Cent. Eur. J. Phys. • 9(5) • 2011 • 1144-1150
DOI: 10.2478/s11534-011-0043-2

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Central European Journal of **Physics**

On the feasibility of charged wormholes

Research Article

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Received 16 December 2010; accepted 20 April 2011

Abstract: While wormhole spacetimes are predictions of the general theory of relativity, specific solutions may not be compatible with quantum field theory. This paper modifies the charged wormhole model of Kim and Lee with the aim of satisfying an extended version of a quantum inequality due to Ford and Roman. The modified metric may be viewed as a solution of the Einstein fields equations representing a charged wormhole that is compatible with quantum field theory.

PACS (2008): 04.20.Jb, 04.20.Gz

Keywords: charged wormholes • quantum inequalities

1. Introduction

Wormholes are handles or tunnels in the geometry of spacetime connecting two distinct regions of our Universe or of completely different universes. The pioneer work of Morris and Thorne [1] has shown that, being solutions of the Einstein field equations, macroscopic wormholes may be actual physical objects that could even be traversed by humanoid travelers. Unlike black holes, which are also predictions of Einstein's theory, wormholes can only be held open by the use of "exotic" matter; such matter violates the weak energy condition.

Because of the close connection between space and time, general relativity is able to tolerate science-fiction type phenomena such as wormholes and even time travel, as exemplified by the Gödel solution. Quantum field theory, on the other hand, is not so forgiving: it places severe restrictions on the existence of traversable wormholes [2–5]. In fact, according to Ford and Roman [4, 5], the wormholes discussed in Ref. [1] could not exist on a macroscopic scale. Interesting exceptions are the wormholes discussed in Refs. [6] and [7], but they are subject to extreme fine-tuning. This fine-tuning became an issue in seeking compatibility with quantum field theory by a suitable extension of the quantum inequalities [8, 9]. Given that exotic matter is rather problematical, the idea behind the extension was to strike a balance between reducing the size of the exotic region and the concomitant fine-tuning of the metric coefficients. One can only be accomplished at the expense of the other.

A particularly interesting generalization of the Morris-Thorne wormhole can be obtained by the addition of an electric charge, as proposed by Kim and Lee [10, 11]. The resulting spacetime is a combination of a Morris-Thorne spherically symmetric static wormhole and a Reissner-Nördstrom spacetime.

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As in the case of black holes, wormholes with an electric charge have been of interest for some time. For example, by adding an electric charge, Gonzales, Guzman, and Sarbach [12] studied the possibility of stabilizing a wormhole supported by a ghost scalar field, discussed in their earlier papers [13, 14].

Rotating and magnetized wormholes supported by phantom scalar fields are discussed in Ref. [15]. (A ghost scalar field is often considered a simple example of phantom energy, which is itself of interest in a wormhole setting since it leads to a violation of the weak energy condition.)

The aim of this paper is to show that a relatively small modification of the metric describing a charged wormhole suffices to satisfy an extended version of the Ford-Roman inequality, thereby making such a wormhole compatible with quantum field theory. The modified model is also a solution of the Einstein field equations.

2. Traversable wormholes

The spacetime geometry of a traversable wormhole can be described by the metric

$$ds^2 = -e^{2\beta(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (1)$$

where $\beta(r) \to 0$ and $\alpha(r) \to 0$ as $r \to \infty$ and where $\alpha(r)$ has a continuous derivative. (We are using units in which $c = G = 1$.) The function $\beta = \beta(r)$ is called the *redshift function*, which must be everywhere finite to prevent an event horizon. The function $\alpha = \alpha(r)$ is related to the *shape function* $b = b(r)$:

$$e^{2\alpha(r)} = \frac{1}{1 - \frac{b(r)}{r}}.$$

So $b(r) = r(1 - e^{-2\alpha(r)})$. (Observe that $b'(r)$ is continuous and that $\frac{b(r)}{r} \to 0$ as $r \to \infty$.) The minimum radius $r = r_0$ is called the *throat* of the wormhole, where $b(r_0) = r_0$. Also, $b'(r_0) \leqslant 1$, referred to as the *flare-out condition* in Ref. [1]. It follows that α has a vertical asymptote at $r = r_0$:

$$\lim_{r \to r_0+} \alpha(r) = +\infty.$$

To hold a wormhole open, the weak energy condition (WEC) must be violated. The WEC states that the stress-energy tensor $T_{\alpha\beta}$ must obey

$$T_{\alpha\beta}\mu^\alpha\mu^\beta \geqslant 0$$

for all time-like vectors and, by continuity, all null vectors.

3. The quantum inequalities

To make this paper reasonably self-contained, we need a brief discussion of the quantum inequalities due to Ford and Roman [5], slightly extended in [8, 9].

In a series of papers, Ford and Roman (see Ref. [5] and references therein) discuss a type of constraint on the violation of the weak energy condition by means of certain quantum inequalities which limit the magnitude and time duration of negative energy. These inequalities place severe restrictions on the dimensions of Morris-Thorne wormholes.

One of these quantum inequalities, applied to different situations, deals with an inertial Minkowski spacetime without boundaries. If u^μ is the observer's four-velocity, that is, the tangent vector to a timelike geodesic, then $\langle T_{\mu\nu}u^\mu u^\nu \rangle$ is the expectation value of the local energy density in the observer's frame of reference. It is shown in Ref. [5] that

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geqslant -\frac{3}{32\pi^2\tau_0^4}. \quad (2)$$

Here τ is the observer's proper time and τ_0 the duration of the sampling time. More precisely, the energy density is sampled in a time interval of duration τ_0 which is centered around an arbitrary point on the observer's worldline so chosen that $\tau = 0$ at this point. (See Ref. [5] for details.)

In a wormhole setting, a more convenient form is inequality (7) below, as we will see. Applied to spherically symmetric traversable wormholes in Ref. [1], it was found that none were able to meet this condition. As a result, the throat sizes could only be slightly larger than Planck length. The inequality was subsequently extended in Refs. [8, 9] to cover an entire region around the throat. It was then shown that it is possible to strike a balance between the size of the exotic region and the amount of fine-tuning required to achieve this reduction.

Before discussing the extended quantum inequality, we need to introduce the following length scales, modeled after the length scales in Ref. [5], which were introduced in Ref. [8]:

$$r_m \equiv \min\left[r, \left|\frac{b(r)}{b'(r)}\right|, \frac{1}{|\beta'(r)|}, \left|\frac{\beta'(r)}{\beta''(r)}\right|\right]. \quad (3)$$

It is shown that if $R_{\max}$ is the magnitude of the maximum curvature, then

$$R_{\max} \leqslant \frac{1}{r_m^2}.$$

So the smallest radius of curvature r_c is

$$r_c \approx \frac{1}{\sqrt{R_{\max}}} \geqslant r_m.$$

Working on this scale, the spacetime is approximately Minkowskian, so that inequality (2) can be applied with an appropriate τ_0.
According to Ref. [5], whenever the density ρ is positive or zero, one should Lorentz transform to a frame of a radially moving geodesic observer moving with velocity v relative to the static frame. In this "boosted frame" the density and maximum curvature are denoted by ρ' and R'_{max}, respectively. We now have

$$r'_c \approx \frac{1}{\sqrt{R'_{\text{max}}}} \geqslant \frac{r_m}{\gamma},$$

where $\gamma = (1-v^2)^{-\frac{1}{2}}$. The suggested sampling time is $\tau_0 = \frac{fr_m}{\gamma}$, where f is a scale factor such that $f \ll 1$. It is shown in Ref. [8] that in this boosted frame,

$$T_{\hat{0}'\hat{0}'} = \rho' = \frac{\gamma^2}{8\pi r^2}\left[b'(r) - v^2\frac{b(r)}{r} + 2v^2 r\beta'(r)\left(1-\frac{b(r)}{r}\right)\right]. \tag{4}$$

In order for ρ' to be negative, v has to be sufficiently large:

$$v^2 > \frac{b'(r)}{\frac{b(r)}{r} - 2r\beta'(r)\left(1-\frac{b(r)}{r}\right)}. \tag{5}$$

Furthermore,

$$\frac{r_m}{r} \leqslant \left(\frac{1}{\frac{v^2 b(r)}{r} - b'(r) - 2v^2 r\beta'(r)\left(1-\frac{b(r)}{r}\right)}\right)^{\frac{1}{4}} \times \frac{\sqrt{\gamma}}{f}\left(\frac{l_p}{r}\right)^{\frac{1}{2}}. \tag{6}$$

At the throat, where $b(r_0) = r_0$, inequality (6) reduces to Eq. (95) in Ref. [5]:

$$\frac{r_m}{r_0} \leqslant \left(\frac{1}{v^2 - b'(r_0)}\right)^{\frac{1}{4}} \frac{\sqrt{\gamma}}{f}\left(\frac{l_p}{r_0}\right)^{\frac{1}{2}}. \tag{7}$$

Observe that inequality (7) is trivially satisfied if $b'(r_0) = 1$, but not necessarily if $b'(r_0) < 1$. More formally, inequality (7) is satisfied whenever $b'(r_0) - \epsilon < 1$ for ϵ sufficiently small. Since r_m includes r_0, the wormhole can be macroscopic. Inequality (6) will be applied in Sec. 6.

Remark 3.1.
To avoid division by zero in Eq. (7), we actually assume that $b'(r_0)$ is extremely close to 1 instead of exactly 1. This also guarantees that the flare-out condition is met at the throat [1]:

$$\frac{b(r_0) - rb'(r_0)}{2[b(r_0)]^2} > 0.$$

Although retained here for convenience, the v^2 in Eq. (6) could actually be omitted [8]. The reason is that, according to Ref. [2], the boosted frame may be replaced by a static observer.

4. The charged wormhole of Kim and Lee

To study a wormhole with a constant electric charge Q, it was proposed by Kim and Lee [10] that the Einstein field equations take on the form

$$G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} = 8\pi\left[T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}\right].$$

In other words, the usual wormhole spacetime $G^{(0)}_{\mu\nu} = 8\pi T^{(0)}_{\mu\nu}$ is to be modified by adding the matter term $T^{(1)}_{\mu\nu}$ to the right side and the corresponding back reaction $G^{(1)}_{\mu\nu}$ to the left side. The proposed metric is

$$ds^2 = -\left(1+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{b(r)}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{8}$$

Comparing metrics (1) and (8), since $e^{2\beta(r)} = 1 + \frac{Q^2}{r^2}$, we have

$$\beta(r) = \frac{1}{2}\ln\left(1+\frac{Q^2}{r^2}\right)$$

and since

$$e^{2\alpha(r)} = \frac{1}{1-\frac{b(r)}{r}}$$

in (1), it follows that the effective shape function b_{eff} is

$$b_{\text{eff}}(r) = b(r) - \frac{Q^2}{r}. \tag{9}$$

Given the conversion factor $\frac{c^2}{\sqrt{G}}$, Q^2 is likely to be small in geometrized units. Accordingly, we will assume that $b(r) - \frac{Q^2}{r} > 0$ in the vicinity of the throat.

5. The modified charged wormhole

For reasons discussed later in this section, we are going to propose the following modified metric for a charged wormhole:

$$ds^2 = -\left(1 + R(r) + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{10}$$

It is assumed that $R(r) \geqslant 0$, thereby avoiding an event horizon, and that $R(r)$ has a continuous derivative. As in metric (8), $b_{\text{eff}}(r) = b(r) - \frac{Q^2}{r}$. Observe that the effective redshift function is

$$\beta(r) = \frac{1}{2}\ln\left(1 + R(r) + \frac{Q^2}{r^2}\right).$$

In the discussion below, ρ is the density, τ the radial tension, and p the transverse pressure. Following Kim and Lee [10], we assume that the matter terms are

$$\rho^{(1)} = \tau^{(1)} = p^{(1)} = \frac{Q^2}{8\pi r^4}. \tag{11}$$

The components of the Einstein tensor in the orthonormal frame are

$$8\pi(\rho^{(0)} + \rho^{(1)}) = \frac{b'}{r^2} + \frac{Q^2}{r^4}, \tag{12}$$

$$8\pi\left(\tau^{(0)} + \tau^{(1)}\right) = \frac{b}{r^3} - \frac{Q^2}{r^4} - \frac{1 - \frac{b}{r} + \frac{Q^2}{r^2}}{r\left(1 + R(r) + \frac{Q^2}{r^2}\right)}\left[R'(r) - \frac{2Q^2}{r^3}\right], \tag{13}$$

and

$$8\pi\left(p^{(0)} + p^{(1)}\right) = \left(1 - \frac{b}{r} + \frac{Q^2}{r^2}\right) \times \left[\beta''(r) - \frac{rb' - b + \frac{2Q^2}{r}}{2r\left(r - b + \frac{Q^2}{r}\right)}\beta'(r) + [\beta'(r)]^2 + \frac{\beta'(r)}{r} - \frac{rb' - b + \frac{2Q^2}{r}}{2r^2\left(r - b + \frac{Q^2}{r}\right)}\right]. \tag{14}$$

According to Ref. [10], employing the effective shape function $b_{\text{eff}}(r) = b(r) - \frac{Q^2}{r}$ and assuming $T^{\text{eff}}_{\mu\nu} = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}$ (total matter) in the original Kim-Lee model, yields a self-consistent solution of a system of equations similar to that of the scalar field case discussed earlier in Ref. [10]. The inclusion of the smooth function $R(r)$ does not alter this conclusion. So the metric (10) may be viewed as a solution of the Einstein field equations representing a wormhole with an electric charge.

Returning to the WEC, if we use the radial outgoing null vector $\mu^{\hat{\alpha}} = (1, 1, 0, 0)$, then $T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = \rho - \tau \geqslant 0$. For the above components, since $\rho^{(1)}$ and $\tau^{(1)}$ drop out, we have $\rho^{(0)} - \tau^{(0)} < 0$ whenever the condition is violated. (We will examine this violation shortly.)

Next, let us assume that $b = b(r)$ is a typical shape function in the sense of Morris and Thorne [1]: if the charge Q is zero, then the wormhole has a throat at $r = r*$, where $b(r*) = r*$. For $r > r*$, we must have $b(r) < r$. It follows that $b'(r*) \leqslant 1$. (See Fig. 1.) The new shape function $b_{\text{eff}}(r)$ [Eq. (9)] has analogous properties: in particular, there is a throat at $r = r_0$, that is,

$$1 - \frac{b(r_0)}{r_0} + \frac{Q^2}{r_0^2} = 0. \tag{15}$$

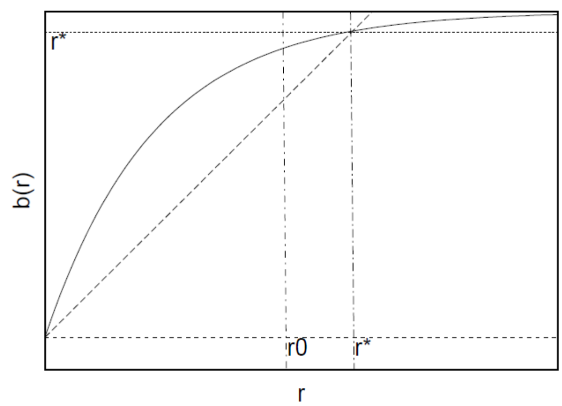

Figure 1. The location of the throat $r = r_0$.

Remark 5.1.

Eq. (15) actually has two roots,

$$r_0 = \frac{1}{2}\left(b(r_0) \pm \sqrt{[b(r_0)]^2 - 4Q^2}\right).$$

In the special case $Q = 0$, we get two possibilities, $r_0 = 0$ and $r_0 = b(r_0)$, showing that the smaller root is meaningless. We will therefore assume that there is only one throat, corresponding to the larger root.

For $r > r_0$, $1 - \frac{b(r)}{r} + \frac{Q^2}{r^2} > 0$, or $\frac{b(r)}{r} - \frac{Q^2}{r^2} < 1$. Hence

$$\frac{b_{\text{eff}}(r)}{r} < 1$$

for $r > r_0$. So we have, once again, $b'_{\text{eff}}(r_0) \leqslant 1$. Finally, the profile curve $z = z(r)$ is such that

$$\frac{dz}{dr} = \pm\sqrt{\frac{\frac{b(r)}{r} - \frac{Q^2}{r^2}}{1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}}}, \tag{16}$$

showing that there is a vertical tangent at the throat $r = r_0$ in the usual embedding diagram. As noted at the end of Sec. 4, the numerator is greater than zero.

Returning to $b = b(r)$, since $b(r) < r$ for $r > r*$, $b(r) > r$ for $r < r*$, and $b(r_0) - r_0 = \frac{Q^2}{r_0} > 0$ by Eq. (15), it follows that $r_0 < r*$. (See Fig. 1. If $b = b(r)$ is indeed a typical shape function, then $b(r) > r$ for $r < r*$.) Since we are using geometrized units, Q^2 is likely to be small, so that r_0 is not much less than $r*$.

The need for $b'_{\text{eff}}(r_0) \leqslant 1$ referred to above can be seen from the exoticity function in Ref. [1]:

$$\frac{b_{\text{eff}}(r_0) - r_0 b'_{\text{eff}}(r_0)}{2[b_{\text{eff}}(r_0)]^2} > 0. \tag{17}$$

In other words, the flare-out condition is met whenever

$$\frac{b(r_0)}{r_0} - b'(r_0) > \frac{2Q^2}{r_0^2} \tag{18}$$

at the throat. This is consistent with the violation of the WEC, $\rho^{(0)} - \tau^{(0)} < 0$:

$$8\pi\left(\rho^{(0)} - \tau^{(0)}\right) = \frac{b'}{r^2} - \frac{b}{r^3} + \frac{2Q^2}{r^4} + \frac{1 - \frac{b}{r} + \frac{Q^2}{r^2}}{r\left(1 + R(r) + \frac{Q^2}{r^2}\right)}\left[R'(r) - \frac{2Q^2}{r^3}\right]. \tag{19}$$

At the throat,

$$\frac{r_0 b'(r_0) - b(r_0) + \frac{2Q^2}{r_0}}{r_0^3} + 0 < 0 \tag{20}$$

by inequality (18).

6. Feasibility

In order to study the feasibility of the charged wormhole, let us denote the redshift function in the Kim and Lee wormhole by $\beta_1(r)$. Thus

$$\beta_1(r) = \frac{1}{2}\ln\left(1 + \frac{Q^2}{r^2}\right). \tag{21}$$

Recall next that inequality (7) is trivially satisfied if $b' = 1$ at the throat. For either of the charged wormholes, if Q is chosen properly, the condition can be met: from Eq. (9),

$$b'_{\text{eff}}(r_0) = b'(r_0) + \frac{Q^2}{r_0^2};$$

so for a proper choice of Q, $b'_{\text{eff}}(r_0) = 1$, even if $b'(r*)$ is less than 1.

To study the problem more closely, let us restate inequalities (5) and (6) for b_{eff}:

$$v^2 > \frac{b'_{\text{eff}}(r)}{\frac{b_{\text{eff}}(r)}{r} - 2r\beta'(r)\left(1 - \frac{b_{\text{eff}}(r)}{r}\right)} \tag{22}$$

and

$$\frac{r_m}{r} \leqslant \left(\frac{1}{\frac{v^2 b_{\text{eff}}(r)}{r} - b'_{\text{eff}}(r) - 2v^2 r\beta'(r)\left(1 - \frac{b_{\text{eff}}(r)}{r}\right)}\right)^{\frac{1}{4}} \times \frac{\sqrt{\gamma}}{f}\left(\frac{l_p}{r}\right)^{\frac{1}{2}}. \tag{23}$$

As before, at the throat, inequality (23) is trivially satisfied.

Problems arise when we move away from the throat. It is shown in Ref. [8] that for any of the typical shape functions (which would include b_{eff}), $\frac{b(r)}{r} - b'(r) > 0$. So for the wormhole in Sec. 4, the denominator on the right side of inequality (23) is no longer 0, since

$$\beta_1'(r) = -\frac{Q^2}{r\left(r^2 + Q^2\right)}$$

is negative. Furthermore, with Q fixed, $\beta(r)$ cannot be altered. It is easy to demonstrate using specific shape functions that the quantum inequalities cannot be met away from the throat.

To salvage the charged wormhole, some modification is evidently needed. With the Reissner-Nördstrom metric in mind, Eq. (10) appears to be a natural generalization, as long as $R(r)$ is not equal to $-\frac{b(r)}{r}$. To distinguish this case from Eq. (21), let us denote the modified redshift function by $\beta_2(r)$:

$$\beta_2(r) = \frac{1}{2}\ln\left(1 + R(r) + \frac{Q^2}{r^2}\right). \tag{24}$$

The situation regarding the quantum inequalities is now quite different:

$$\beta_2'(r) = \frac{1}{2}\frac{1}{1 + R(r) + \frac{Q^2}{r^2}}\left[R'(r) - \frac{2Q^2}{r^3}\right], \tag{25}$$

which is positive for a proper choice of $R(r)$. Also, $\beta_2'(r)$ is continuous if, and only if, $R'(r)$ is continuous. The question now is whether $\beta_2(r)$ can be constructed or adjusted to meet inequality (23) away from the throat. This amounts to asking whether for a proper choice of $R(r)$,

$$\frac{b_{\text{eff}}(r)}{r} - b_{\text{eff}}'(r) - 2r\beta_2'(r)\left(1 - \frac{b_{\text{eff}}(r)}{r}\right) \tag{26}$$

is 0 or close to 0. If such an adjustment can be made, then $v = 1$ or close to 1, according to inequality (22). Consequently, inequality (23) is once again trivially satisfied. To fix ideas, suppose that for $v = 1$ or close to 1,

$$\frac{v^2 b_{\text{eff}}(r)}{r} - b_{\text{eff}}'(r) - 2rv^2\beta_2'(r)\left(1 - \frac{b_{\text{eff}}(r)}{r}\right)$$

indeed 0 or nearly 0. Then we must find a function $R = R(r)$ such that $\beta_2'(r)$ is nearly equal to

$$\frac{\frac{v^2 b_{\text{eff}}(r)}{r} - b_{\text{eff}}'(r)}{2rv^2\left(1 - \frac{b_{\text{eff}}(r)}{r}\right)},$$

to be denoted by $\beta'(r)$, which is continuous for $r > r_0$. Substituting in Eq. (25), we get after rearranging,

$$R'(r) - 2\beta'(r)R(r) = 2\beta'(r) + 2\beta'(r)\frac{Q^2}{r^2} + \frac{2Q^2}{r^3}, \tag{27}$$

where (nearly)

$$\beta'(r) = \frac{\frac{v^2 b_{\text{eff}}(r)}{r} - b_{\text{eff}}'(r)}{2rv^2\left(1 - \frac{b_{\text{eff}}(r)}{r}\right)}. \tag{28}$$

The solution of the differential equation is

$$R(r) = e^{2\beta(r)}\int_{r_0}^{r} e^{-2\beta(r')}\left(2\beta'(r') + 2\beta'(r')\frac{Q^2}{(r')^2} + \frac{2Q^2}{(r')^3}\right)dr'. \tag{29}$$

The continuity of $\beta'(r)$ is sufficient to guarantee that $R(r)$ is a solution. Using integration by parts, the solution can be written

$$R(r) = -1 - \frac{Q^2}{r^2} + e^{2\beta(r)}e^{-2\beta(r_0)}\left(1 + \frac{Q^2}{r_0^2}\right),$$

showing that $R(r_0) = 0$. So it is in principle possible to determine $R(r)$ such that inequality (23) is satisfied. The resulting wormhole is therefore compatible with quantum field theory.

Finally, it follows that

$$\beta_2(r) = \frac{1}{2}\ln\left[e^{2\beta(r)}e^{-2\beta(r_0)}\left(1 + \frac{Q^2}{r_0^2}\right)\right].$$

Particularly noteworthy is that $\beta_2'(r) = \beta'(r)$, which can be easily computed from the shape function and helps determine the tidal constraints when discussing traversability.

7. Assigning various parameters-traversability

There are several parameters that come into play when describing the wormhole geometry. In particular, $b_{\text{eff}}(r_0) = r_0$ and $b_{\text{eff}}'(r_0) = 1$ lead to

$$b(r_0) - \frac{Q^2}{r_0} = r_0 \tag{30}$$

and

$$b'(r_0) + \frac{Q^2}{r_0^2} = 1. \tag{31}$$

If $b(r)$ and Q are known, we can determine r_0. It is also possible to fix r_0 at some desired (macroscopic) value and determine $b(r)$ and Q. As a simple example, suppose $b(r)$ has the form $b(r) = Ar^B$, $B < 1$. Then from Eqs. (30) and (31), we find that for nonzero Q,

$$A = \frac{2}{1+B}r_0^{1-B}$$

and

$$Q^2 = \frac{1-B}{1+B}r_0^2.$$

B can be so chosen that Q^2 is relatively small, as desired in our geometrized units. This also confirms our earlier assertion that r_0 cannot be much smaller than $r*$ without making Q^2 unrealistically large.

As a check on the traversability by humanoid travelers (as in Ref. [1]), consider the proper distance $\ell(r)$ from the throat to a point away from the throat:

$$\ell(r) = \int_{r_0}^{r} \frac{dr'}{\sqrt{1 - \frac{b'_{\text{eff}}(r')}{r'}}}.$$

For $b(r) = Ar^B$ and $Q^2 > 0$, this distance is finite. For example, if $r_0 = 5$ m, and $Q^2 = 0.1$, then $\ell(6) \approx 140$ m. However, if $Q \to 0$, then $\ell(r) \to \infty$ for this particular shape function, so that the wormhole would not be traversable.

According to Ref. [1], the space station has to be far enough away from the throat so that $|\beta_2'(r)| \leqslant (10^8 \text{ m})^{-2}$. As noted at the end of the previous section, the condition can be readily determined from $\beta'(r)$. The simple shape function $b(r) = Ar^B$ being considered flares out too slowly, however, to satisfy this condition, unless r is excessively large. As in Ref. [8], some distance away from the throat one must therefore join $b = b(r)$ smoothly to a function that causes this constraint to be met.

Remark 7.1.
Since $R(r)$ depends only on Q and $b(r)$, modifying $b(r)$ away from the throat does not affect the conclusion. However, as in most wormhole models, we are primarily interested in the vicinity of the throat.

8. Conclusion

In this paper the charged wormhole described by the metric

$$ds^2 = -\left(1 + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

due to Kim and Lee is extended to

$$ds^2 = -\left(1 + R(r) + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$

where $R(r) \geqslant 0$ has a continuous derivative at the throat, and $R(r_0) = 0$. The main objective was to show that $R(r)$ can be so chosen that the quantum inequality (23) is satisfied. The extended model is therefore compatible with quantum field theory.

It is also shown that the flare-out condition has been satisfied and that an event horizon has been avoided. Various combinations of $b(r)$, Q, and $r = r_0$ are possible and may be chosen to make the wormhole traversable by humanoid travelers; in particular, the tidal constraints can be computed from the shape function. The particular model discussed shows that Q may have to be nonzero for the wormhole to be traversable.

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