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ISOMETRIES AND DIRECT DECOMPOSITIONS OF PSEUDO MV-ALGEBRAS

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ABSTRACT. In the paper isometries in pseudo MV-algebras are investigated. It is shown that for every isometry f in a pseudo MV-algebra $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ there exists an internal direct decomposition $\mathcal{A} = \mathcal{B}^0 \times \mathcal{C}^0$ of $\mathcal{A}$ with $\mathcal{C}^0$ commutative such that $f(0) = 1_{C^0}$ and $f(x) = x_{B^0} \oplus (1_{C^0} \odot (x_{C^0})^-) = x_{B^0} \oplus (1_{C^0} - x_{C^0})$ for each $x \in A$.

On the other hand, if $\mathcal{A} = \mathcal{P}^0 \times \mathcal{Q}^0$ is an internal direct decomposition of a pseudo MV-algebra $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ with $\mathcal{Q}^0$ commutative, then the mapping $g\colon A \to A$ defined by $g(x) = x_{P^0} \oplus (1_{Q^0} - x_{Q^0})$ is an isometry in $\mathcal{A}$ and $g(0) = 1_{Q^0}$.

Isometries in abelian lattice ordered groups were introduced and investigated by Swamy in [27]. Jakubík [7], [8] studied isometries in non-abelian lattice ordered groups and proved that for every isometry g in a lattice ordered group G there exists a uniquely determined direct decomposition $G = A \times B$ of G with B abelian such that $g(x) = x_A - x_B + g(0)$ for each $x \in G$. Further, he showed that if $G = A \times B$ is a direct decomposition of a lattice ordered group G with B abelian and b is an element of G, then the mapping g defined by $g(x) = x_A - x_B + b$ is an isometry in G and $b = g(0)$. Isometries in some types of partially ordered groups have been investigated in [14], [15], [16], [23].

The notion of an MV-algebra was introduced by Chang [1] as an algebraic model of infinite valued logic. In [22] Mundici showed that any MV-algebra is an interval of an abelian lattice ordered group with a strong unit. Isometries in MV-algebras were dealt with by Jakubík [11], [12].

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Georgescu and Iorgulescu [4] introduced pseudo MV-algebras as a non-commutative generalization of MV-algebras. Dvurečenskij [2] proved that any pseudo MV-algebra is an interval of a lattice ordered group with a strong unit. A completely different proof of this important result was given by Dvurečenskij and Vetterlein in [3]. Non-commutative MV-algebras were also introduced independently by Rachůnek [26]. His notion of a non-commutative MV-algebra is equivalent to the notion of a pseudo MV-algebra. Further, Rachůnek showed that non-commutative MV-algebras and hence also pseudo MV-algebras are a special kind of bounded DRl-monoids.

DRl-monoids were studied in [17], [19], [20], [21], [24], [25], [29] and isometries in commutative DRl-monoids (called DRl-semigroups) have been investigated in [13], [18], [28].

We recall the definition and some basic properties of a pseudo MV-algebra from [4].

A pseudo MV-algebra is an algebra $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ with an additional binary operation $\odot$ defined by $y \odot x = (x^- \oplus y^-)^\sim$ such that following axioms hold for all $x, y, z \in A$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$, $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

(In [4] instead of $\odot$ the symbol $\cdot$ is used.)

Any pseudo MV-algebra $\mathcal{A}$ can be ordered by the relation $\leq$ defined by $x \leq y$ iff $x^- \oplus y = 1$. Then $(A, \leq)$ is a distributive lattice with the least element 0 and the greatest element 1. For the join $x \vee y$ and the meet $x \wedge y$ of two elements x and y the following statements are valid:
$x \vee y = x \oplus (x^\sim \odot y)$, $x \wedge y = x \odot (x^- \oplus y)$.

Let $(G, +, \vee, \wedge)$ be a lattice ordered group, u a positive element of G and A the interval $[0, u]$ of G. Then $(A, \oplus, ^-, ^\sim, 0, u)$ where

$$x \oplus y = (x + y) \wedge u, \quad x^- = u - x, \quad x^\sim = -x + u$$

is a pseudo MV-algebra which will be denoted by $\Gamma(G, u)$. Moreover, $x \odot y = (x - u + y) \vee 0$.

D v u r e č e n s k i j [2] defined a partial binary operation $+$ on a pseudo MV-algebra $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ by putting $x+y = x \oplus y$ iff $x \leq y^-$. Having used this partial operation $+$ he proved that for each pseudo MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$. The partial operation $+$ on $\mathcal{A}$ coincides with the operation $+$ as defined in G. Further, the partial order $\leq$ on A is that induced from the partial order in G.

The direct product of pseudo MV-algebras is defined in the usual way, see e.g. [5].

The internal direct decomposition of an MV-algebra was defined and studied by J a k u b í k in [9]. Analogously, we can define the two-factor internal direct decomposition of a pseudo MV-algebra.

Let

$$\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1),\ \mathcal{B} = (B, \oplus_B, ^{-_B}, ^{\sim_B}, 0_B, 1_B),\ \mathcal{C} = (C, \oplus_C, ^{-_C}, ^{\sim_C}, 0_C, 1_C)$$

be pseudo MV-algebras.

An isomorphism φ of $\mathcal{A}$ onto the direct product $\mathcal{B} \times \mathcal{C}$ is called a direct decomposition of $\mathcal{A}$.

For $x \in A$ we denote by x_B (x_C) the component of x in $\mathcal{B}$ ($\mathcal{C}$, respectively) with respect to the isomorphism φ.

We denote $B^0 = \{x \in A:\ x_C = 0_C\}$, $C^0 = \{x \in A:\ x_B = 0_B\}$. Then B^0 and C^0 are subsets of A containing 0. Since φ is an isomorphism, for $x, y \in B^0$ we have $(x \oplus y)_C = 0_C$. Thus $x \oplus y \in B^0$. Analogously, $z \oplus t \in C^0$ for each $z, t \in C^0$. Hence the sets B^0 and C^0 are closed under the operation $\oplus$.

In a natural way, we introduce the following operations $^{-_{B^0}}$, $^{\sim_{B^0}}$, 1_{B^0} on the set B^0. Let $b \in B^0$ and let $d \in A$ be such that $d_B = (b^-)_B$ and $d_C = 0_C$. Then $d \in B^0$ and we put $b^{-_{B^0}} = d$. Analogously, for $c \in B^0$ we put $c^{\sim_{B^0}} = e$, where e is an element of A such that $e_B = (c^\sim)_B$, $e_C = 0_C$. Clearly, $e \in B^0$. Further, 1_{B^0} is an element of A such that $(1_{B^0})_B = 1_B$, $(1_{B^0})_C = 0_C$.

Similarly we define the operations $^{-_{C^0}}$, $^{\sim_{C^0}}$, 1_{C^0} on C^0.

Then $\mathcal{B}^0 = (B^0, \oplus, ^{-_{B^0}}, ^{\sim_{B^0}}, 0, 1_{B^0})$ and $\mathcal{C}^0 = (C^0, \oplus, ^{-_{C^0}}, ^{\sim_{C^0}}, 0, 1_{C^0})$ are pseudo MV-algebras.

In general, $\mathcal{B}^0$ and $\mathcal{C}^0$ need not be subalgebras of $\mathcal{A}$.

Now, we define a mapping $\varphi^B\colon B \to B^0$. For $t \in B$ there exists an element $z \in A$ such that $z_B = t$ and $z_C = 0_C$. Thus $z \in B^0$ and we put $\varphi^B(t) = z$. Then φ^B is an isomorphism of $\mathcal{B}$ onto $\mathcal{B}^0$. Analogously defined mapping φ^C of C into C^0 is an isomorphism of $\mathcal{C}$ onto $\mathcal{C}^0$.

Then the mapping φ^0 of A into $B^0 \times C^0$ given by $\varphi^0(x) = \big(\varphi^B(x_B), \varphi^C(x_C)\big)$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}^0 \times \mathcal{C}^0$. This isomorphism φ^0 is called an internal direct decomposition of $\mathcal{A}$ and we write $\mathcal{A} = B^0 \times \mathcal{C}^0$ in this case. $\mathcal{B}^0$ and $\mathcal{C}^0$ are called internal direct factors of $\mathcal{A}$.

For $x \in A$ we denote by x_{B^0} (x_{C^0}) the component of x in B^0 (C^0, respectively) with the respect to the isomorphism φ^0. Hence $x_{B^0} = \varphi^B(x_B)$, $x_{C^0} = \varphi^C(x_C)$, $\varphi(x_{B^0}) = (x_B, 0_C)$, $\varphi(x_{C^0}) = (0_B, x_C)$.

If $x \in B^0$ and $y \in C^0$, then $x \oplus y = y \oplus x$.

For each $x \in A$, $x = x_{B^0} \oplus x_{C^0}$ and if $x = x_1 \oplus x_2$ where $x_1 \in B^0$ and $x_2 \in C^0$, then $x_1 = x_{B^0}$ and $x_2 = x_{C^0}$.

Further, if $x, y \in A$, then $x \le y$ iff $x_{B^0} \le y_{B^0}$ and $x_{C^0} \le y_{C^0}$. B^0 and C^0 are convex subsets of A. For each $x, y \in A$, $(x \wedge y)_{B^0} = x_{B^0} \wedge y_{B^0}$, $(x \wedge y)_{C^0} = x_{C^0} \wedge y_{C^0}$, $(x \vee y)_{B^0} = x_{B^0} \vee y_{B^0}$, $(x \vee y)_{C^0} = x_{C^0} \vee y_{C^0}$.

Throughout the paper $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ will be a pseudo MV-algebra. Further, we suppose that $(G, +, \vee, \wedge)$ is a lattice ordered group with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$ (it is clear that $u = 1$). Then the above mentioned operations $\vee$ and $\wedge$ on A coincide with the lattice operations in G (reduced to the interval $[0, u]$) and for all $x, y \in A$ we have:

$$x \oplus y = (x + y) \wedge u, \quad x^- = u - x, \quad x^\sim = -x + u.$$

Further, if $x, y \in A$ and $x \le y$, then $y - x, -x + y \in A$.

We shall apply these assertions without special references.

For basic properties of lattice ordered groups, see e.g. [6].

Lemma 1. *Let x, $y \in A$, $x \le y$. Then the following statements are valid.*

(i) $(y - x) \oplus x = y$, $x \oplus (-x + y) = y$.

(ii) *Let* $P_x^y = \{z \in A : \ z \oplus x = y\}$, $Q_x^y = \{t \in A : \ x \oplus t = y\}$. *Then* $y \odot x^- = y - x$ *is the least element of* P_x^y *and* $x^\sim \odot y = -x + y$ *is the least element of* Q_x^y.

(iii) *If* $y - x = 1$, *then* $y = 1$, $x = 0$.

(iv) *If* $-x + y = 1$, *then* $y = 1$, $x = 0$.

P r o o f. Let $x, y \in A$ and $x \le y$.

(i) Clearly, $(y - x) \oplus x = [(y - x) + x] \wedge 1 = y \wedge 1 = y$. Analogously, $x \oplus (-x + y) = y$.

(ii) Since $(y \odot x^-) \oplus x = y \vee x = y$ and $x \oplus (x^\sim \odot y) = x \vee y = y$, we obtain $y \odot x^- \in P_x^y$ and $x^\sim \odot y \in Q_x^y$. Let $z, t \in A$, $z \oplus x = y$, $x \oplus t = y$. By [4, Proposition 1.12(d)], $z \ge y \odot x^-$, $t \ge x^\sim \odot y$. Therefore $y \odot x^- = (y - 1 + 1 - x) \vee 0$

$= y - x$ is the least element of P^y_x and $x^\sim \odot y = (-x + 1 - 1 + y) \vee 0 = -x + y$ is the least element of Q^y_x.

(iii) If $y - x = y \odot x^- = 1$, then $y = y \vee x = (y \odot x^-) \oplus x = 1 \oplus x = 1$ whence $x^- = 1$ and so $x = 0$.

(iv) Let $-x + y = x^\sim \odot y = 1$. Then $y = x \vee y = x \oplus (x^\sim \odot y) = x \oplus 1 = 1$. This yields $x^\sim = 1$. Hence $x = 0$. □

Georgescu and Iorgulescu [4] defined the distance function $d\colon A \times A \to A$ for a pseudo MV-algebra $\mathcal{A}$ by $d(x, y) = (x \odot y^-) \oplus (y \odot x^-)$.

Further, it was shown that this distance function has the following properties [4, Proposition 1.35].

(P_0) $d(x, y) = (x \odot y^-) \vee (y \odot x^-)$,

(P_1) $d(x, y) = 0$ iff $x = y$,

(P_2) $d(x, 0) = x$,

(P_3) $d(x, 1) = x^-$,

(P_4) $d(x, y) = d(y, x)$,

(P_5) $d(x, z) \le d(x, y) \oplus d(y, z) \oplus d(x, y)$,

(P_6) $d(x, z) \le d(y, z) \oplus d(x, y) \oplus d(y, z)$.

Jakubík [11] defined an autometrization of an MV-algebra $\mathcal{D}$ with the underlying set D as a mapping $\rho\colon D \times D \to D$ such that $\rho(x, y) = (x \vee y) - (x \wedge y)$ for each $x, y \in D$.

The following lemma shows that Jakubík's autometrization $\rho(x, y)$ coicides with the distance function $d(x, y)$ of Georgescu and Iorgulescu in any pseudo MV-algebra.

Lemma 2. *For each $x, y \in A$, $(x \vee y) - (x \wedge y) = (x \odot y^-) \oplus (y \odot x^-)$.*

Proof. Let $x, y \in A$. In view of Lemma 1, (P_0) and [4, Propositions 1.23, 1.16, 1.7(7)] we have $(x \vee y) - (x \wedge y) = (x \vee y) \odot (x \wedge y)^- = (x \vee y) \odot (x^- \vee y^-) = (x \odot (x^- \vee y^-)) \vee (y \odot (x^- \vee y^-)) = (x \odot x^-) \vee (x \odot y^-) \vee (y \odot x^-) \vee (y \odot y^-) = 0 \vee (x \odot y^-) \vee (y \odot x^-) \vee 0 = (x \odot y^-) \vee (y \odot x^-) = (x \odot y^-) \oplus (y \odot x^-)$. □

We can use Jakubík's definition of an isometry in an MV-algebra from [11] also for a pseudo MV-algebra $\mathcal{A}$.

A bijection $f\colon A \to A$ is said to be an isometry in $\mathcal{A}$ if the relation $d(f(x), f(y)) = d(x, y)$ identically holds.

An isometry f is called 2-periodic if $f(f(x)) = x$ for each $x \in A$.

We shall write $f^2(x)$ instead of $f(f(x))$.

Lemma 3. *Let $x, y \in A$, $x \leq y$. Then $d(x,y) = y - x$.*

Proof. The proof is obvious. $\square$

Throughout the rest of the paper let f be an isometry in $\mathcal{A}$.

Lemma 4. *Let $x \in A$. Then*

(i) $f^2(x) = x$.

(ii) $f(x) = (f(0) \vee x) - (f(0) \wedge x)$.

Proof.

(i) First we prove that $f^2(0) = 0$. Since f is a bijection, there exists $z \in A$ such that $f(z) = 0$. In view of (P_2) and (P_4) we get $z = d(z,0) = d(f(z), f(0)) = d(0, f(0)) = f(0)$. From this we obtain $f^2(0) = f(z) = 0$.

Let $x \in A$. According to (P_2), $x = d(x,0) = d(f^2(x), f^2(0)) = d(f^2(x), 0) = f^2(x)$.

(ii) Let $x \in A$. From (i) and (P_2) it follows that $f(x) = d(f(x),0)) = d(f^2(x), f(0)) = d(x, f(0)) = (f(0) \vee x) - (f(0) \wedge x)$. $\square$

From Lemma 4 it follows that any isometry in a pseudo MV-algebra is 2-periodic and uniquely determined by the element $f(0)$. Lemma 4(i) generalizes assertion (β) from [12].

Further, from Lemma 4 we immediately obtain the following corollary.

Corollary 1. $f(1) = 1 - f(0)$.

Lemma 5.

(i) $f(0) \vee f(1) = 1$, $f(0) \wedge f(1) = 0$.

(ii) *For each $x \in A$, $x \wedge f(1) = (x \vee f(0)) - f(0)$.*

(iii) *For each $x \in A$, $f(x) = (x \wedge f(1)) + f(0) - (x \wedge f(0))$.*

Proof.

(i) By (P_2) and (P_4), $1 = d(0,1) = d(f(0), f(1)) = (f(0) \vee f(1)) - (f(0) \wedge f(1))$. Then Lemma 1(iii) yields $f(0) \vee 1 = 1$, $f(0) \wedge 1 = 0$.

(ii) Let $x \in A$. By (i), $(x \wedge f(1)) \wedge f(0) = x \wedge (f(1) \wedge f(0)) = x \wedge 0 = 0$. Then (i) and [2, Proposition 2.1(X)] yield $(x \wedge f(1)) + f(0) = (x \wedge f(1)) \vee f(0) = (x \vee f(0)) \wedge (f(1) \vee f(0)) = (x \vee f(0)) \wedge 1 = x \vee f(0)$. Hence $x \wedge f(1) = (x \vee f(0)) - f(0)$.

(iii) Let $x \in A$. In view of (ii) and Lemma 4 we have $f(x) = (x \vee f(0)) - f(0) + f(0) - (x \wedge f(0)) = (x \wedge f(1)) + f(0) - (x \wedge f(0))$. $\square$

Since the lattice $(A, \leq)$ is distributive, from Lemma 5 we obtain the following corollary.

Corollary 2. *$f(1)$ is the uniquely determined complement of $f(0)$ in the lattice $(A, \leq)$.*

Lemma 6. *Let $x \in A$.*

(i) *If $x \leq f(0)$, then $f(x) = f(0) - x$, $f(x) \oplus x = f(0)$.*

(ii) *If $f(0) \leq x$, then $f(x) = x - f(0)$, $f(x) \oplus f(0) = x$.*

(iii) *If $f(x) \leq f(0)$, then $x = f(0) - f(x)$, $x \oplus f(x) = f(0)$.*

(iv) *If $f(0) \leq f(x)$, then $x = f(x) - f(0)$, $f(x) = x \oplus f(0)$.*

Proof.

(i) Let $x \in A$, $x \leq f(0)$. By (P_2), Lemmas 3 and 4, $f(0) - x = d(x, f(0)) = d(f(x), f^2(0)) = d(f(x), 0) = f(x)$. Then clearly, $f(0) = f(x) + x = f(x) \oplus x$.

Proofs of (ii), (iii) and (iv) are analogous. □

Let $B = \{x \in A : \ x \leq f(1)\}$, $C = \{x \in A : \ x \leq f(0)\}$.

Lemma 7. *Let $x \in B$ and $y \in C$. Then $x \wedge y = 0$, $x + y = x \oplus y = x \vee y = y \oplus x = y + x$.*

Proof. Let $x \in B$, $y \in C$. Then from Lemma 5 we get $0 = f(1) \wedge f(0) \geq x \wedge y \geq 0$. Hence $x \wedge y = 0$. Then [4, Proposition 1.26(ii)] implies $x \oplus y = x \vee y = y \oplus x$. According to [2, Proposition 2.1(X)] , $x + y = x \vee y = y + x$. □

Lemma 8. *For each $x \in B$, $f(x) = x + f(0) = x \oplus f(0)$.*

Proof. Let $x \in B$. By Lemmas 3, 4 and Corollary 1, $1 - (x + f(0)) = (1 - f(0)) - x = f(1) - x = d(x, f(1)) = d(f(x), f^2(1)) = d(f(x), 1) = 1 - f(x)$. This yields $f(x) = x + f(0) = x \oplus f(0)$. □

Lemma 9. *Let $x \in A$. Then $x \in C$ iff $f(x) \leq f(0)$.*

Proof. Let $x \in C$. According to Lemma 6(i), $f(x) = f(0) - x \leq f(0)$.

Let $x \in A$, $f(x) \leq f(0)$. By Lemma 6(iii), $x = f(0) - f(x) \leq f(0)$. Hence $x \in C$. □

In [10] Jakubík showed that if e is an element of a pseudo MV-algebra $\mathcal{A}$ which has a complement e' in the lattice $(A, \leq)$, then there exists a direct decomposition of $\mathcal{A}$. Since the lattice $(A, \leq)$ is distributive, e' is uniquely determined.

From Lemma 5 it follows that $f(0)$ is a complement of $f(1)$ in the lattice $(A, \leq)$. Hence we have $e = f(1)$, $e' = f(0)$ in our case.

Lemma 10. *The sets B and C are closed with respect to the operation $\oplus$.*

Proof. From [10, Lemma 3.6] it follows that the set B is closed under the operation $\oplus$. Similarly we can show that C is closed. □

Lemma 11. *For each $x \in C$, $f(x) = f(0) - x$, $x + f(0) = f(0) + x$, $x \oplus f(0) = f(0) \oplus x$.*

Proof. Let $x \in C$. Hence $x \leq f(0)$. By Lemma 9, $f(x) \leq f(0)$. Then from Lemma 6(i) and (iii) it follows that $f(x) = f(0) - x$, $x = f(0) - f(x)$. Thus we get $x = f(0) + x - f(0)$ and hence $x + f(0) = f(0) + x$. Then clearly $x \oplus f(0) = f(0) \oplus x$. □

Lemma 12.

(i) *For each $x \in C$, $x + 1 = 1 + x$.*

(ii) *If $x, y \in C$, then $x \oplus y = y \oplus x$.*

Proof.

(i) Let $x \in C$. By Lemmas 7, 11 and Corollary 1, $x + 1 - f(0) = x + f(1) = f(1) + x = 1 - f(0) + x = 1 + x - f(0)$. This implies $x + 1 = 1 + x$.

(ii) Let $x, y \in C$. Since $x \oplus y \geq y$, $f(0) - y \geq f(0) - (x \oplus y)$, in view of Lemmas 3, 10 and 11 we have $(x \oplus y) - y = d(x \oplus y, y) = d(f(x \oplus y), f(y)) = d(f(0) - (x \oplus y), f(0) - y) = f(0) - y - [f(0) - (x \oplus y)] = -y + (x \oplus y)$. From this and (i) we get $x \oplus y = y + (x \oplus y) - y = y + [(x+y) \wedge 1] - y = (y+x) \wedge (y+1-y) = (y + x) \wedge 1 = y \oplus x$. □

Lemma 13.

(i) (Cf. [10, Lemmas 3.3 and 3.4]) *For each element $x \in A$ there exist uniquely determined elements $x_1 \in B$ and $x_2 \in C$ such that $x = x_1 \oplus x_2$. Moreover, $x_1 = x \wedge f(1)$ and $x_2 = x \wedge f(0)$.*

(ii) *Let $x \in B$, $y \in C$. Then $f(x \oplus y) = x \oplus (f(0) - y) = x \oplus (f(0) \odot y^-)$.*

Proof.

(ii) Let $x \in B, y \in C$. By (i), $x = (x \oplus y)_1 = (x \oplus y) \wedge f(1)$, $y = (x \oplus y)_2 = (x \oplus y) \wedge f(0)$. Then Lemmas 1, 5, 7, 9 and 11 yield $f(x \oplus y) = (x \oplus y) \wedge f(1) + f(0) - ((x \oplus y) \wedge f(0)) = x + (f(0) - y) = x \oplus (f(0) - y) = x \oplus (f(0) \odot y^-)$. □

Theorem 1. *For each $x \in A$, $f(x) = [f(0) - (x \wedge f(0))] \vee (f(1) \wedge x)$.*

Proof. Let $x \in A$. Then $x_1 = f(1) \wedge x \in B$, $x_2 = f(0) \wedge x \in C$. From Lemmas 9 and 11 it follows that $f(0) - x_2 \in C$. Then Lemmas 5 and 7 yield $f(x) = x_1 + (f(0) - x_2) = x_1 \vee (f(0) - x_2) = [f(0) - (f(0) \wedge x)] \vee (f(1) \wedge x)$. □

In [12] it was shown that the assumption of 2-periodicity of isometry in [11, Proposition 4.4] can be omitted. Theorem 1 with Corollary 2 generalize [11, Proposition 4.4] without the assumption of 2-periodicity of isometry.

We define the unary operations ${}^{-_e}$, ${}^{\sim_e}$ on B by putting $x^{-_e} = f(1) - x$, $x^{\sim_e} = -x + f(1)$ for each $x \in B$.

Analogously we define the unary operations ${}^{-_{e'}}$, ${}^{\sim_{e'}}$ on C. For each $x \in C$ we put $x^{-_{e'}} = f(0) - x$, $x^{\sim_{e'}} = -x + f(0)$.

From Lemma 1 it follows that these operations are defined as in [10, p. 135] ($X_1 = B$, $X_2 = C$ in our case).

Theorem 2. $\mathcal{B} = (B, \oplus, {}^{-_e}, {}^{\sim_e}, 0, f(1))$ *is a pseudo MV-algebra,* $\mathcal{C} = (C, \oplus, {}^{-_{e'}}, {}^{\sim_{e'}}, 0, f(0))$ *is a commutative pseudo MV-algebra.*

Proof. By [10, Corollary 4.2], $\mathcal{B}$ is a pseudo MV-algebra. Analogously it can be shown that $\mathcal{C}$ is also a pseudo MV-algebra. The commutativity of $\mathcal{C}$ follows from Lemma 12. □

Theorem 3. *If for each $x \in A$ we put $\varphi(x) = (x \wedge f(1), x \wedge f(0))$, then φ is an isomorphism of $\mathcal{A}$ onto the direct product $\mathcal{B} \times \mathcal{C}$.*

Proof. It follows from [10, Proposition 4.3]. □

Hence φ is a direct decomposition of $\mathcal{A}$. In view of the definition of an internal direct decomposition we conclude that φ is also an internal direct decomposition of $\mathcal{A}$. (Clearly, $\mathcal{B}^0 = \mathcal{B}$, $\mathcal{C}^0 = \mathcal{C}$.) Hence, $x_{B^0} = x_B = x \wedge f(1)$, $x_{C^0} = x_C = x \wedge f(0)$, $x = x_B \oplus x_C$ for each $x \in A$.

Theorem 4. *Let $\mathcal{A} = (A, \oplus, {}^-, {}^\sim, 0, 1)$ be a pseudo MV-algebra and f an isometry in $\mathcal{A}$. Let $\mathcal{B}$ and $\mathcal{C}$ be as in* Theorem 2*. Then $\mathcal{A} = B \times C$, $1_C = f(0)$ and $f(x) = x_B \oplus (f(0) - x_C) = x_B \oplus (f(0) \odot (x_C)^-)$ for each $x \in A$.*

Proof. It follows from Theorems 3 and Lemma 13. □

Theorem 5. *Let $\mathcal{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra, $\varphi\colon \mathcal{A} \to P \times Q$ a direct decomposition of $\mathcal{A}$ with $\mathcal{Q}$ commutative and $\varphi^0\colon \mathcal{A} \to \mathcal{P}^0 \times \mathcal{Q}^0$ an internal direct decomposition of $\mathcal{A}$. Let P^0 (Q^0) be the underlying set of $\mathcal{P}^0$ ($\mathcal{Q}^0$, respectively). Then*

(i) *$\mathcal{Q}^0$ is a commutative pseudo MV-algebra,*

(ii) *for every $x \in P^0$ and $y \in Q^0$, $x + y$ is defined in $\mathcal{A}$,*

(iii) *for each $x, y \in A$, $d(x, y) = d(x_{P^0}, y_{P^0}) \oplus d(x_{Q^0}, y_{Q^0})$,*

(iv) *if we put $g(x) = x_{P^0} \oplus (1_{Q^0} - x_{Q^0})$ for each $x \in A$, then g is an isometry in $\mathcal{A}$ and $f(0) = 1_{Q^0}$.*

Proof.

(i) It is obvious.

(ii) Let $x \in P^0$ and $y \in Q^0$. Since $x \wedge y = 0$, from [2, Proposition 2.1(X)] it follows that $x + y$ is defined in $\mathcal{A}$.

(iii) Let $x, y \in A$. Then $d(x,y) = (x \vee y) - (x \wedge y) = (x_{P^0} \vee y_{P^0}) + (x_{Q^0} \vee y_{Q^0}) - [(x_{P^0} \wedge y_{P^0}) + (x_{Q^0} \wedge y_{Q^0})] = (x_{P^0} \vee y_{P^0}) - (x_{P^0} \wedge y_{P^0}) + (x_{Q^0} \vee y_{Q^0}) - (x_{Q^0} \wedge y_{Q^0}) = d(x_{P^0}, y_{P^0}) \oplus d(x_{Q^0}, y_{Q^0})$.

(iv) Let $x, y \in A$. Then $d(g(x), g(y)) = d(x_{P^0} \oplus (1_{Q^0} - x_{Q^0}), y_{P^0} \oplus (1_{Q^0} - y_{Q^0}))$ $= d(x_{P^0}, y_{P^0}) \oplus d(1_{Q^0} - x_{Q^0}, 1_{Q^0} - y_{Q^0}) = d(x_{P^0}, y_{P^0}) \oplus [((1_{Q^0} - x_{Q^0}) \vee (1_{Q^0} - y_{Q^0}))$ $- ((1_{Q^0} - x_{Q^0}) \wedge (1_{Q^0} - y_{Q^0}))] = d(x_{P^0}, y_{P^0}) \oplus [(1_{Q^0} - (x_{Q^0} \wedge y_{Q^0})) - (1_{Q^0} - (x_{Q^0} \vee y_{Q^0}))] = d(x_{P^0}, y_{P^0}) \oplus [(x_{Q^0} \vee y_{Q^0}) - (x_{Q^0} \wedge y_{Q^0})] = d(x_{P^0}, y_{P^0}) \oplus d(x_{Q^0}, y_{Q^0}) = d(x, y)$. Therefore g is an isometry. Clearly, $g(0) = 1_{Q^0}$. □

Theorems 4 and 5 show that there exists a one-to-one correspondence between isometries in A and internal direct decompositions of A with commutative second factor and that isometries in pseudo MV-algebras can be described similarly as isometries in lattice ordered groups.

Unlike isometries in pseudo MV-algebras, those in lattice ordered groups need not be 2-periodic. An isometry g in a lattice ordered group is 2-periodic iff $g(g(0)) = 0$.

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