Cent. Eur. J. Phys. • 11(1) • 2013 • 49-58
DOI: 10.2478/s11534-012-0127-7

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Central European Journal of **Physics**

Bäcklund transformation for the first flows of the relativistic Toda hierarchy and associated properties

Research Article

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Received 28 December 2011; accepted 03 September 2012

Abstract: In this communication we study a class of one parameter dependent auto-Bäcklund transformations for the first flow of the relativistic Toda lattice and also a variant of the usual Toda lattice equation. It is shown that starting from the Hamiltonian formalism such transformations are canonical in nature with a well defined generating function. The notion of spectrality is also analyzed and the separation variables are explicitly constructed.

PACS (2008): 02.90.+p, 05.45.Yv

Keywords: Lax matrix • Bäcklund transformations • generating function • separation of variables

1. Introduction

Bäcklund transformations (BT) have played a central role in the study of integrable systems both continuous as well as semi-discrete.

At the classical level the study of Bäcklund transformations is motivated by the possibility of being able to obtain a wide class of solutions starting from a fairly simple or in some cases even a trivial solution.

More recently in order to arrive at a fully quantized mechanism for quantum nonlinear integrable systems attempts have been made to derive BT's which can be derived from a suitable generating function, so that they may be viewed as a kind of canonical transformation, in order to subsequently quantize them. This has helped to achieve an integral representation of Baxter's Q-operator [1]. Quantum analogs of the relativistic Toda lattice were also considered by the authors of [2].

In our previous works we have pursued the issue of BT's for the standard Toda lattice and derived the corresponding integral representation of its associated Q-operator following the procedure devised by Sklyanin in [3, 4]. We have also studied the dimer self-trapping (DST) and the D_n-type Toda lattices under open and dynamical boundary conditions respectively [5, 6]. The former system is often used for studying quasi particle motion on a dimer. Therefore in continuation of the programme in this communication we address the issue of deriving canonical BT's for a family of semi-discrete integrable lattices related to the relativistic Toda hierarchy, which were first introduced by Suris in a series of papers dealing with the flows of the relativistic Toda hierarchy which appeared in connection with the discretization of continuous integrable systems [7–9].

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In case of semi-discrete flows it usually involves a transformation of the space part of the corresponding Lax pair while retaining the zero-curvature condition. Starting with the linear system

$$\Psi_{n+1} = \ell_n(\lambda)\Psi_n, \quad \dot{\Psi}_n = M_n(\lambda)\Psi_n, \tag{1}$$

with n being the lattice index and the over dot denoting derivative with respect to the continuous temporal variable t; their consistency yields the following zero-curvature equation for a semi-discrete system, *viz*

$$\dot{\ell}_n(\lambda) = M_{n+1}(\lambda)\ell_n(\lambda) - \ell_n(\lambda)M_n(\lambda), \tag{2}$$

from which the equation of motion is assumed to follow. As an example for the standard Toda lattice [10] which has the following equation of motion

$$\ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}, \quad n = 1, ..., N, \tag{3}$$

it may be verified that this follows from (2) when the Lax pair is given by,

$$\ell_n(\lambda) = \begin{pmatrix} \lambda + p_n & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}, \quad M_n(\lambda) = \begin{pmatrix} -\lambda & -e^{q_n} \\ e^{-q_{n-1}} & 0 \end{pmatrix}. \tag{4}$$

We begin by briefly introducing the relativistic Toda hierarchy.

1.1. First flow of the relativistic Toda hierarchy

The simplest flow of the relativistic Toda hierarchy (RTH) [7, 8] is given by

$$\dot{d}_k = d_k(c_k - c_{k-1}), \quad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}). \tag{5}$$

It can be derived from the Hamiltonian

$$H_+^{(1)} = \frac{1}{2}\sum_k (d_k + c_{k-1})^2 + \sum_k (d_k + c_{k-1})c_k \tag{6}$$

with the following Poisson structure:

$$\{c_k, d_{k+1}\}_1 = -c_k, \quad \{c_k, d_k\}_1 = c_k, \quad \{d_k, d_{k+1}\}_1 = c_k, \tag{7}$$

as can be easily verified. On the other hand when the Hamiltonian is taken as

$$H_+^{(2)} = \sum_{k=1}^{N} (d_k + c_k), \tag{8}$$

along with the quadratic Poisson brackets

$$\{c_k, c_{k+1}\}_2 = -c_k c_{k+1}, \quad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1}, \quad \{c_k, d_k\}_2 = c_k d_k \tag{9}$$

one is again led to equations (5). Thus we conclude that it possesses a bi-Hamiltonian structure.
Under the following transformation

$$d_k = p_k - e^{q_k - q_{k-1}}, \quad c_k = e^{q_{k+1}-q_k}, \tag{10}$$

the Hamiltonian (6), i.e., $H_+^{(1)}$ is mapped, in terms of the canonical variables $(q_k, p_k)_{k=1}^N$, to

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} p_k e^{q_k - q_{k-1}} \quad \text{with} \quad \{p_n, q_m\} = \delta_{nm}. \tag{11}$$

The Hamiltons equations of motion are then given by

$$\dot{q}_n = \frac{\partial H}{\partial p_n} = p_n + e^{q_n - q_{n-1}}, \tag{12}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} = p_{n+1}e^{q_{n+1}-q_n} - p_n e^{q_n - q_{n-1}}, \tag{13}$$

so that the corresponding Newtonian equation of motion appears as

$$\ddot{q}_n = \dot{q}_{n+1}e^{q_{n+1}-q_n} - \dot{q}_{n-1}e^{q_n - q_{n-1}} - e^{2(q_{n+1}-q_n)} + e^{2(q_n - q_{n-1})}. \tag{14}$$

1.2. Second flow of the RTH

The second flow of the RTH is given by

$$\dot{d}_k = d_k\left(\frac{c_k}{d_k d_{k+1}} - \frac{c_{k-1}}{d_{k-1}d_k}\right), \quad \dot{c}_k = c_k\left(\frac{1}{d_k} - \frac{1}{d_{k-1}}\right). \tag{15}$$

This also possess a bi-Hamiltonian structure with the Hamiltonians being

$$H_-^{(1)} = -\sum_{k=1}^{N} \log d_k, \quad H_-^{(2)} = \sum_{k=1}^{N} \frac{d_k + c_k}{d_k d_{k+1}}, \tag{16}$$

corresponding to the two Poisson structures (7) and (9) respectively. The transformation (10) causes the Hamiltonian $H_-^{(1)}$ to appear as

$$H_-^{(1)} = -\sum_{k=1}^{N} \log(p_k - e^{q_k - q_{k-1}}), \tag{17}$$

and in turn leads to the following Hamiltons equations of motion,

$$\dot{q}_n = \frac{\partial H_-^{(1)}}{\partial p_n} = -\frac{1}{p_n - e^{q_n - q_{n-1}}}, \quad (18)$$

$$\dot{p}_n = -\frac{\partial H_-^{(1)}}{\partial q_n} = \frac{e^{q_{n+1}-q_n}}{p_{n+1} - e^{q_{n+1}-q_n}} - \frac{e^{q_n - q_{n-1}}}{p_n - e^{q_n - q_{n-1}}}. \quad (19)$$

Solving for p_n from the first of these equations and using it in the second leads to the following Newtonian equation of motion,

$$\ddot{q}_n = \dot{q}_n^2 \left(\dot{q}_{n+1} e^{q_{n+1}-q_n} - \dot{q}_{n-1} e^{q_n - q_{n-1}} \right). \quad (20)$$

2. Lax formulation of first RTH flow

Recall that the first Newtonian equation belonging to the RTH is given by (14) and is equivalent to the following semi-discrete system (time being the continuous variable)

$$\dot{q}_n = p_n + e^{q_n - q_{n-1}}, \quad (21)$$
$$\dot{p}_n = p_{n+1} e^{q_{n+1}-q_n} - p_n e^{q_n - q_{n-1}}. \quad (22)$$

It may be derived from the zero-curvature condition (2) with the spectral parameter, λ, dependent local Lax pair

$$\ell_n(\lambda) = \begin{pmatrix} \lambda + p_n & -p_n e^{q_n} \\ e^{-q_n} & -1 \end{pmatrix}, \; M_n(\lambda) = \begin{pmatrix} 0 & p_n e^{q_n} \\ -e^{-q_{n-1}} & \lambda \end{pmatrix}, \quad n = 1, \ldots, N. \quad (23)$$

2.1. The classical r-matrix algebra of first RTH flow

It is easy to verify that $\ell_n(\lambda)$ given in (23) satisfies the Sklyanin quadratic algebra

$$\{\ell_n^1(\lambda), \ell_m^2(\mu)\} = [r(\lambda-\mu), \ell_n^1(\lambda)\ell_m^2(\mu)]\delta_{nm}, \quad (24)$$

where $\ell_n^1(\lambda) = \ell_n(\lambda) \otimes I$ and $\ell_n^2(\mu) = I \otimes \ell_n(\mu)$ are the standard tensor products of $\ell_n(\lambda)$ with the 2×2 unit matrix I and

$$r(\lambda-\mu) = \frac{\mathcal{P}}{\lambda-\mu} := \frac{1}{\lambda-\mu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (25)$$

where $\mathcal{P}$ denotes the permutation matrix. One defines the monodromy matrix in the usual manner as

$$T_N(\lambda) := \prod_{n=1}^{\overleftarrow{N}} \ell_n(\lambda) = \ell_N(\lambda)\ell_{N-1}(\lambda)\ldots\ell_1(\lambda)$$
$$:= \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}. \quad (26)$$

It is well known that the trace of the monodromy matrix is a generator of the conserved quantities. Let

$$t(\lambda) = \mathrm{tr}(T_N(\lambda)) = A_N(\lambda) + D_N(\lambda). \quad (27)$$

A direct calculation then shows that the elements of the monodromy matrix are polynomials in λ having the general form

$$T_N(\lambda) = \begin{pmatrix} \lambda^N + \lambda^{N-1}P + O(\lambda^{N-2}) & -\lambda^{N-1}p_1 e^{q_1} + O(\lambda^{N-2}) \\ \lambda^{N-1}e^{-q_N} + O(\lambda^{N-2}) & O(\lambda^{N-2}) \end{pmatrix}, \quad (28)$$

where $P = \sum_{k=1}^N p_k$ represents the total momentum of the system. Hence

$$t(\lambda) = A_N(\lambda) + D_N(\lambda)$$
$$= \lambda^N + \lambda^{N-1}P + \lambda^{N-2}\sum_{k=1}^{N}(p_{k+1}p_k - p_k e^{q_k - q_{k-1}}) + \cdots. \quad (29)$$

Since $t(\lambda)$ is a constant of motion it follows that the coefficients of the different powers of λ are conserved. Denoting these by $C_i (i = 1, \ldots)$ we have

$$C_1 = P, \quad (30)$$

$$C_2 = \sum_{k=1}^{N} \left(p_{k+1}p_k - p_k e^{q_k - q_{k-1}} \right), \quad (31)$$

and so on. It is easy to check that the Hamiltonian H is a combination of the C_i's, namely

$$H = \frac{1}{2}C_1^2 - C_2. \quad (32)$$

It may be mentioned that it is also possible to obtain the equation of motion for the first relativistic Toda hierarchy from an Euler-Lagrange perspective with Lagrangian given by

$$L = \frac{1}{2}\sum_{k=1}^{N} \left(\dot{q}_k - e^{q_k - q_{k-1}} \right)^2. \quad (33)$$

3. Canonical Bäcklund transformation for first RTL flow

In this section we will construct a Bäcklund transformation (BT) for the system given by (21) and (22) based upon the Hamiltonian approach. The method proposed in [1, 4] relies firstly on the ability to find an invertible matrix $G_n(\lambda,\xi)$ satisfying the following gauge transformation:

$$G_{n+1}(\lambda,\xi)\ell_n(\lambda;q_n,p_n) = \ell_n(\lambda;\tilde{q}_n,\tilde{p}_n)G_n(\lambda,\xi), \qquad (34)$$

which serves as an auxiliary matrix for the purpose. Eqn. (34) serves as a similarity transformation for the local Lax operator $\ell_n(\lambda,q_n,p_n)$ and hence of the monodromy matrix $T_N(\lambda)$ as defined in (26). This local transformation

$$b_\xi^{(n)} : \ell_n(\lambda,q_n,p_n) \longrightarrow \ell_n(\lambda,\tilde{q}_n,\tilde{p}_n), \quad n=1,\ldots,N, \qquad (35)$$

depends on the parameter ξ, while the auxiliary matrix $G_n(\lambda,\xi)$ which induces the BT is assumed to be a non-singular matrix obeying the quadratic algebra (24). In the following we employ the local Lax operator of the DST model as the auxiliary matrix $G_n(\lambda,\xi)$ in order to induce the BT as it satisfies the algebra (24).

3.1. With the DST model as auxiliary matrix

The Lax operator for the DST model [11] has the following form

$$G_n(\lambda,\xi) = \begin{pmatrix} \lambda-\xi+s_nS_n & s_n \\ S_n & 1 \end{pmatrix}. \qquad (36)$$

We assume that the BT is formally defined by

$$\begin{pmatrix} \lambda-\xi+s_nS_n & s_n \\ S_n & 1 \end{pmatrix}\begin{pmatrix} \lambda+p_n & -p_ne^{q_n} \\ e^{-q_n} & -1 \end{pmatrix} = \begin{pmatrix} \lambda+\tilde{p}_n & -\tilde{p}_ne^{\tilde{q}_n} \\ e^{-\tilde{q}_n} & -1 \end{pmatrix}\begin{pmatrix} \lambda-\xi+t_nT_n & t_n \\ T_n & 1 \end{pmatrix}. \qquad (37)$$

Upon equating the coefficients of the different powers of λ we arrive at the following set of relations:

$$\begin{aligned}
p_n(s_nS_n-\xi)+s_ne^{-q_n} &= \tilde{p}_n(t_nT_n-\xi)-T_n\tilde{p}_ne^{\tilde{q}_n}, &(38)\\
p_n+s_nS_n &= t_nT_n+\tilde{p}_n, &(39)\\
p_ne^{q_n} &= -t_n, &(40)\\
(s_nS_n-\xi)p_ne^{q_n}+s_n &= \tilde{p}_n(e^{\tilde{q}_n}-t_n), &(41)\\
S_n &= e^{-\tilde{q}_n}, &(42)\\
S_np_n+e^{-q_n} &= e^{-\tilde{q}_n}(t_nT_n-\xi)-T_n, &(43)\\
S_np_ne^{q_n} &= -t_ne^{-\tilde{q}_n}, &(44)
\end{aligned}$$

which yield the following solutions:

$$\begin{aligned}
\tilde{p}_n &= -\left(e^{\tilde{q}_n-q_n}+T_ne^{\tilde{q}_n}+\xi\right)+e^{-\tilde{q}_n}s_n, &(45)\\
p_n &= -\frac{\left(e^{\tilde{q}_n-q_n}+T_ne^{\tilde{q}_n}+\xi\right)}{1+T_ne^{q_n}}, &(46)\\
t_n &= e^{\tilde{q}_n}+\frac{\xi}{T_n+e^{-q_n}}, &(47)
\end{aligned}$$

along with S_n which is already given by (42). These relations may be derived from a local generating function $f_\xi^{(n)}$ such that

$$p_n = \frac{\partial f_\xi^{(n)}}{\partial q_n},\quad \tilde{p}_n = -\frac{\partial f_\xi^{(n)}}{\partial \tilde{q}_n},\quad S_n = \frac{\partial f_\xi^{(n)}}{\partial s_n},\quad t_n = \frac{\partial f_\xi^{(n)}}{\partial T_n}, \qquad (48)$$

where

$$\begin{aligned}
f_\xi^{(n)}(q_n,\tilde{q}_n,T_n,s_n) &= e^{\tilde{q}_n-q_n}+T_ne^{\tilde{q}_n}\\
&+\xi\tilde{q}_n+e^{-\tilde{q}_n}s_n + \xi\log(T_n+e^{-q_n}). \qquad (49)
\end{aligned}$$

In order to be consistent with (34) we now impose the conditions that

$$T_n = S_{n-1}, \quad s_n = t_{n+1}. \qquad (50)$$

This causes the elimination of the auxiliary variables from (45) and (46) which are then given by

$$\tilde{p}_n = e^{\tilde{q}_{n+1}-\tilde{q}_n}-e^{\tilde{q}_n-\tilde{q}_{n-1}}-\left[e^{-q_n+\tilde{q}_n}+\frac{\xi e^{-q_{n+1}}}{e^{-\tilde{q}_n}+e^{-q_{n+1}}}\right], \qquad (51)$$

$$p_n = -\left[e^{-q_n+\tilde{q}_n}+\frac{\xi e^{-q_n}}{e^{-\tilde{q}_{n-1}}+e^{-q_n}}\right]. \qquad (52)$$

Eqs. (51) and (52) define the required one parameter auto-Bäcklund transformations and it may be checked that their generating function is given by

$$F_\xi = \sum_{n=1}^{N}\left[\xi\log(e^{\tilde{q}_n-q_n}+e^{\tilde{q}_n-\tilde{q}_{n-1}})+e^{\tilde{q}_n}\left(e^{-q_n}+e^{-\tilde{q}_{n-1}}\right)\right] + \text{const.}, \qquad (53)$$

with

$$p_n = \frac{\partial F_\xi}{\partial q_n}, \quad \tilde{p}_n = -\frac{\partial F_\xi}{\partial \tilde{q}_n}.$$

Note that we may assume without loss of generality that the constant of integration depends on the parameter ξ.

3.2. With a different auxiliary matrix

It is quite obvious that there is a certain amount of freedom in the choice of the auxiliary matrix used to induce the Bäcklund transformations as defined in (34). One requirement is that the auxiliary matrix used must have the same r-matrix algebra as the Lax operator of the model whose BT is being studied [4]. For instance if we were to choose a different auxiliary matrix then the formal appearance of the BT may differ significantly as is illustrated below. We assume that the matrix $G_i(\lambda,\xi)$ to be given by

$$G_i(\lambda,\xi)=\begin{pmatrix}\lambda-\xi+s_iS_i & s_i^2S_i-2\xi s_i\\ S_i & -(\lambda+\xi)+s_iS_i\end{pmatrix}. \qquad (54)$$

Substituting the expressions for $\ell_i(\lambda,q_i,p_i)$ and $G_i(\lambda,\xi)$ from (24) and (54) respectively into (34) we have after equating the different coefficients of powers of λ the following expressions:

$$p_i=\frac{(2\xi s_i-s_i^2S_i)+s_{i+1}e^{\bar q_i}(e^{-q_i}+e^{-\bar q_i})}{(e^{q_i}+e^{\bar q_i})}, \qquad (55)$$

$$\bar p_i=\frac{(2\xi s_i-s_i^2S_i)-s_{i+1}e^{q_i}(e^{-q_i}+e^{-\bar q_i})}{(e^{q_i}+e^{\bar q_i})}, \qquad (56)$$

$$S_{i+1}=e^{-q_i}+e^{-\bar q_i}, \qquad (57)$$

$$s_{i+1}=\left\{\left(\frac{\xi}{e^{-q_i}+e^{-\bar q_i}}\right)-\left(\frac{e^{\bar q_{i+1}}-e^{q_{i+1}}}{2}\right)\right\} \pm\left[\left(\frac{\xi}{e^{-q_i}+e^{-\bar q_i}}\right)^2+\left(\frac{e^{\bar q_{i+1}}-e^{q_{i+1}}}{2}\right)^2+e^{q_{i+1}+\bar q_{i+1}}\right]^{1/2}. \qquad (58)$$

It is clear that by using the expressions for s_i and S_i as obtained from the last two equations by the replacement $(i\to i-1)$, in the first two equations, we obtain the values of p_i and $\bar p_i$ entirely in terms of the set $\{q_i\}$ and the parameter ξ. This gives another set of one-parameter auto-Bäcklund transformation relations.

4. A variant of the Toda lattice model

In this section we will consider a model which is closely related to the standard Toda lattice. It was introduced by Suris in [12] and has the following equation of motion:

$$\ddot q_n=\dot q_n(e^{q_{n+1}-q_n}-e^{q_n-q_{n-1}}). \qquad (59)$$

It may be written as the following equivalent system, *viz*

$$\dot q_n=p_n,\quad \dot p_n=p_n(e^{q_{n+1}-q_n}-e^{q_n-q_{n-1}}), \qquad (60)$$

and can be derived from the zero curvature condition (2) with the following Lax pair

$$\ell_n(\lambda)=\begin{pmatrix}\lambda+p_n & -e^{q_n}\\ -\lambda e^{-q_n} & 1\end{pmatrix},\quad M_n(\lambda)=\begin{pmatrix}e^{q_n-q_{n-1}} & e^{q_n}\\ \lambda e^{-q_{n-1}} & \lambda\end{pmatrix}. \qquad (61)$$

Note that the system of equations (60) can also be derived from (2) using an alternate form of the Lax pair given by:

$$\ell_n(\lambda)=\begin{pmatrix}\lambda e^{p_n}-\lambda^{-1} & e^{q_n}\\ -e^{-q_n} & \lambda\end{pmatrix},$$
$$M_n(\lambda)=\begin{pmatrix}\lambda^{-2}+e^{q_n-q_{n-1}} & -\lambda^{-1}e^{q_n}\\ \lambda^{-1}e^{-q_{n-1}} & 0\end{pmatrix}. \qquad (62)$$

This indicates the non uniqueness of the Lax pair for the equation of motion (59). In the latter case the equations of motion are given by

$$\dot p_n=e^{q_{n+1}-q_n}-e^{q_n-q_{n-1}}, \qquad (63)$$
$$\dot q_n=e^{p_n}, \qquad (64)$$

and one can easily verify that elimination of p_n leads once again to (59), which incidentally is closest to the standard Toda lattice equation. The Hamiltonian of (59) is given by

$$H=\sum_{n=1}^{N}(e^{p_n}+e^{q_n-q_{n-1}}). \qquad (65)$$

The corresponding Lagrangian being

$$L=\sum_{n=1}^{N}[\dot q_n\log\dot q_n-\dot q_n-e^{q_n-q_{n-1}}]. \qquad (66)$$

Proposition 4.1.
The Lax pair given in (62) admits the following quadratic r-matrix algebra

$$\{\ell_n^1(\lambda),\ell_m^2(\mu)\}=\left[r(\lambda,\mu),\ell_n^1(\lambda)\ell_m^2(\mu)\right]\delta_{nm} \qquad (67)$$

with

$$r(\lambda,\mu)=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & (a-\frac{\lambda^2}{\lambda^2-\mu^2}) & \frac{\lambda\mu}{\lambda^2-\mu^2} & 0\\ 0 & \frac{\lambda\mu}{\lambda^2-\mu^2} & (a-\frac{\mu^2}{\lambda^2-\mu^2}) & 0\\ 0 & 0 & 0 & a\end{pmatrix}.$$

Proof: By direct calculation.
Setting $a = (\lambda^2+\mu^2)/2(\lambda^2-\mu^2)$ this r-matrix may be written as

$$r(\lambda,\mu)=\begin{pmatrix} \frac{\lambda^2+\mu^2}{2(\lambda^2-\mu^2)} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\lambda\mu}{\lambda^2-\mu^2} & 0 \\ 0 & \frac{\lambda\mu}{\lambda^2-\mu^2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{\lambda^2+\mu^2}{2(\lambda^2-\mu^2)} \end{pmatrix}. \quad (68)$$

In [13] a version of the relativistic Toda lattice having the following Lax matrix was analyzed in the context of Bäcklund transformation, namely;

$$\ell_k(\lambda) = \begin{pmatrix} \lambda e^{\alpha p_k}-\lambda^{-1} & \alpha e^{q_k} \\ -\alpha e^{-q_k+\alpha p_k} & 0 \end{pmatrix},$$

$$M_k(\lambda) = \begin{pmatrix} 0 & -\lambda e^{q_{k+1}-\alpha p_k} \\ \lambda e^{-q_k} & \frac{\lambda^2-1}{\alpha}+\alpha e^{q_{k+1}-q_k-\alpha p_{k+1}} \end{pmatrix}. \quad (69)$$

This Lax matrix also satisfies the r-matrix algebra (67) with an r-matrix given by (68). This makes it a potential candidate for inducing the BT of the Lax matrix given in (62). Indeed it follows from the defining relation of the BT, namely

$$\begin{pmatrix} \frac{\lambda}{\xi}e^{\alpha S_{k+1}}-\frac{\xi}{\lambda} & \alpha e^{s_{k+1}} \\ -\alpha e^{-s_{k+1}+\alpha S_{k+1}} & 0 \end{pmatrix}\begin{pmatrix} \lambda e^{p_k}-\lambda^{-1} & e^{q_k} \\ -e^{-q_k} & \lambda \end{pmatrix}$$
$$=\begin{pmatrix} \lambda e^{\tilde p_k}-\lambda^{-1} & e^{\tilde q_k} \\ -e^{-\tilde q_k} & \lambda \end{pmatrix}\begin{pmatrix} \frac{\lambda}{\xi}e^{\alpha S_k}-\frac{\xi}{\lambda} & \alpha e^{s_k} \\ -\alpha e^{-s_k+\alpha S_k} & 0 \end{pmatrix}, \quad (70)$$

(where ξ is the Backlund parameter) that the BT is now given by

$$e^{p_n} = \frac{1}{\alpha^2}e^{q_n-\tilde q_{n-1}}+e^{\tilde q_n-\tilde q_{n-1}}, \quad (71)$$
$$e^{\tilde p_n} = \frac{1}{\alpha^2}e^{q_{n+1}-\tilde q_n}+e^{q_{n+1}-q_n}. \quad (72)$$

It may be verified that the above BT is derivable from the generating function

$$F=\sum_{n=1}^{N}\Big[\frac{1}{2}q_n^2 + \frac{1}{2}\tilde q_n^2 - q_n\tilde q_{n-1}+2(\tilde q_n-q_n)\log\alpha - \int \log(1+\alpha^2 e^x)\,dx\Big|_{x=\tilde q_n-q_n}\Big], \quad (73)$$

that is

$$p_n=\frac{\partial F}{\partial q_n},\quad \tilde p_n=-\frac{\partial F}{\partial \tilde q_n}.$$

Interestingly although the auxiliary matrix used here depended explicitly on the parameter ξ, the latter does not appear in the BT given by (71) and (72) and therefore also in the expression for the corresponding generating function.
It is pertinent to mention here that the Lax pair (62) is actually a reduced form of the Lax pair of the system given below, obtained by setting the parameter $\alpha=0$. For this model the equation of motion is given by

$$\ddot x_k = \dot x_k\Big[(e^{x_{k+1}-x_k}-e^{x_k-x_{k-1}}+\frac{\alpha e^{x_{k+1}-x_k}}{1+\alpha e^{x_{k+1}-x_k}}\dot x_{k+1} - \frac{\alpha e^{x_k-x_{k-1}}}{1+\alpha e^{x_k-x_{k-1}}}\dot x_{k-1}\Big], \quad (74)$$

and is derivable from the following Lax pair

$$\ell_n(\lambda) = \begin{pmatrix} \lambda e^{p_n}-\lambda^{-1} & e^{q_n} \\ -(1+\alpha e^{p_n})e^{-q_n} & \lambda \end{pmatrix},$$

$$M_n(\lambda) = \begin{pmatrix} \lambda^{-2}+(1+\alpha e^{p_{n-1}})e^{q_n-q_{n-1}} & -\lambda^{-1}e^{q_n} \\ \lambda^{-1}(1+\alpha e^{p_n})e^{-q_{n-1}} & 0 \end{pmatrix}. \quad (75)$$

Clearly setting $\alpha=0$ causes it to reduce to (62).

Proposition 4.2.

The above Lax pair satisfies the algebra

$$\{\ell_n^1(\lambda),\ell_m^2(\mu)\} = [r(\lambda,\mu),\ell_n^1(\lambda)\ell_m^2(\mu)]\delta_{nm},$$

where the classical r-matrix is given by (68).

Proof: By a direct calculation.

5. Spectrality

An integrable Hamiltonian system with a Lax pair is said to be separable if it possesses a suitable set of Darboux coordinates (λ_j,μ_j), $j=1,\ldots,N$ satisfying, in most cases, a common associated spectral curve $\Gamma(\lambda,\mu,I_1,\ldots,I_N)=0$ where $I_1,\ldots,I_N$ are conserved quantities in involution. Given a Lax pair with the monodromy matrix having the form stated in (26) the affine part of the spectral curve is defined by

$$\det(\mu - T_N(\lambda)) = 0. \quad (76)$$

It will be recalled that the defining relation for the BT is of the form

$$G_{i+1}(\lambda,\xi)\ell_i(p,q;\lambda) = \ell_i(\tilde p,\tilde q,\lambda)G_i(\lambda,\xi) \quad (77)$$

From the definition of the monodromy matrix it is now easy to see that

$$G_1 T_N(p,q,\lambda) = \tilde{T}_N(\tilde{p},\tilde{q},\lambda)G_1. \tag{78}$$

Furthermore one can verify that as $\det G(\lambda = \xi) = 0$, there exists a vector w_i such that

$$G_i(\lambda,\xi)\Big|_{\lambda=\xi} w_i = 0 \ \forall \ i = 1,\dots,N, \tag{79}$$

which in turn implies

$$G_1 T_N(p,q,\lambda)\Big|_{\lambda=\xi} w_1 = \tilde{T}_N(\tilde{p},\tilde{q},\lambda)\Big|_{\lambda=\xi} G_1 w_1 = 0, \tag{80}$$

so that

$$G_1\left[T_N(p,q,\lambda)\Big|_{\lambda=\xi} w_1\right] = 0. \tag{81}$$

A comparison with $G_1 w_1 = 0$, then clearly indicates that we must have

$$T_N(p,q,\lambda)\Big|_{\lambda=\xi} w_1 = \Lambda w_1. \tag{82}$$

But it also follows from (77) that

$$\ell_i(p,q,\lambda)\Big|_{\lambda=\xi} w_i = g_i w_{i+1}, \tag{83}$$

so that

$$T_N(p,q,\lambda)\Big|_{\lambda=\xi} w_1 = \left(\prod_{i=1}^{N} g_i\right) w_{N+1} = \left(\prod_{i=1}^{N} g_i\right) w_1, \tag{84}$$

where we have assumed periodicity $w_{k+N} = w_k$. Consequently from (82) and (84) it follows that

$$\Lambda = \prod_{i=1}^{N} g_i. \tag{85}$$

In case of the first flow of the RTL the auxiliary matrix G_n when evaluated at $\lambda = \xi$ is given by

$$G_n(\lambda,\xi)\Big|_{\lambda=\xi} = \begin{pmatrix} T_n t_n & t_n \\ T_n & 1 \end{pmatrix}, \tag{86}$$

where

$$T_n = e^{-\tilde{q}_{n-1}}, \quad t_n = \left(e^{\tilde{q}_n} + \frac{\xi e^{\tilde{q}_{n-1}}}{1 + e^{\tilde{q}_{n-1}-q_n}}\right). \tag{87}$$

Since $\det[G_n(\lambda,\xi)|_{\lambda=\xi}] = 0$ we have upon setting $G_n(\lambda,\xi)|_{\lambda=\xi} w_n = 0$

$$w_n = \begin{pmatrix} e^{\tilde{q}_{n-1}} \\ -1 \end{pmatrix}. \tag{88}$$

It now follows from (83) that

$$\begin{aligned} g_n &= -(1 + e^{-q_n+\tilde{q}_{n-1}}) \\ \text{and} \quad p_n &= -e^{-q_n}\left[e^{\tilde{q}_n} + \frac{\xi}{e^{-\tilde{q}_{n-1}} + e^{-q_n}}\right], \end{aligned} \tag{89}$$

the expression for p_n being the same as obtained earlier in (52). Hence from (84) we find that the eigenvalue of the monodromy matrix is given by

$$\Lambda = (-1)^N \prod_{n=1}^{N}(1 + e^{-q_n+\tilde{q}_{n-1}}). \tag{90}$$

Recalling that the generating function of the BT for the first flow of the RTH is given by (53) it immediately follows upon setting the constant of integration to be $iN\pi\xi$ that

$$\mu = \frac{\partial F_\xi}{\partial \xi} = \sum_n \log(e^{\tilde{q}_n-q_n} + e^{\tilde{q}_n-\tilde{q}_{n-1}}) + iN\pi, \tag{91}$$

so that

$$e^\mu = (-1)^N \prod_{n=1}^{N}(1 + e^{-q_n+\tilde{q}_{n-1}}) = \Lambda. \tag{92}$$

6. Separation of variables for the first RTH flow

It will be recalled that the Lax pair for the first flow of the RTH as given in (23) satisfies the r-matrix algebra (24) with the $r-$ matrix having the form stated in (25). Introducing the shifting $\lambda \to \lambda - c_n$ where c_n are parameters at each of the lattice sites our Lax operator assumes the form

$$\ell_n(\lambda) = \begin{pmatrix} \lambda - c_n + p_n & -p_n e^{q_n} \\ e^{-q_n} & -1 \end{pmatrix}. \tag{93}$$

The monodromy matrix now depends on the parameters $\{c_n\}_{n=1}^N$ and is defined in the usual way by

$$T_N(\lambda) := \overleftarrow{\prod_{n=1}^{N}} \ell_n(\lambda, c_n). \tag{94}$$

It satisfies the following algebra

$$\{T_N^1(\lambda), T_N^2(\mu)\} = [r(\lambda-\mu), T_N^1(\lambda)T_N^2(\mu)], \qquad (95)$$

where $T_N^1(\lambda) = T_N(\lambda)\otimes I$ and $T_N^2(\mu) = I\otimes T_N(\mu)$. Writing the monodromy matrix as a 2×2 matrix as stated in (2.6) it follows from (94) that its elements have the following expansions:

$$A_N(\lambda) = \lambda^N + \sum_{i=1}^{N}(p_i - c_i)\lambda^{N-1} + \mathcal{O}(\lambda^{N-2}), \qquad (96)$$

$$B_N(\lambda) = \lambda^{N-1}(-p_1 e^{q_1}) + \mathcal{O}(\lambda^{N-2}), \qquad (97)$$

$$C_N(\lambda) = \lambda^{N-1}e^{-q_N} + \mathcal{O}(\lambda^{N-2}), \qquad (98)$$

$$D_N(\lambda) = \mathcal{O}(\lambda^{N-2}). \qquad (99)$$

The determinant of the monodromy matrix is given by

$$\det T_N(\lambda) = \prod_{i=1}^{N}\det \ell_i(\lambda, c_i) = (-1)^N\prod_{i=1}^{N}(\lambda - c_i). \qquad (100)$$

In the separation representation of the quadratic algebra (95) we look for N canonical pairs of variables (λ_i, μ_i) $i = 1,\ldots,N$ having standard Poisson brackets

$$\{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0, \quad \{\lambda_i, \mu_j\} = \delta_{ij}. \qquad (101)$$

The precise choices of the pairs $(\lambda_i, \mu_i), i = 1,\ldots,N$ will now be made. As $C_N(\lambda)$ is a polynomial of degree $(N-1)$ if we denote the zeros of $C_N(\lambda)$ by λ_i, i.e.,

$$C_N(\lambda_i) = 0, \quad \forall\, i = 1,\ldots,N-1$$

then the monodromy matrix when evaluated at the zeros of $C_N(\lambda)$ reduces to

$$T_N(\lambda=\lambda_i) = \begin{pmatrix} A_N(\lambda_i) & B_N(\lambda_i) \\ 0 & D_N(\lambda_i)\end{pmatrix}, \qquad (102)$$

and its eigenvalues are obviously given by the diagonal elements. As mentioned earlier one can associate with the monodromy matrix the following spectral curve defined by

$$\det(\mu I - T_N(\lambda)) = \mu^2 - P(\lambda)\mu + Q(\lambda) = 0, \qquad (103)$$

where I denotes the 2×2 unit matrix. Clearly it follows that

$$Q(\lambda) = \det T_N(\lambda) = A_N(\lambda)D_N(\lambda) - B_N(\lambda)C_N(\lambda),$$
$$P(\lambda) = \operatorname{tr} T_N(\lambda). \qquad (104)$$

Let us introduce a new variable $v_i \quad i = 1,\ldots,n$ which is defined by

$$\mu_i := e^{-v_i} := D_N(\lambda_i), \quad i = 1,\ldots,N-1. \qquad (105)$$

This provides a set of $N-1$ pairs of variables. As for the remaining pair we define the variables μ_N and λ_N as follows:

$$\mu_N := q_N, \quad \lambda_N := \sum_{n=1}^{N}(p_n - c_n). \qquad (106)$$

Clearly it follows that their Poisson bracket $\{\lambda_N, \mu_N\} = 1$. Then from (98) we have

$$C_N(\lambda) = e^{-v_N}\prod_{i=1}^{N-1}(\lambda - \lambda_i), \qquad (107)$$

and from (96) and (99) the ratios

$$\frac{A_N(\lambda)}{C_N(\lambda)} = G_1(\lambda) + \sum_{i=1}^{N-1}\frac{A_N(\lambda_i)}{C_N'(\lambda_i)(\lambda-\lambda_i)}, \qquad (108)$$

$$\frac{D_N(\lambda)}{C_N(\lambda)} = \sum_{i=1}^{N-1}\frac{D_N(\lambda_i)}{C_N'(\lambda_i)(\lambda-\lambda_i)}, \qquad (109)$$

where $G_1(\lambda)$ is linear in λ. Here $C_N'(\lambda_i)$ represents the derivative of $C_N(\lambda)$ with respect to λ evaluated at $\lambda = \lambda_i$. Let $G_1(\lambda) = a\lambda + b$, then it follows that

$$A_N(\lambda) = (a\lambda + b)e^{-v_N} \prod_{i=1}^{N-1}(\lambda - \lambda_i) + e^{-v_N} \prod_{j=1}^{N-1}(\lambda - \lambda_j) \sum_i \frac{A_N(\lambda_i)}{C'_N(\lambda_i)(\lambda - \lambda_i)},$$
$$= e^{-v_N}\left[a\lambda^N - (a\sum_{i=1}^{N-1}\lambda_i)\lambda^{N-1} + b\lambda^{N-1} + \mathcal{O}(\lambda^{N-2})\right]. \quad (110)$$

Next comparing (110) and (96) we immediately see that

$$a = e^{v_N}, \quad b = e^{v_N}\left[\sum_{j=1}^{N}(p_j - c_j) + \sum_{i=1}^{N-1}\lambda_i\right] := e^{v_N}S. \quad (111)$$

Therefore we have finally

$$\frac{A_N(\lambda)}{C_N(\lambda)} = e^{v_N}(\lambda + S) + \sum_{i=1}^{N-1}\frac{A_N(\lambda_i)}{C'_N(\lambda_i)(\lambda - \lambda_i)}. \quad (112)$$

It follows from (100) and (104) that

$$B_N(\lambda) = \frac{A_N(\lambda)D_N(\lambda) - (-1)^N\prod_{i=1}^{N}(\lambda - c_i)}{C_N(\lambda)}, \quad (113)$$

while

$$C_N(\lambda) = e^{-v_N}\prod_{i=1}^{N-1}(\lambda - \lambda_i). \quad (114)$$

Thus (109) together with (112)-(114) complete the determination of the elements of the monodromy matrix in terms of the separation variables.
It may be proved that the representation of these elements in terms of the separation variables is a faithful representation of the algebra (95). We illustrate this below for the particular case of the Poisson bracket $\{C_N(\lambda), A_N(\mu)\}$ and show explicitly that

$$\{C_N(\lambda), A_N(\mu)\} = \frac{1}{\lambda - \mu}\left[A_N(\lambda)C_N(\mu) - A_N(\mu)C_N(\lambda)\right]. \quad (115)$$

Using (112) it follows that the left hand side is

$$\begin{aligned} &\{C_N(\lambda), C_N(\mu)e^{v_N}(\mu + S)\} \\ + &\left\{C_N(\lambda), C_N(\mu)\sum_{i=1}^{N-1}\frac{A_N(\lambda_i)}{C'_N(\lambda_i)(\mu - \lambda_i)}\right\}. \end{aligned}$$

Denoting the first Poisson bracket by t_1 we have using the expression for $C_N(\lambda)$ as given in (114)

$$\begin{aligned} t_1 &= \left\{e^{-v_N}\prod_{i=1}^{N-1}(\lambda - \lambda_i), e^{-v_N}\prod_{j=1}^{N-1}(\mu - \lambda_j)e^{v_N}(\mu + S)\right\} \\ &= \prod_{i=1}^{N-1}(\lambda - \lambda_i)\left\{e^{-v_N}, \sum_{k=1}^{N}(p_k - c_k)\right\} e^{-v_N}\prod_{j=1}^{N-1}(\mu - \lambda_j)e^{v_N} \\ &= -e^{-v_N}\prod_{i=1}^{N-1}(\lambda - \lambda_i).\{v_N, u_N\}.e^{-v_N}\prod_{j=1}^{N-1}(\mu - \lambda_j)e^{v_N} \\ &= e^{-v_N}\prod_{i=1}^{N-1}(\lambda - \lambda_i).e^{-v_N}\prod_{j=1}^{N-1}(\mu - \lambda_j).e^{v_N} \\ &= C_N(\lambda)C_N(\mu)e^{v_N}. \end{aligned} \quad (116)$$

In arriving at this relation we have made use of the fact that from (111), $S = \sum_{j=1}^{N}(p_j - c_j) + \sum_{i=1}^{N-1}\lambda_i$ while $\lambda_N = \sum_{j=1}^{N}(p_j - c_j)$, together with the Poisson bracket $\{\lambda_N, \mu_N\} = 1$ which follows from (106).

Next consider the term

$$t_2 = \left\{C_N(\lambda), C_N(\mu)\sum_{i=1}^{N-1}\frac{A_N(\lambda_i)}{C'_N(\lambda_i)(\mu - \lambda_i)}\right\}. \quad (117)$$

Note that from (104) we have setting $\lambda = \lambda_i$

$$A_N(\lambda_i)e^{-v_i} = \det(T_N(\lambda_i)) = (-1)^N\prod_{j=1}^{N}(\lambda_j - c_j), \quad (118)$$

which implies

$$A_N(\lambda_i) = (-1)^N e^{v_i}\prod_{k=1}^{N}(\lambda_k - c_k). \quad (119)$$

Using (114) and (119) the right hand side may be written as

$$t_2 = e^{-2v_N} \prod_{i=1}^{k-1} \sum_{k=1}^{N-1} \left\{ \lambda - \lambda_k, (-1)^N e^{v_k} \frac{\prod_{i=1}^{N}(\lambda_k - c_i)}{C_N'(\lambda_k)(\mu - \lambda_k)} \right\} \prod_{i=1}^{N-1} (\mu - \lambda_i)$$
$$= C_N(\lambda) C_N(\mu) \sum_{k=1}^{N-1} \{v_k, \lambda_k\} \frac{A_N(\lambda_k)}{C_N'(\lambda_k)(\lambda - \lambda_k)(\mu - \lambda_k)}$$
$$= \frac{C_N(\lambda) C_N(\mu)}{(\mu - \lambda)} \sum_{k=1}^{N-1} (-1) \left(\frac{A_N(\lambda_k)}{C_N'(\lambda_k)(\lambda - \lambda_k)} - \frac{A_N(\lambda_k)}{C_N'(\lambda_k)(\mu - \lambda_k)} \right), \quad (120)$$

where it has been assumed that $\{v_k, \lambda_j\} = -\delta_{kj}$. Using now the expression for $A_N(\lambda)$ as given by (112) we find that t_2 may be simplified to yield

$$t_2 = \frac{1}{\lambda - \mu} \left(A_N(\lambda) C_N(\mu) - A_N(\mu) C_N(\lambda) \right) - e^{v_N} C_N(\lambda) C_N(\mu). \quad (121)$$

Hence finally we arrive at (115), namely

$$\{C_N(\lambda), A_N(\mu)\} = t_1 + t_2 = \frac{1}{\lambda - \mu} (A_N(\lambda) C_N(\mu) - A_N(\mu) C_N(\lambda)).$$

Thus we have at our disposal two sets of N pairs of canonical variables, namely (p_i, q_i) with $i = 1, \dots, N$ and a second set defined by the zeros of $C_N(\lambda)$, viz λ_i and v_i and hence μ_i defined by $e^{-v_i} = D_N(\lambda_i)$, $i = 1, \dots, N$, together with $\mu_N = q_n$ and $\lambda_N = \sum_{n=1}^{N} (p_n - c_n)$ both of which reproduce the r matrix algebra. In the second case the variables have the Poisson brackets $\{\lambda_N, \mu_N\} = 1$ and $\{\lambda_k, v_k\} = 1$, with $k = 1, \dots, N-1$.

7. Conclusion

In this article we have focussed on a class of semi-discrete integrable systems related to the relativistic Toda hierarchy and have explicitly obtained a canonical Bäcklund transformation (BT) for the first flow of the relativistic Toda hierarchy. We have also investigated the property of spectrality for this particular flow, since it is closely related to the issue of its separability. We have therefore derived the separation variables explicitly. In addition we have studied a variant of the standard Toda lattice and have derived a set of BT's for it. This model is characterized by a different $r-$matrix algebra and it would be interesting to analyze the problem of deriving the corresponding separation representation for it.

Acknowledgement

AGC would like to thank Professor Y B Suris for drawing his attention to the systems described here and wishes to thank the referees for their valuable comments.

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