3 \subset G_2 \subset G_1 \subset G$, $H_2 \subset H_1 \subset H$, $F \cap G_1^- \neq \emptyset \neq F_1^+ \cap G$, $G_1 \cap H_1^- \neq \emptyset \neq G_2^+ \cap H$, $F_1 \cap G_3^- \neq \emptyset \neq F_2^+ \cap G_2$ and

$G_3 \cap H_2^- \neq \emptyset \neq G_4^+ \cap H_1$. Let $a \in G_2^+ \cap H$, $b \in F_1 \cap G_3^-$, $c \in F_2^+ \cap G_2$ and $d \in G_3 \cap H_2^-$. Then $F_2$ is bounded above by $c$ and $H_2$ is bounded below by $d$. Since $c \leq a, b \leq d$ and $F_1 \subset F$, we have that $a \in F_2^+ \cap H$ and $b \in H_2^- \cap F$. It follows that for any $F \in \mathcal{F}$ and $H \in \mathcal{H}$, there exist convex $F_2 \in \mathcal{F}$ and convex $H_2 \in \mathcal{H}$ such that $F \cap H_2^- \neq \emptyset \neq F_2^+ \cap H$, i.e., $\mathcal{F} \preceq \mathcal{H}$. Consequently $\preceq$ is a partial order on $X$.

To show that $\preceq$ extends $\leq$ we need to show that for every $x, y \in X$, $x \leq y$ if, and only if, $\mathcal{N}_x^* \preceq \mathcal{N}_y^*$.

Let $x, y \in X$ such that $x \leq y$, $x \neq y$, and let $F \in \mathcal{N}_x^*$, $G \in \mathcal{N}_y^*$. By Proposition 2.1, $\tau_{\mathfrak{U}^*}$ has a convex base, and so there exist convex sets $F' \in \mathcal{N}_x^*$ and $G' \in \mathcal{N}_y^*$ such that $F' \subset F$ and $G' \subset G$. Since $\theta_{\mathrm{oi}} \subset \theta_{\mathrm{r}} \subset \tau_{\mathfrak{U}^*}$, $]\leftarrow, y[\, \in \mathcal{N}_x^*$ and $]x, \rightarrow[\, \in \mathcal{N}_y^*$. Choose $F_1 = F' \cap\, ]\leftarrow, y[\, \in \mathcal{N}_x^*$ and $G_1 = G' \cap\, ]x, \rightarrow[\, \in \mathcal{N}_y^*$. Evidently, $F_1$ and $G_1$ are convex sets, $F_1 \subset F$ and $G_1 \subset G$. Since $x \in F \cap G_1^-$ and $y \in F_1^+ \cap G$, $F \cap G_1^- \neq \emptyset \neq F_1^+ \cap G$. Consequently $\mathcal{N}_x^* \preceq \mathcal{N}_y^*$. Note that if $x = y$, then $\mathcal{N}_x^* = \mathcal{N}_y^*$.

Let $x, y \in X$ such that $\mathcal{N}_x^* \preceq \mathcal{N}_y^*$. Since $\tau_{\mathfrak{U}^*}$ is $T_2$, $x = y$ whenever $\mathcal{N}_x^* = \mathcal{N}_y^*$; thus assume that $\mathcal{N}_x^* \neq \mathcal{N}_y^*$. Suppose that $x \not\leq y$. Since $\theta_{\mathrm{i}} \subset \theta_{\mathrm{r}} \subset \tau_{\mathfrak{U}^*}$, $G = X \setminus [x, \rightarrow[\, \in \mathcal{N}_y^*$ and $F = X \setminus ]\leftarrow, y] \in \mathcal{N}_x^*$. By definition there exist convex sets $F_1 \in \mathcal{N}_x^*$ and $G_1 \in \mathcal{N}_y^*$ such that $F_1 \subset F$, $G_1 \subset G$, and $F \cap G_1^- \neq \emptyset \neq F_1^+ \cap G$. Let $a \in G \cap F_1^+$, then $x \not\leq a$ and $x \leq a$, which is a contradiction. Consequently $x \leq y$.

Next we show that $(\widetilde{X}, \widetilde{\mathfrak{U}}, \preceq)$ is a PO-quasi-uniform space. As explained above, $\{\widetilde{U} : U \in \mathfrak{U}\}$ is a base for $\widetilde{\mathfrak{U}}$, where $\widetilde{U} = \{(\mathcal{F}, \mathcal{G}) \in \widetilde{X} \times \widetilde{X} :$ there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \times G \subset U\}$ for every $U \in \mathfrak{U}$. Since $\mathfrak{U}$ is a biconvex quasi-uniformity on $(X, \leq)$, one can take $\{\widetilde{U} : U$ is a biconvex member of $\mathfrak{U}\}$ as a base for $\widetilde{\mathfrak{U}}$.

Let $U$ be a biconvex member of $\mathfrak{U}$. To show that $\widetilde{U}(\mathcal{F})$ is convex for every $\mathcal{F} \in \widetilde{X}$, let $\mathcal{G}, \mathcal{H} \in \widetilde{U}(\mathcal{F})$ and $\mathcal{E} \in \widetilde{X}$ such that $\mathcal{G} \preceq \mathcal{E} \preceq \mathcal{H}$. Then there exist $F_1, F_2 \in \mathcal{F}$, $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $F_1 \times G \subset U$ and $F_2 \times H \subset U$; evidently for $F_3 = F_1 \cap F_2 \in \mathcal{F}$, $F_3 \times G \subset U$ and $F_3 \times H \subset U$. Let $E \in \mathcal{E}$; since $\mathcal{G} \preceq \mathcal{E} \preceq \mathcal{H}$, there exist convex $E_1, E_2 \in \mathcal{E}$, $G_1 \in \mathcal{G}$, $H_1 \in \mathcal{H}$ such that $E_2 \subset E_1 \subset E$, $G_1 \subset G$, $H_1 \subset H$ and $E_1^- \cap G \neq \emptyset \neq G_1^+ \cap E$, $E_1 \cap H_1^- \neq \emptyset \neq E_2^+ \cap H$. Let $a \in G \cap E_1^-$ and $b \in E_2^+ \cap H$; evidently $a \in G \cap E_2^-$. To show that $F_3 \times E_2 \subset U$, let $(p, q) \in F_3 \times E_2$. Then $a \leq q \leq b$. Since $a, b \in U(p)$ and $U(p)$ is convex, $q \in U(p)$. Thus $F_3 \times E_2 \subset U$ and consequently $\mathcal{E} \in \widetilde{U}(\mathcal{F})$. Furthermore, $U^{-1}(x)$ is convex for every $x \in X$, and so one can apply the same argument as the one above to show that $\widetilde{U}^{-1}(\mathcal{F})$ is convex for every $\mathcal{F} \in \widetilde{X}$. It follows that $\widetilde{\mathfrak{U}}$

is a biconvex quasi-uniformity on $(\widetilde{X}, \preceq)$, i.e., $(\widetilde{X}, \widetilde{\mathfrak{U}}, \preceq)$ is a PO-quasi-uniform space. □

Recall that a quasi-uniform space $(X, \mathfrak{U})$ is said to be complete if every $\mathfrak{U}$-Cauchy filter on $X$ has a $\tau_{\mathfrak{U}}$-cluster point. A completion of a quasi-uniform space $(X, \mathfrak{U})$ is a complete $T_1$ quasi-uniform space $(Y, \mathfrak{V})$ that has a dense subspace quasi-unimorphic to $(X, \mathfrak{U})$.

Unlike bicompletions, not every quasi-uniform space has a completion. The following result is known ([8]).

**Theorem 3.2.** *Let $(X, \mathfrak{U})$ be a $T_1$ quasi-uniform space. The following statements are equivalent:*

(a) $(X, \mathfrak{U})$ *has a $T_1$ completion.*

(b) *For each $\mathfrak{U}$-Cauchy filter $\mathcal{F}$,* $\mathrm{adh}_{\tau_{\mathfrak{U}^{-1}}} \mathcal{F} \subset \mathrm{adh}_{\tau_{\mathfrak{U}}} \mathcal{F}$.

(c) *For each $\mathfrak{U}$-Cauchy filter $\mathcal{F}$, if* $\mathrm{adh}_{\tau_{\mathfrak{U}^{-1}}} \mathcal{F} \neq \emptyset$, *then* $\mathrm{adh}_{\tau_{\mathfrak{U}}} \mathcal{F} \neq \emptyset$.

The completion, once it exists, is constructed as follows. Let $\mathbb{F} = \{\mathcal{F} : \mathcal{F}$ is a $\mathfrak{U}$-Cauchy filter on $X$ that has no cluster point$\}$ and let $\widetilde{X} = X \cup \mathbb{F}$. Let $\Phi$ be the collection of all choice functions that pick a member of each filter in $\mathbb{F}$, that is $\Phi = \{\phi : \ \phi \colon \mathbb{F} \to 2^X$ and $\phi(\mathcal{F}) \in \mathcal{F}\}$. For every $U \in \mathfrak{U}$ and every $\phi \in \Phi$ we define a set $S(U, \phi) = U \cup \Delta \cup \{(\mathcal{F}, x) \in \mathbb{F} \times X : \ x \in U(\phi(\mathcal{F}))\}$. It is shown that $\{S(U, \phi) : \ U \in \mathfrak{U}$ and $\phi \in \Phi\}$ is a base for a complete $T_1$ quasi-uniformity $\widetilde{\mathfrak{U}}$ on $\widetilde{X}$ and that $(\widetilde{X}, \widetilde{\mathfrak{U}})$ contains $(X, \mathfrak{U})$ as a dense open subset. With the following natural extension of the partial order similar to the one defined in Theorem 3.1 one would have hoped for a similar result to that obtained in the same theorem, unfortunately one does not. To be more precise:

Let $(X, \mathfrak{U}, \leq)$ be a PO-quasi-uniform space such that $\theta_{\mathrm{r}} \subset \tau_{\mathfrak{U}}$. If the completion $(\widetilde{X}, \widetilde{\mathfrak{U}})$ of $(X, \mathfrak{U})$ exists (see Proposition 3.2), then one can define a partial order $\preceq$ on $\widetilde{X}$ which extends $\leq$ in the following way.

Let $\widetilde{X} = X \cup \mathbb{F}$ as above. For $\mathfrak{U}$-Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ on $X$ we say that $\mathcal{F} \preceq \mathcal{G}$ if, and only if, $\mathcal{F} = \mathcal{G}$ or for any two sets $F \in \mathcal{F}$, $G \in \mathcal{G}$, there exist convex sets $F_1 \in \mathcal{F}$ and $G_1 \in \mathcal{G}$ such that $F_1 \subset F$, $G_1 \subset G$, and $F \cap G_1^- \neq \emptyset \neq F_1^+ \cap G$. For $x, y \in X$, we say that $x \preceq y$ if, and only if, $x \leq y$. For $x \in X$ and $\mathcal{G} \in \mathbb{F}$, we say that $x \preceq \mathcal{G}$ ($\mathcal{G} \preceq x$) if, and only if, $\mathcal{N}_x \preceq \mathcal{G}$ ($\mathcal{G} \preceq \mathcal{N}_x$), where $\mathcal{N}_x$ is the (unique) $\tau_{\mathfrak{U}}$-neighborhood filter of $x$.

The proof that $\preceq$ is a partial order on $\widetilde{X}$ is analogous to the statement in the proof of Proposition 3.1 and evidently, the partial order $\preceq$ on $\widetilde{X}$ extends the partial order $\leq$ on $X$.

Unfortunately, due to the way the completion is constructed, even if the completion $(\widetilde{X}, \widetilde{\mathfrak{U}})$ of $(X, \mathfrak{U})$ exists, $(\widetilde{X}, \widetilde{\mathfrak{U}}, \preceq)$ does not have to be a PO-quasi-uniform space. This is because $\widetilde{\mathfrak{U}}$ does not have to be biconvex as the following example shows. One must also add that this counterexample does not depend on the extension of the partial order and therefore no extension of the partial order to the completion in this case makes $(\widetilde{X}, \widetilde{\mathfrak{U}}, \preceq)$ a PO-quasi-uniform space.

*Example* 3.3. Let $Y = \mathbb{R} \setminus \{0\}$ and $Y_q = \,]-q, q[$ for every $q \in \mathbb{Q}^+$. Let $X$ be the disjoint union $Y \cup \bigcup_{q\in\mathbb{Q}^+} Y_q$ with the following partial order: The order in $Y$ and $Y_q$ is the standard linear order induced from $\mathbb{R}$. Also,

$x \not\leq y$ and $y \not\leq x$ for all $x \in Y_{q_1}$, $y \in Y_{q_2}$, where $q_1, q_2 \in \mathbb{Q}^+$, $q_1 \neq q_2$;

$x \not\leq y$ and $y \not\leq x$ for all $x \in Y_q$, $y \in Y$, where $q \in \mathbb{Q}^+$, $-q < y < q$ in $\mathbb{R}$;

$x < y$ for all $x \in Y_q$, $y \in Y$, where $q \in \mathbb{Q}^+$, $q \leq y$ in $\mathbb{R}$;

$y < x$ for all $x \in Y_q$, $y \in Y$, where $q \in \mathbb{Q}^+$, $y \leq -q$ in $\mathbb{R}$.

It can be easily seen that $(X, \leq)$ is a poset.

Let $\mathcal{B}_{\theta_{\mathrm{r}}}$ be the collection of all convex sets $A \in \theta_{\mathrm{r}}$ such that either $A \cap Y \neq \emptyset$ or $A \cap Y_q = \emptyset$ for all $q > x$ and some $x \in \mathbb{R}$. Then $\mathcal{B}_{\theta_{\mathrm{r}}}$ is a convex base for $\theta_{\mathrm{r}}$ and $\theta_{\mathrm{oi}} \subset \theta_{\mathrm{r}}$, so that $G^{\leftarrow}, G^{\rightarrow} \in \theta_{\mathrm{r}}$ for every $G \in \mathcal{B}_{\theta_{\mathrm{r}}}$. By Proposition 2.2, $\mathfrak{S} = \{S(G) : \; G \in \mathcal{B}_{\theta_{\mathrm{r}}}\}$ is a subbase for a quasi-uniformity $\mathfrak{U}$ on $X$ which is transitive, biconvex and compatible with $\theta_{\mathrm{r}}$, i.e., $\tau_{\mathfrak{U}} = \theta_{\mathrm{r}}$.

Let $\mathcal{F}$ be the filter in $X$ generated by the filter base $\mathcal{B}_{\mathcal{F}} = \Big\{]-q, q[\setminus \bigcup_{i\in\mathbb{Q}^+} Y_i : q \in \mathbb{Q}^+ \cap Y\Big\}$. It can be easily seen that $\mathcal{F}$ is $\mathfrak{U}$-Cauchy and that it has no $\theta_{\mathrm{r}}$ cluster points. Indeed, $Y_q$ is open for every $q \in \mathbb{Q}^+$ and $Y_q \cap B = \emptyset$ for all $B \in \mathcal{B}_{\mathcal{F}}$ so that no point in $\bigcup_{q\in\mathbb{Q}^+} Y_q$ is a cluster point of $\mathcal{F}$. For every $x \in Y \setminus \mathbb{R}^-$ there exists $q \in \mathbb{Q}^+ \cap Y$ such that $q < x$, so that $x \in \,]q, \rightarrow[$. But $]\leftarrow, q[\, \in \mathcal{F}$ and therefore, $x$ is not a cluster point of $\mathcal{F}$. Similarly no $x \in Y \setminus \mathbb{R}^+$ is a cluster point of $\mathcal{F}$. To show that it is $\mathfrak{U}$-Cauchy one need only consider $S(G)(x)$ with $G \in \mathcal{B}_{\theta_{\mathrm{r}}}$ and suitable $x \in X$. If $X \setminus (G^{\leftarrow} \cup G^{\rightarrow}) \neq \emptyset$, then $S(G)(x) = X$ for every $x \in X \setminus (G^{\leftarrow} \cup G^{\rightarrow})$ so that $S(G)(x) \in \mathcal{F}$. Let $X \setminus (G^{\leftarrow} \cup G^{\rightarrow}) = \emptyset$. If $G \cap Y_q = \emptyset$ for all $q > x$ and some $x \in \mathbb{R}$, then $G \cap Y$ is either not bounded below or not bounded above in $\mathbb{R}$. Hence there exists some $x \in G^{\leftarrow} \setminus G$ or $x \in G^{\rightarrow} \setminus G$ so that $S(G)(x) = G^{\leftarrow}$ or $S(G)(x) = G^{\rightarrow}$. In either case $S(G)(x) \in \mathcal{F}$. Finally, say $G \cap Y \neq \emptyset$. If $G \cap Y = Y$, then $G = X$ and $S(G)(x) = X$ for every $x \in X$. Otherwise, there must be some $x \in \mathbb{R}^+ \cap Y \cap G$ and $y \in (\mathbb{R}^- \cap Y) \setminus G$, or $x \in \mathbb{R}^- \cap Y \cap G$ and $y \in (\mathbb{R}^+ \cap Y) \setminus G$. In the first case $S(G)(y) = G^{\leftarrow} \in \mathcal{F}$ and in the second case $S(G)(y) = G^{\rightarrow} \in \mathcal{F}$.

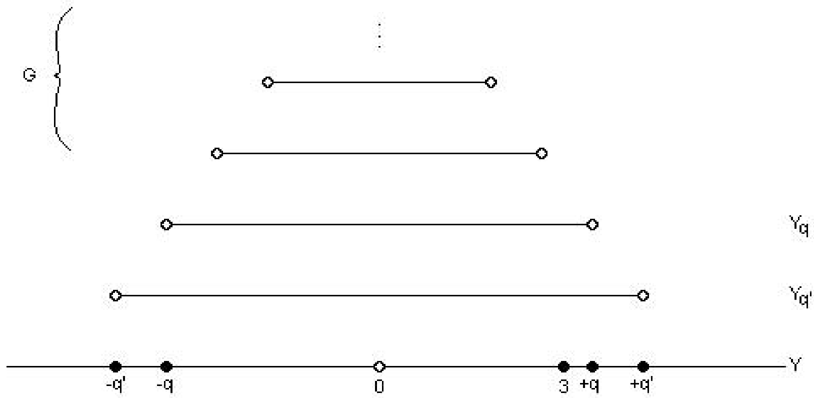


FIGURE 2

Now let $G = \bigcup\{Y_q : \ q \in \mathbb{Q}^+, \ q < 3\}$, where in fact 3 can be any constant greater than 0, see Figure 2. Then $G$ is a $\theta_{\rm r}$ open convex set in $X$ and $G \cap Y_q = \emptyset$ for all $q > 3$ so that $G \in \mathcal{B}_{\theta_{\rm r}}$. Take any $F \in \mathcal{B}_{\mathcal{F}}$. For $x \in F$ we have either $x \in G^{\rightarrow} \setminus G$ or $x \in G^{\leftarrow} \setminus G$, and therefore, either $S(G)(x) = G^{\rightarrow}$ or $S(G)(x) = G^{\leftarrow}$ and either $S^{-1}(G)(x) = X \setminus G^{\leftarrow}$ or $S^{-1}(G)(x) = X \setminus G^{\rightarrow}$. Consequently, $S(G)(F) = \bigcup\limits_{x \in F} S(G)(x) = G^{\leftarrow} \cup G^{\rightarrow} = G \cup Y$ and $S^{-1}(G)(F) = \bigcup\limits_{x \in F} S^{-1}(G)(x) = X \setminus G^{\leftarrow} \cup X \setminus G^{\rightarrow} = Y \cup \bigcup\{Y_q : \ q \in \mathbb{Q}^+, \ q \geq 3\}$ which are both not convex. Thus we have shown that $S(G)(F)$ and $S^{-1}(G)(F)$ are not convex for any $F \in \mathcal{B}_{\mathcal{F}}$. One can also note that if $U = S^{-1}(G_1) \cap \dots \cap S^{-1}(G_n) \subset S^{-1}(G)$, then $F \subset U(F) \subset S^{-1}(G)(F)$ and therefore $U(F)$ cannot be convex since it does not include $Y_q$ for any $q < 3$.

Finally, one can prove that $\theta_{\rm r} \subset \tau_{\mathfrak{U}^{-1}}$ and therefore, by Theorem 3.2, the completion exists. One only needs to show that all sets of the form $X \setminus [x, \rightarrow[$, $]x, \rightarrow[$, $X \setminus ]\leftarrow, x]$ and $]\leftarrow, x[$ are in $\tau_{\mathfrak{U}^{-1}}$ which although tedious is straightforward.

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