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VECTOR VALUED PARANORMED STATISTICALLY CONVERGENT DOUBLE SEQUENCE SPACES

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ABSTRACT. In this article we introduce the vector valued paranormed sequence spaces ${}_2\bar{c}(q,p)$, ${}_2\bar{c}_0(q,p)$, $({}_2\bar{c})^B(q,p)$, $({}_2\bar{c}_0)^B(q,p)$, $({}_2\bar{c})^R(q,p)$ and $({}_2\bar{c}_0)^R(q,p)$ defined over a seminormed space (X,q). We study their different properties like completeness, solidness, symmetry, convergence freeness etc. We prove some inclusion results.

1. Introduction

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast [2] and Schoenberg [11] independently. Later on it was further investigated by Fridy and Orhan [3], Šalát [10], Rath and Tripathy [9], Tripathy [13], Tripathy and Sen [15] and many others. The idea depends on the notion of *density* of subsets of $\mathbb{N}$. Throughout the paper, χ_E denotes the *characteristic function* of E. *A* subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists.

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Throughout the paper, $w(q)$, $\ell_\infty(q)$, $c(q)$, $c_0(q)$, $\bar{c}(q)$, $\bar{c}_0(q)$ denote the classes of *all*, *bounded*, *convergent*, *null*, *statistically convergent* and *statistically null* X-valued sequence spaces respectively, where X is a seminormed space, seminormed by q.

A sequence $(x_k) \in \bar{c}(q)$ if for every $\varepsilon > 0$, there exists $L \in X$ such that $\delta\big(\{k \in \mathbb{N} : \; q(x_k - L) \geq \varepsilon\}\big) = 0$. We write $\text{stat-}\lim x_k = L$.

Two sequences (x_k) and (y_k) are said to be equal for *allmost all k* (in short a.a.k.) if $\delta\big(\{k \in \mathbb{N} : \; x_k \neq y_k\}\big) = 0$.

The studies on paranormed sequences were initiated by Nakano [7] and Simmons [12] at the initial stage. Later on they were studied by Maddox [5], Nanda [7], Tripathy and Sen [15] and many others.

Let $p = (p_k)$ be a sequence of positive real numbers and $H = \sup\limits_k p_k < \infty$. Then for (a_k) and (b_k) two sequences of complex terms, we have the following well known inequality

$$|a_k + b_k|^{p_k} \leq C\big(|a_k|^{p_k} + |b_k|^{p_k}\big)\,, \qquad \text{where} \quad C = \max\big(1, 2^{H-1}\big)\,.$$

On generalizing the sequence space $\bar{c}$ and $\bar{c}_0$, Tripathy and Sen [15] introduced the following sequence spaces of complex terms:

$$\bar{c}(p) = \big\{(x_k) \in w : \; \text{stat-}\lim |x_k - L|^{p_k} = 0 \; \text{ for some } \; L \in C\big\}$$

and

$$\bar{c}_0(p) = \big\{(x_k) \in w : \; \text{stat-}\lim |x_k|^{p_k} = 0\big\}\,.$$

The spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$, $\bar{c}(p) \cap \ell_\infty(p)$ and $\bar{c}_0(p) \cap \ell_\infty(p)$ are paranormed by $g(x) = \sup\limits_k |x_k|^{\frac{p_k}{M}}$, where $M = \max(1, H)$.

2. Definitons and preliminaries

Some works on double sequences is done by Hardy [4] and Moricz [6], Tripathy [14] and others. A double sequence $\langle a_{nk} \rangle$ is said to be *convergent to L in Pringsheim's sense* if $\lim\limits_{n,k\to\infty} a_{nk} = L$, where n and k tend to ∞ independent of each other. The notion of *regular convergence* for double sequences was introduced by Hardy [4]. A double sequence $\langle a_{nk} \rangle$ is said to be *regularly convergent* if it converges in the Pringsheim's sense and the following limits exist:

$$\lim_{n\to\infty} a_{nk} = L_k \qquad \text{for each} \quad k \in \mathbb{N}$$

and

$$\lim_{k\to\infty} a_{nk} = J_n \qquad \text{for each} \quad n \in \mathbb{N}\,.$$

The notion of *asymptotic density* for subsets of $\mathbb{N}\times\mathbb{N}$ was introduced by T r i p a t h y [14]. A subset E of $\mathbb{N}\times\mathbb{N}$ is said to have density $\rho(E)$ if

$$\rho(E)=\lim_{p,q\to\infty}\frac{1}{pq}\sum_{n\le p}\sum_{k\le q}\chi_E(n,k)$$

exists.

T r i p a t h y [14] introduced the notion of statistically convergent double sequences. A double sequence $\langle a_{nk}\rangle$ is said to be *statistically convergent to L in Pringsheim's sense* if for every $\varepsilon>0$, $\rho\big(\{(n,k):\ |a_{nk}-L|\ge\varepsilon\}\big)=0$.

A double sequence $\langle a_{nk}\rangle$ is said to be *regularly statistically convergent* if it is statistically convergent in the Pringsheim's sense and the following statistical limits exist

$$\text{stat-}\lim_{n\to\infty}a_{nk}=L_k \qquad \text{for each} \quad k\in\mathbb{N}$$

and

$$\text{stat-}\lim_{k\to\infty}a_{nk}=J_n \qquad \text{for each} \quad n\in\mathbb{N}\,.$$

Throughout the article ${}_2w(q)$, ${}_2\ell_\infty(q)$, ${}_2c(q)$, ${}_2c_0(q)$, ${}_2c^R(q)$, ${}_2c_0^R(q)$, ${}_2c^B(q)$, ${}_2c_0^B(q)$, ${}_2\bar{c}(q)$, ${}_2\bar{c}_0(q)$, $({}_2\bar{c})^R(q)$, $({}_2\bar{c}_0)^R(q)$, $({}_2\bar{c})^{BR}(q)$, $({}_2\bar{c}_0)^{BR}(q)$, $({}_2\bar{c})^B(q)$, $({}_2\bar{c}_0)^B(q)$ denote the spaces of *all*, *bounded*, *convergent in Pringsheim's sense*, *null in Pringsheim's sense*, *regularly convergent*, *regularly null*, *bounded and convergent in Pringsheim's sense*, *bounded and null in Pringsheim's sense*, *statitstically convergent in Pringsheim's sense*, *statitstically null in Pringsheim's sense*, *regularly statitstically convergent*, *regularly statitstically null*, *bounded regularly convergent*, *bounded regularly null*, *bounded statitstically convergent in Pringsheim's sense*, *bounded null in Pringsheim's sense* X-valued double sequences respectively, where X is a seminormed space, seminormed by q.

Let $\langle a_{nk}\rangle$ and $\langle b_{nk}\rangle$ be two double sequences, then we say that $a_{nk}=b_{nk}$ for *almost all n and k* (in short a.a.n & k) if $\rho\big(\{(n,k):\ a_{nk}\ne b_{nk}\}\big)=0$.

Let $p=(p_{nk})$ be a double sequence of positive real numbers. The notion of paranormed double sequences was introduced by C o l a k and T u r k m e n o g l u [1] and further investigated by T u r k m e n o g l u [16].

A double sequence space E is said to be *solid* if $\langle\alpha_{nk}a_{nk}\rangle\in E$, whenever $\langle a_{nk}\rangle\in E$ and for all sequences $\langle\alpha_{nk}\rangle$ of scalars with $|\alpha_{nk}|\le 1$ for all $n,k\in\mathbb{N}$.

A double sequence space E is said to be *symmetric* if $\langle a_{\pi(n)\pi(k)}\rangle\in E$, whenever $\langle a_{nk}\rangle\in E$, where $\pi(n)$, $\pi(k)$ are permutations of $\mathbb{N}$.

A double sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

A double sequence space E is said to be *convergence free* if $\langle b_{nk}\rangle \in E$, whenever $\langle a_{nk}\rangle \in E$ and $b_{nk} = \theta$, whenever $a_{nk} = \theta$, where θ is the zero element of X.

The zero double sequence is denoted by ${}_2\theta = \langle\theta\rangle$ and the zero single sequence by $\bar{\theta} = (\theta, \theta, \theta, \theta \dots)$.

We introduce the following paranormed double sequence spaces.

$$ {}_2(\bar{c})(q,p) = \big\{\langle a_{nk}\rangle \in {}_2w:\ \text{stat-}\lim(q(a_{nk}-L))^{p_{nk}} = 0 \ \text{ for some } \ L \in X\big\}, $$

$$ {}_2(\bar{c}_0)(q,p) = \big\{\langle a_{nk}\rangle \in {}_2w:\ \text{stat-}\lim(q(a_{nk}))^{p_{nk}} = 0\big\}, $$

$\langle a_{nk}\rangle \in ({}_2\bar{c}\,)^R(q,p)$ if $\langle a_{nk}\rangle \in ({}_2\bar{c}\,)(q,p)$ and the following statistical limits hold:

$$ \text{stat-}\lim_{n\to\infty}\big(q(a_{nk}-L_k)\big)^{p_{nk}} = 0 \qquad \text{for each} \quad k \in \mathbb{N}\,, \tag{1} $$

$$ \text{stat-}\lim_{k\to\infty}(q(a_{nk}-J_n))^{p_{nk}} = 0 \qquad \text{for each} \quad n \in \mathbb{N}\,. \tag{2} $$

We have $\langle a_{nk}\rangle \in ({}_2\bar{c}_0)^R(q,p)$ if $\langle a_{nk}\rangle \in {}_2\bar{c}_0(q,p)$ and equation (1) and (2) hold with $L_k = J_n = \theta$ for each $n, k \in \mathbb{N}$.

$$ {}_2\ell_\infty(q,p) = \Big\{\langle a_{nk}\rangle:\ \sup_{n,k}\big(q(a_{nk})\big)^{p_{nk}} < \infty\Big\}. $$

We define ${}_2m(q,p) = {}_2\bar{c}(q,p) \cap {}_2\ell_\infty(q,p)$, ${}_2m_0(q,p) = {}_2\bar{c}_0(q,p) \cap {}_2\ell_\infty(q,p)$, ${}_2m^R(q,p) = ({}_2\bar{c}\,)^R(q,p) \cap {}_2\ell_\infty(q,p)$ and ${}_2m_0^R(q,p) = ({}_2\bar{c}\,)^R(q,p) \cap {}_2\ell_\infty(q,p)$. Let $p = (p_{nk})$ be a sequence of positive real numbers. Then the double sequence $\langle a_{nk}\rangle$ is said to be strongly (p)-*Cesaro summable to* L, i.e. $\langle a_{nk}\rangle \in {}_2w_{(p)}(q)$ if

$$ \lim_{u,v\to\infty}\frac{1}{uv}\sum_{n=1}^{u}\sum_{k=1}^{v}\big(q(a_{nk}-L)\big)^{p_{nk}} = 0\,. $$

The following results will be used for establishing some results of this article.

Lemma 1. *If a sequence space is solid, then it is monotone.*

We procure the following result of T r i p a t h y [14]. He proved it for $X = C$.

Lemma 2. (T r i p a t h y [14, Theorem 1]) *The following are equivalent:*

(i) *The double sequence $\langle a_{nk}\rangle$ is statistically convergent to L.*

(ii) *The double sequence $\langle a_{nk} - L\rangle$ is statistically convergent to* 0.

(iii) *There exists a sequence $\langle b_{nk}\rangle \in {}_2c$ such that $a_{nk} = b_{nk}$ for a.a.n & k.*

(iv) *There exists a subset $M = \big\{(n_i, k_j) \in \mathbb{N}\times\mathbb{N}:\ i, j \in \mathbb{N}\big\}$ of $\mathbb{N}\times\mathbb{N}$ such that $\rho(M) = 1$ and $\langle a_{n_ik_i}\rangle \in {}_2c$.*

(v) *There exists two sequences $\langle x_{nk}\rangle$ and $\langle y_{nk}\rangle$ such that $a_{nk} = x_{nk} + y_{nk}$ for all $n, k \in \mathbb{N}$, where $\langle x_{nk}\rangle$ converges to L and $\langle y_{nk}\rangle \in {}_2\bar{c}_0$.*

3. Main results

In this section we prove the results of this article. The proof of the following result is a routine verification.

Theorem 1. *Let $\langle p_{nk}\rangle \in {}_2\ell_\infty$, then the class of sequences ${}_2\bar{c}(q,p)$, ${}_2\bar{c}_0(q,p)$, $({}_2\bar{c}\,)^R(q,p)$, $({}_2\bar{c}_0)^R(q,p)$, ${}_2m(q,p)$, ${}_2m_0(q,p)$, ${}_2m^R(q,p)$ and $({}_2m_0)^R(q,p)$ are linear spaces.*

We prove the following decomposition theorem.

Theorem 2. *The following are equivalent:*

(i) *The double sequence $\langle a_{nk}\rangle \in {}_2\bar{c}(q,p)$, i.e. there exists $L \in X$ such that* $\text{stat-}\lim(q(a_{nk}-L))^{p_{nk}} = 0$.

(ii) *The double sequence $\langle a_{nk}-L\rangle \in {}_2\bar{c}_0(q,p)$.*

(iii) *There exists a sequence $\langle b_{nk}\rangle \in {}_2c(q,p)$ such that $a_{nk}=b_{nk}$ for a.a.n & k.*

(iv) *There exists a subset $M = \big\{(n_i,k_j)\in \mathbb{N}\times\mathbb{N}:\ i,j\in\mathbb{N}\big\}$ of $\mathbb{N}\times\mathbb{N}$ such that $\rho(M)=1$ and $\langle a_{n_ik_i}\rangle \in {}_2c(q,t)$, where $t=(p_{n_ik_j})$.*

(v) *There exists two sequences $\langle x_{nk}\rangle$ and $\langle y_{nk}\rangle$ such that $a_{nk}=x_{nk}+y_{nk}$ for all $n,k\in\mathbb{N}$, where* $\text{stat-}\lim\big(q(a_{nk}-L)\big)^{p_{nk}}=0$ *and $\langle y_{nk}\rangle \in {}_2\bar{c}_0(q,p)$.*

P r o o f. Let $z_{nk}=\big(q(a_{nk}-L)\big)^{p_{nk}}$ for all $n,k\in\mathbb{N}$. Then $\text{stat-}\lim z_{nk}=0$ and the result follows from Lemma 2. □

Theorem 3. *Let $0<\inf p_{nk}\le \sup p_{nk}<\infty$, then the spaces $Z(q,p)$ for $Z=({}_2\bar{c}\,)^{BR}$, $({}_2\bar{c}_0)^{BR}$, $({}_2\bar{c}\,)^{B}$, ${}_2\ell_\infty(q,p)$ and $({}_2\bar{c}_0)^B$ are paranormed spaces (not necessarily totally), paranormed by*

$$g\big(\langle a_{nk}\rangle\big)=\sup_{n,k}\big(q(a_{nk})\big)^{\frac{p_{nk}}{H}},$$

where $H=\max\Big(1,\sup_{n,k} p_{nk}\Big)$.

P r o o f. Clearly $g({}_2\overline{\theta}\,)=0$, $g(-A)=g(A)$, where $A=\langle a_{nk}\rangle$ and $g(A+B)\le g(A)+(B)$. Now we verify the continuity of scalar multiplication.

Let $A\to {}_2\overline{\theta}$, then $g(A)\to 0$. We have for a given scalar λ,

$$g(\lambda A)=\sup_{n,k}\big(q(\lambda a_{nk})\big)^{\frac{p_{nk}}{H}}\le \max\big(1,|\lambda|\big)\cdot g(A)\to 0 \qquad \text{as} \quad A\to {}_2\overline{\theta}\,.$$

Next let $\lambda \to 0$. Without loss of generality, let $|\lambda| < 1$. Then for a given $A = \langle a_{nk} \rangle$, we have

$$g(\lambda A) = \sup_{n,k} \big(q(\lambda a_{nk})\big)^{\frac{p_{nk}}{H}} \leq |\lambda|^{\frac{h}{H}} \cdot g(A) \to 0 \qquad \text{as} \quad \lambda \to 0\,,$$

$$\text{where} \quad h = \inf_{n,k} p_{nk} > 0\,.$$

The case when $\lambda \to 0$ and $A \to {}_2\overline{\theta}$ implies $g(\lambda A) \to 0$ follows similarly.

Hence the spaces are paranormed by g. □

Theorem 4. *Let $p = \langle p_{nk} \rangle \in {}_2\ell_\infty$. Then the spaces $Z(q,p)$ for $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, $({}_2\bar{c}_0)^R$, $({}_2\bar{c})^R$, $({}_2\bar{c}\,)^{BR}$, $({}_2\bar{c}_0)^{BR}$, $({}_2\bar{c}\,)^B$ and $({}_2\bar{c}_0)^B$ are sequence algebras.*

Proof. Consider the space ${}_2\bar{c}(q,p)$. Let $\langle a_{nk} \rangle, \langle b_{nk} \rangle \in {}_2\bar{c}(q,p)$. Then there exists $K_1, K_2 \subset \mathbb{N} \times \mathbb{N}$ with $\rho(K_1) = \rho(K_2) = 1$ such that

$$\lim_{\substack{n,k\to\infty\\(n,k)\in K_1}} \big(q(a_{nk} - L)\big)^{p_{nk}} = 0 \qquad \text{and} \qquad \lim_{\substack{n,k\to\infty\\(n,k)\in K_2}} \big(q(a_{nk} - \xi)\big)^{p_{nk}} = 0$$

$$\text{for some} \quad L, \xi \in X\,.$$

Let $K = K_1 \cap K_2$, then $\rho(K) = 1$. Now it follows that

$$\lim_{\substack{n,k\to\infty\\(n,k)\in K}} \big(q(a_{nk}b_{nk} - L\xi)\big)^{p_{nk}} = 0\,.$$

Thus $\langle a_{nk}b_{nk} \rangle \in {}_2\bar{c}(q,p)$.

Similarly it can be shown that the other spaces are also sequence algebras. □

Theorem 5. *The spaces ${}_2\bar{c}_0(q,p)$, $({}_2\bar{c}_0)^B(q,p)$, $({}_2\bar{c}_0)^R(q,p)$ and $({}_2\bar{c}_0)^{BR}(q,p)$ are solid. Hence are monotone.*

Proof. Let $\langle a_{nk} \rangle \in {}_2\bar{c}_0(q,p)$ or $({}_2\bar{c}_0)^B(q,p)$ or $({}_2\bar{c}_0)^R(q,p)$ or $({}_2\bar{c}_0)^{BR}(q,p)$. Let $\langle \alpha_{nk} \rangle$ be a double sequence of scalars with $|\alpha_{nk}| \leq 1$ for all $n,k \in \mathbb{N}$. Then the solidness of the above spaces follows from the following inequality $\big(q(\alpha_{nk}a_{nk})\big)^{p_{nk}} \leq \big(q(a_{nk})\big)^{p_{nk}}$ for all $n,k \in \mathbb{N}$.

The rest follows from Lemma 1. □

Corollary 6. *The spaces ${}_2\bar{c}(q,p)$, $({}_2\bar{c}\,)^B(q,p)$, $({}_2\bar{c})^R(q,p)$ and $({}_2\bar{c}\,)^{BR}(q,p)$ are not monotone, as such are not solid.*

Proof. The proof follows from the following example and Lemma 1. □

Example 1. Let $X = \ell_\infty$ and $p_{nk} = 1$ for all $n,k \in \mathbb{N}$. Let $\langle a_{nk} \rangle \in {}_2\bar{c}(q,p)$ be defined by $a_{nk} = e = (1,1,1,1,---)$ for all $n,k \in \mathbb{N}$. Let $a_{nk} = (a^i_{nk})$ and $q\big((a^i_{nk})\big) = \sup_{i\geq 2} |a^i_{nk}|$ for all $n,k \in \mathbb{N}$. Consider the Jth step space

$\big((_2\bar{c})^{BR}(q,p)\big)_J$ defined by $\langle b_{nk}\rangle \in \big((_2\bar{c})^{BR}(q,p)\big)_J$ implies $b_{nk} = a_{nk}$ for n even and all $k \in \mathbb{N}$ and $b_{nk} = \bar{\theta}$, otherwise. Then $\langle b_{nk}\rangle \notin (_2\bar{c})^{BR}(q,p)$. Hence $(_2\bar{c})^{BR}(q,p)$ is not monotone.

From this example, it follows that the other spaces are not monotone, too.

Theorem 7. *The spaces $Z(q,p)$ for $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, $(_2\bar{c}_0)^R$, $(_2\bar{c})^R$, $(_2\bar{c})^{BR}$, $(_2\bar{c}_0)^{BR}$, $(_2\bar{c})^B$ and $(_2\bar{c}_0)^B$ are not convergence free.*

Proof. The proof follows from the following example. □

Example 2. Let $X = \ell_\infty$, $q\big((x^i)\big) = \sup_{i\ge 2}|x^i|$, $p_{nk} = 1$ for k odd and for all $n \in \mathbb{N}$ and $p_{nk} = 2$ otherwise. Consider the sequence $\langle a_{nk}\rangle \in {}_2\bar{c}(q,p)$ defined by $a_{1k} = \theta = a_{n1}$ for all $n,k \in \mathbb{N}$. $a_{nk} = (2,2,2,2,2,----)$, otherwise.

Consider $\langle b_{nk}\rangle$ defined as

$$\begin{aligned} b_{1k} &= \theta = b_{n1} && \text{for all } n,k \in \mathbb{N},\\ b_{nk} &= e && \text{for all } k \text{ even and all } n > 1,\\ &= 2e && \text{otherwise.} \end{aligned}$$

Then $\langle a_{nk}\rangle \in {}_2\bar{c}(q,p)$, but $\langle b_{nk}\rangle \notin {}_2\bar{c}(q,p)$. Hence ${}_2\bar{c}(q,p)$ is not convergence free. This example shows that the spaces $Z(q,p)$ for $Z = (_2\bar{c})^R$, $(_2\bar{c})^{BR}$, $(_2\bar{c})^B$ are not convergence free, too.

Example 3. Let $p_{nk} = 1$ for all $n,k \in \mathbb{N}$, $X = C$ and $q(x) = |x|$.

Consider the double sequence $\langle a_{nk}\rangle$ defined as

$$a_{nk} = \begin{cases} 0 & \text{for } n \text{ even and for all } k \in \mathbb{N},\\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

Consider the sequence $\langle b_{nk}\rangle$ defined as

$$b_{nk} = \begin{cases} 0 & \text{for } n \text{ even and for all } k \in \mathbb{N},\\ 1 & \text{otherwise.} \end{cases}$$

Then clearly $\langle a_{nk}\rangle \in Z(q,p)$, but $\langle b_{nk}\rangle \notin Z(q,p)$ for $Z = {}_2\bar{c}_0$, $(_2\bar{c}_0)^R$, $(_2\bar{c}_0)^{BR}$, $(_2\bar{c}_0)^B$.

Hence the spaces are not convergence free.

Proposition 8. *The spaces $Z(q,p)$ for $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, $(_2\bar{c}_0)^R$, $(_2\bar{c})^R$, $(_2\bar{c})^{BR}$, $(_2\bar{c}_0)^{BR}$, $(_2\bar{c})^B$ and $(_2\bar{c}_0)^B$ are not symmetric.*

Proof. The proof follows from the following example. □

Example 4. Let $p_{nk} = 1$ for n even and all $k \in \mathbb{N}$ and $p_{nk} = 2$ otherwise. Let $X = C$, and $q(x) = |x|$. Consider the sequence $\langle a_{nk} \rangle$ defined by $a_{n1} = 1 = a_{1k}$, for all $n = i^2 = k$, $i \in \mathbb{N}$, and $a_{nk} = 0$, otherwise.

Then $\langle a_{nk} \rangle \in Z(q,p)$ for $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, $({}_2\bar{c}_0)^R$, $({}_2\bar{c})^R$, $({}_2\bar{c})^{BR}$, $({}_2\bar{c}_0)^{BR}$, $({}_2\bar{c})^B$ and $({}_2\bar{c}_0)^B$. Consider its rearranged sequence $\langle b_{nk} \rangle$ defined by

$$b_{nk} = \begin{cases} 1 & \text{for all } n \text{ even and all } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle b_{nk} \rangle \notin Z(q,p)$ for $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, $({}_2\bar{c}_0)^R$, $({}_2\bar{c})^R$, $({}_2\bar{c})^{BR}$, $({}_2\bar{c}_0)^{BR}$, $({}_2\bar{c})^B$ and $({}_2\bar{c}_0)^B$. Hence the spaces $Z(q,p)$ for $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, $({}_2\bar{c}_0)^R$, $({}_2\bar{c})^R$, $({}_2\bar{c})^{BR}$, $({}_2\bar{c}_0)^{BR}$, $({}_2\bar{c})^B$ and $({}_2\bar{c}_0)^B$ are not symmetric .

Theorem 9. *For two sequences $p = \langle p_{nk} \rangle$ and $t = \langle t_{nk} \rangle$ we have $({}_2\bar{c}_0)^B(q,p) \supseteq ({}_2\bar{c}_0)^B(q,t)$ if and only if $\liminf\limits_{\substack{n,k\to\infty \\ (n,k)\in K}} \left(\frac{p_{nk}}{t_{nk}}\right) > 0$ where $K \subset \mathbb{N}\times\mathbb{N}$ such that $\rho(K) = 1$.*

P r o o f. Suppose that

$$\liminf_{\substack{n,k\to\infty \\ (n,k)\in K}} \left(\frac{p_{nk}}{t_{nk}}\right) > 0 \,. \tag{3}$$

Then there exists $\alpha > 0$ such that $p_{nk} > \alpha t_{nk}$ for sufficiently large pair $(n,k) \in K$. Let $\langle a_{nk} \rangle \in ({}_2\bar{c}_0)^B(q,t)$, then for $\varepsilon > 0$ we have $\big(q(a_{nk})\big)^{q_{nk}} < \varepsilon$ for all $(n,k) \in L \subseteq \mathbb{N}\times\mathbb{N}$, where $L = \big\{(n,k) \in \mathbb{N}\times\mathbb{N} :\ \big(q(a_{nk})\big)^{q_{nk}} < \varepsilon\big\}$, such that $\rho(L) = 1$.

Let $J = K \cap L$. Then $\rho(J) = 1$.

Now $\big(q(a_{nk})\big)^{p_{nk}} \le \Big(\big(q(a_{nk})\big)^{t_{nk}}\Big)^{\alpha}$. This implies $\langle a_{nk} \rangle \in ({}_2\bar{c}_0)^B(q,t)$.

Next let $({}_2\bar{c}_0)^B(q,p) \supseteq ({}_2\bar{c}_0)^B(q,t)$, but there is no $K \subset \mathbb{N}\times\mathbb{N}$ with $\rho(K) = 1$ such that (3) holds. Then there exists $\big\{(n_i,k_j) :\ i,j \in \mathbb{N}\big\} \subset \mathbb{N}\times\mathbb{N}$ with $\rho\Big(\big\{(n_i,k_j) :\ i,j \in \mathbb{N}\big\}\Big) \ne 0$ such that $ip_{n_ik_j} < q_{n_ik_j}$. Define the sequence $\langle a_{nk} \rangle$ by

$$a_{nk} = \begin{cases} \left(\frac{1}{i}\right)^{\frac{1}{q_{n_ik_j}}} & \text{if } k = k_j,\ n = n_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle a_{nk} \rangle \in ({}_2\bar{c}_0)^B(q,t)$. But $(a_{n_ik_i})^{p_{n_ik_j}} > \exp\left(\frac{-\log i}{i}\right)$ Hence we arrive at a contradiction. $\square$

Theorem 10. *Let $0 < \inf\limits_{n,k} p_{nk} \le \sup\limits_{n,k} p_{nk} < \infty$. Then*

$${}_2w_{(p)}(q) \cap {}_2\ell_\infty(q,p) = {}_2\bar{c}(q,p) \cap {}_2\ell_\infty(q,p) \,.$$

P r o o f. Let $A = \langle a_{nk} \rangle \in {}_2w_{(p)}(q) \cap {}_2\ell_\infty(q,p)$ and $H = \sup\limits_{n,k} p_{nk}$, $r = \max\{1, H\}$. Then taking $b_{nk} = \big(q(a_{nk} - L)\big)^{p_{nk}}$ for all $n, k \in \mathbb{N}$, we have the result, which follows from [14, Theorem 4] of T r i p a t h y. $\square$

The following result is a consequence of Theorem 10.

Corollary 11. *For any two sequences of real numbers $p = (p_{nk})$ and $t = (t_{nk})$ satisfying the condition in the hypothesis of* Theorem 10*, we have*

$$ {}_2w_{(p)}(q) \cap {}_2\ell_\infty(q,p) = {}_2w_{(t)}(q) \cap {}_2\ell_\infty(q,t). $$

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