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Super fidelity and related metrics

Research Article

Zhi-Hua Chen[1], Zhihao Ma[2*], Fu-Lin Zhang[3], Jing-Ling Chen[4†]

1 *Department of Science, Zhijiang college, Zhejiang University of Technology, Hangzhou, 310024, P.R. China,*

2 *Department of Mathematics, Shanghai Jiaotong University, Shanghai, 200240, P.R. China*

3 *Department of Physics,Tianjin University, Tianjin, 300072, P.R. China*

4 *Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin, 300071, P.R. China*

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Abstract: We report a new metric of quantum states. This metric is built up from super-fidelity, which has a deep connection with the Uhlmann-Jozsa fidelity and plays an important role in quantum information processing. We find that the new metric possesses some interesting properties.

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1. Introduction

In quantum information theory distinguishing two quantum states is a fundamental task. One of the main tools used in distinguishability theory is a trace metric; another closely related tool is quantum fidelity [1–4]. Both are widely used by the quantum information science community and have found applications in a number of problems such as quantifying entanglement [5, 6], quantum error correction [7], quantum chaos [8], and quantum phase transitions [9, 10].

*E-mail: ma9452316@gmail.com (Corresponding author)
†E-mail: chenjl@nankai.edu.cn

Suppose ρ and σ are two quantum states, then the Uhlmann-Jozsa fidelity [1–4] between ρ and σ is given by

$$F(\rho,\sigma) = \left[\mathrm{Tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}\right]^2. \tag{1}$$

We know that for the case of qubits, the Uhlmann-Jozsa fidelity has a simple form. From the Bloch sphere representation of quantum states, a qubit is described by a density matrix as:

$$\rho(\mathbf{u}) = \frac{1}{2}(\mathbf{I} + \sigma\cdot\mathbf{u}), \tag{2}$$

where $\mathbf{I}$ is the 2×2 unit matrix and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Assume $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ are two states of one qubit, then they can be represented by two vectors

u and **v** in the Bloch sphere. The Uhlmann-Jozsa fidelity for qubits has an elegant form:

$$F(\rho(\mathbf{u}), \rho(\mathbf{v})) = \frac{1}{2}\left[1 + \mathbf{u}\cdot\mathbf{v} + \sqrt{1-|\mathbf{u}|^2}\sqrt{1-|\mathbf{v}|^2}\right], \quad (3)$$

where $\mathbf{u}\cdot\mathbf{v}$ is the inner product of **u** and **v**, and $|\mathbf{u}|$ is the magnitude of **u**.

We know that for general quantum states, unlike the case for qubits, the Uhlmann-Jozsa fidelity has no simple form. To use the simple form of fidelity, we note that in [11], the authors introduced a new fidelity, called super-fidelity, defined as

$$G(\rho_1, \rho_2) := \mathrm{Tr}\rho_1\rho_2 + \sqrt{(1-\mathrm{Tr}\rho_1^2)(1-\mathrm{Tr}\rho_2^2)}. \quad (4)$$

This quantity is called super-fidelity, since it is always larger than, or equal to, the Uhlmann-Jozsa fidelity. It was proved that when ρ_1 and ρ_2 are two qubits, super-fidelity $G(\rho_1, \rho_2)$ coincides with Uhlmann-Jozsa fidelity $F(\rho_1, \rho_2)$. The super-fidelity $G(\rho_1, \rho_2)$ has some appealing properties [11–13]. Let

$$\rho_u = \frac{1}{N}\left(I + \sqrt{\frac{N(N-1)}{2}}\vec{\lambda}.\mathbf{u}\right)$$

be the density matrix of a qunit ($N \times N$ quantum state), where I is the $N\times N$ unit matrix, $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots \lambda_{N^2-1})$ are the generators of $SU(N)$, and **u** is the (N^2-1)-dimensional Bloch vector. Then super-fidelity can be rewritten as

$$G(\rho_u, \rho_v) = \frac{1}{N}\Big[1 + (N-1)\times \mathbf{u}.\mathbf{v} + (\mathbf{N}-\mathbf{1}) \times \sqrt{(1-|\mathbf{u}|^2)(1-|\mathbf{v}|^2)}\Big].$$

This shows that super-fidelity only depends on the magnitudes of $\mathbf{u}, \mathbf{v}$ and the angle between them (that is, $\mathbf{u}.\mathbf{v}$). This property makes super-fidelity easy to calculate, and has a clear geometrical interpretation.

Moreover, very recently, it was found that super-fidelity plays an important role in quantifying entanglement [14]. So it is natural to study the property of super-fidelity in further step.

Recall that super-fidelity by itself is not a metric. It is a measure of the "closeness" of two states. If we say a function $d(x, y)$ defined on the set of quantum states is a metric, it should satisfy the following four axioms:

Axiom 1.1.
$d(x, y) \geqslant 0$ for all states x and y;

Axiom 1.2.
$d(x, y) = 0$ if and only if $x = y$;

Axiom 1.3.
$d(x, y) = d(y, x)$ for all states x and y;

Axiom 1.4.
The triangle inequality: $d(x, y) \leqslant d(x, z) + d(y, z)$ for all states x, y and z.

For super-fidelity, the following three functions were introduced in [13]:

$$A(\rho, \sigma) := \arccos\sqrt{G(\rho, \sigma)}, \quad (5)$$

$$B(\rho, \sigma) := \sqrt{2 - 2\sqrt{G(\rho, \sigma)}}, \quad (6)$$

$$C(\rho, \sigma) := \sqrt{1 - G(\rho, \sigma)}. \quad (7)$$

It was proved in [13] that $C(\rho, \sigma)$ is a genuine metric, that is, it satisfies the axioms 1.1-1.4, while $A(\rho, \sigma)$ and $B(\rho, \sigma)$ do not preserve the metric properties.

The purpose of this paper is to introduce a novel method to define a metric of quantum states based on super-fidelity. Surprisingly, we find the metric induced by the new method coincides with the metric introduced in [13] for the qubits case, and the new metrics have deep connections with the spectral metric. Also we find that the new metrics possess some appealing properties which make these metrics very useful in quantum information theory. The paper is organized as follows: In Sec. 2, two new metrics are defined, and the metric character of the metrics is established. In Sec. 3, intrinsic properties of the two metrics are discussed. Conclusions and discussion are presented in the last section.

2. Metric induced by super-fidelity

The most widely used metric is the trace metric, which is defined as

$$D_{tr}(\rho, \sigma) = \frac{1}{2}\mathrm{Tr}|\rho - \sigma|. \quad (8)$$

Alternatively, one can define other types of distance measures for quantum states, and these also have their own advantages, see [1, 11, 13–20].

Let us define a new metric of states as follows:

$$D_G(\rho, \sigma) = \max_{\tau} |G(\rho, \tau) - G(\sigma, \tau)|, \quad (9)$$

where the maximization is obtained by taking all quantum states, τ, (mixed or pure). We will call $D_G(\rho, \sigma)$

the *G-metric*, and the state τ which attained the maximal will be called the *optimal state* for the metric $D_G(\rho, \sigma)$.
The above definition of this metric may be not easy to calculate, so we can change its definition slightly. If τ is a pure state, then super-fidelity can be simplified as $G(\rho, \tau) = \mathrm{Tr}(\rho\tau)$, hence one can define another version of the metric as follows:

$$D_{PG}(\rho, \sigma) = \max_{\tau} |G(\rho, \tau) - G(\sigma, \tau)|, \tag{10}$$

where the maximization is found by taking all pure states of τ. We will call this metric ($D_{PG}(\rho, \sigma)$) the *PG-metric*, and the pure state τ that attained the maximal value will be called the *optimal pure state*.
First we consider the case of qubits.

Proposition 2.1 ([20]).

For the qubit case, $D_{PG}(\rho, \sigma)$ is equal to the trace metric, namely $D_{PG}(\rho, \sigma) = D_{tr}(\rho, \sigma) = \frac{1}{2}\mathrm{Tr}|\rho - \sigma|$.

We can connect our metric with the metric introduced in [13] as follows:

Proposition 2.2 ([20]).

For the qubit case, $D_G(\rho, \sigma) = C(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$.

Now we come to discuss the case of qunits (i.e., $N \times N$ quantum states). In this case, if τ is a pure state, then the super-fidelity has a simple form: $G(\rho, \tau) = \mathrm{Tr}(\rho\tau)$; this makes the PG-metric easy to study. So we first show the metric character of $D_{PG}(\rho, \sigma)$, where the optimal state τ is restricted to the pure state, and then turn to show the metric character of $D_G(\rho, \sigma)$.
We need the following concepts: For two quantum states ρ and σ, let λ_i ($i = 1, 2, 3, \ldots, n$) be all eigenvalues of $\rho - \sigma$, arranged as $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. Define $E(\rho, \sigma) := \max \lambda_i$.
We can give an interpretation of $E(\rho, \sigma)$ as follows: Let ρ and σ be two quantum states, then the following is well known (see, for example, [23]):

$$E(\rho, \sigma) = \max_{\tau} \mathrm{Tr}[\tau(\rho - \sigma)], \tag{11}$$

where the maximization is obtained by taking over all pure states τ.
Note that generally $E(\rho, \sigma)$ is not a metric, since $E(\rho, \sigma)$ may not be equal to $E(\sigma, \rho)$, but we can symmetrize it as:

$$D_S(\rho, \sigma) := \max[E(\rho, \sigma), E(\sigma, \rho)] = \max |\lambda_i|, \tag{12}$$

where $|\lambda_i|$ is the absolute value of λ_i. From matrix analysis, we get that $D_S(\rho, \sigma)$ is equal to the spectral metric between ρ and σ, which is defined as the largest singular value of $\rho - \sigma$, hence we know that $D_S(\rho, \sigma)$ is in fact the spectral metric. Moreover, we have the following:

Proposition 2.3 ([20]).

For quantum states ρ and σ, $D_{PG}(\rho, \sigma) = D_S(\rho, \sigma)$, that is, the PG-metric is nothing but the spectral metric.

Now we know that the PG-metric is in fact the spectral metric, so it is a true metric. In the following we shall prove that the G-metric is also a true metric.

Theorem 2.1.

The G-metric $D_G(\rho, \sigma)$ as shown in Eq. (9) is truly a metric, i.e., it satisfies conditions 1.1-1.4.

Proof. From the definition, it is easy to prove conditions 1.1 and 1.3 hold. What we need to do is to prove conditions 1.2 and 1.4. If $\rho = \sigma$, then of course $D_G(\rho, \sigma) = 0$. If $D_G(\rho, \sigma) = 0$, we will prove $\rho = \sigma$. From the definition, we know that $D_G(\rho, \sigma) \geqslant D_{PG}(\rho, \sigma)$, so we get $D_{PG}(\rho, \sigma) = 0$. Since $D_{PG}(\rho, \sigma)$ is a true metric, we get $\rho = \sigma$.
Now we come to prove 1.4, the triangle inequality $D_G(\rho, \sigma) \leqslant D_G(\rho, \tau) + D_G(\sigma, \tau)$. $D_G(\rho, \sigma) = \max_{\tau} |G(\rho, \tau) - G(\sigma, \tau)|$, and suppose τ is the optimal state that attains the maximal, so $D_G(\rho, \sigma) = |G(\rho, \tau) - G(\sigma, \tau)|$. Assume that $|G(\rho, \tau) - G(\sigma, \tau)| = G(\rho, \tau) - G(\sigma, \tau)$, then we get $G(\rho, \tau) - G(\sigma, \tau) = G(\rho, \tau) - G(w, \tau) + G(w, \tau) - G(\sigma, \tau) \leqslant |G(\rho, \tau) - G(w, \tau)| + |G(w, \tau) - G(\sigma, \tau)| \leqslant D_G(\rho, w) + D_G(w, \sigma)$. Thus one finally has $D_G(\rho, \sigma) \leqslant D_G(\rho, w) + D_G(\sigma, w)$. Theorem is proved. □

3. Properties of D_G and D_{PG}

We know that for qubits, D_G has a clear form: $D_G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$. What about higher dimensions?
For the qunit case, the relation $D_G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$ (from Proposition 2.2) does not hold.
However, for qunits ρ and σ, the following relation remains:

$$D_G(\rho, \sigma) \leqslant \sqrt{\frac{2 \times (N-1)}{N}} \times \sqrt{1 - G(\rho, \sigma)}. \tag{13}$$

Proof. Let $\rho = \rho(\mathbf{u})$, $\sigma = \sigma(\mathbf{v})$ and $\tau = \tau(\mathbf{w})$, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the corresponding Bloch vectors of the states

ρ, σ, τ, then one obtains

$$\begin{aligned}
&|G(\rho,\tau)-G(\sigma,\tau)|\times\frac{N}{N-1}\\
&=\left|(\mathbf{u}-\mathbf{v})\cdot\mathbf{w}+\sqrt{1-|\mathbf{w}|^2}\left(\sqrt{1-|\mathbf{u}|^2}-\sqrt{1-|\mathbf{v}|^2}\right)\right|\\
&\leqslant|\mathbf{u}-\mathbf{v}||\mathbf{w}|+\sqrt{1-|\mathbf{w}|^2}\left|\sqrt{1-|\mathbf{u}|^2}-\sqrt{1-|\mathbf{v}|^2}\right|\\
&\leqslant\sqrt{|\mathbf{u}-\mathbf{v}|^2+\left|\sqrt{1-|\mathbf{u}|^2}-\sqrt{1-|\mathbf{v}|^2}\right|^2}\\
&=\sqrt{\left[2-2\mathbf{u}.\mathbf{v}-2\sqrt{1-|\mathbf{u}|^2}\sqrt{1-|\mathbf{v}|^2}\right]}\\
&=\sqrt{\frac{2\times N}{N-1}}\times\sqrt{1-G(\rho(\mathbf{u}),\sigma(\mathbf{v}))}.
\end{aligned} \tag{14}$$

Now we will discuss the inequality (13) in more detail. When $N=2$, i.e. in the case of qubits, we find that the inequality (13) in fact becomes an equality, that is, $D_G(\rho,\sigma)=\sqrt{1-G(\rho,\sigma)}$.
When the equality sign holds, i.e.,

$$D_G(\rho(\mathbf{u}),\sigma(\mathbf{v}))=\sqrt{\frac{2\times(\mathrm{N}-1)}{\mathrm{N}}}\sqrt{1-\mathbf{G}(\rho(\mathbf{u}),\sigma(\mathbf{v}))},$$

then the inequality (14) needs to be quality, that means the optimal state $\tau:=\tau(\mathbf{w}_0)$ is always attained, where $\mathbf{w}_0$ is a vector parallel to $\mathbf{u}-\mathbf{v}$, and the modulus of $\mathbf{w}_0$ satisfies

$$|\mathbf{w}_0|=\frac{\sqrt{\mathrm{N}-1}|\mathbf{u}-\mathbf{v}|}{\sqrt{2\times\mathrm{N}}\sqrt{1-\mathbf{G}(\rho(\mathbf{u}),\rho(\mathbf{v}))}}.$$

We can always find such an optimal state for cases of qubits and qunits. For qubits we know that every 2×2 density matrix has one-to-one correspondence to a Bloch vector [1, 21]. From the above formula we can see that τ is a density matrix. So for the qunit case, we conclude that:

$$D_G(\rho(\mathbf{u}),\sigma(\mathbf{v}))=\sqrt{\frac{2\times(\mathrm{N}-1)}{\mathrm{N}}}\sqrt{1-\mathbf{G}(\rho(\mathbf{u}),\sigma(\mathbf{v}))}.$$

Now we will study the intrinsic properties of the G-metric D_G and PG-metric D_{PG}. We are interested in the following properties:

Property 3.1 (contractive under quantum operation).

Suppose T is a quantum operation (i.e. a completely positive trace preserving (CPT) map), and ρ and σ are density operators; we say a metric $D(\rho,\sigma)$ is contractive under quantum operation if the following holds:

$$D(T(\rho),T(\sigma))\leqslant D(\rho,\sigma). \tag{15}$$

This property has a physical interpretation [17]: a quantum process acting on two quantum states can not increase their distinguishability.

Property 3.2 (jointly convex property).

We say that the metric $D(\rho,\sigma)$ has the jointly convex property. If p_j are probabilities then

$$D\left(\sum_j p_j\rho_j,\sum_j p_j\sigma_j\right)\leqslant\sum_j p_jD(\rho_j,\sigma_j). \tag{16}$$

The jointly convex property also has a physical interpretation [17]: the distinguishability between the states $\sum_j p_j\rho_j$ and $\sum_j p_j\sigma_j$, where p_j is not known, can never be greater than the average distinguishability when p_j is known.
We know that the Uhlmann-Jozsa fidelity $F(\rho,\sigma)$ has the CPT expansive property:

Property 3.3 (CPT expansive property).

If ρ and σ are density operators, Φ is a CPT map, then

$$F(\Phi(\rho),\Phi(\sigma))\geqslant F(\rho,\sigma). \tag{17}$$

We may guess that the super-fidelity $G(\rho,\sigma)$ also has the CPT expansive property. The following counterexample shows that this property actually does not hold.

Example 3.1 ([13]).

Let

$$A=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},\qquad B=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

Define $\Phi(\gamma)=A\gamma A^{+}+B\gamma B^{+}$, where γ is an arbitrary density operator, then we defined a completely positive trace preserving map.

Let ρ and σ be the density operators defined by

$$\rho=\begin{pmatrix}\frac{1}{2}&0&0&0\\0&\frac{1}{2}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \sigma=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&\frac{1}{2}&0\\0&0&0&\frac{1}{2}\end{pmatrix}.$$

Then

$$\Phi(\rho)=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \Phi(\sigma)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

One then easily obtains $G(\rho,\sigma) > G(\Phi(\rho),\Phi(\sigma))$, which shows that the CP expansive property does not hold for super-fidelity.
So we get that the metric $C(\rho,\sigma) := \sqrt{1-G(\rho,\sigma)}$ introduced in [13] is not contractive under quantum operation. However, we can prove the following:

Theorem 3.1.

The PG-metric $D_{PG}(\rho,\sigma)$ is contractive under quantum operation, that is, $D_{PG}(\phi(\rho),\phi(\sigma)) \leqslant D_{PG}(\rho,\sigma)$.

Proof. Suppose γ is the optimal pure state for quantum states $\phi(\rho),\phi(\sigma)$, so we get $D_{PG}(\phi(\rho),\phi(\sigma)) = |G(\phi(\rho),\gamma) - G(\phi(\sigma),\gamma)| = |(\mathrm{Tr}\phi(\rho)\gamma) - (\mathrm{Tr}\phi(\sigma)\gamma)|$.
Let ϕ be a quantum operation, and denote $\gamma^{'} := \phi^*(\gamma)$. Then we have

$$\begin{aligned} D_{PG}(\phi(\rho),\phi(\sigma)) &= |(\mathrm{Tr}\phi(\rho)\gamma) - (\mathrm{Tr}\phi(\sigma)\gamma)| \\ &= |(\mathrm{Tr}\rho\phi^*(\gamma)) - (\mathrm{Tr}\sigma\phi^*(\gamma))| \\ &= |(\mathrm{Tr}\rho\gamma^{'}) - (\mathrm{Tr}\sigma\gamma^{'})| \\ &\leqslant D_{PG}(\rho,\sigma). \end{aligned}$$

Theorem is proved. □

Note that the PG-metric is, in fact, the spectral metric, and it was proved in [25] that the spectral metric is contractive under quantum operation; here we give an elementary proof, our method is quite different from that of [25].
How about the G-metric? Numerical experiment shows that the G-metric $D_G(\rho,\sigma)$ is not contractive under quantum operation.
Now we discuss the jointly convex property.

Proposition 3.1 (joint convexity of the PG-metric).

Let $\{p_i\}$ be probability distributions over an index set, and let ρ_i and σ_i be density operators with the indices from the same index set. Then

$$D_{PG}\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \leqslant \sum_i p_i D_{PG}(\rho_i,\sigma_i)\,. \tag{18}$$

We know that

$$D_{PG}(\rho,\sigma) = D_S(\rho,\sigma) = \max(E(\rho,\sigma), E(\sigma,\rho)),$$

so we only need to prove that the following holds:

$$E\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \leqslant \sum_i p_i E(\rho_i,\sigma_i),$$

since $E(\rho,\sigma) = \max_\gamma \mathrm{Tr}(\gamma(\rho-\sigma))$, where the maximization in the right-hand side is taken over all pure states γ, then there exists a pure state γ such that

$$\begin{aligned} E\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) &= \sum_i p_i \mathrm{Tr}(\gamma(\rho_i-\sigma_i)) \\ &\leqslant \sum_i p_i E(\rho_i,\sigma_i). \end{aligned}$$

The proof is complete. □

Note that the PG-metric is a kind of partitioned trace distance, and in [30] the author proved that the partitioned trace distance shows strong convexity, which leads to joint convexity.
We also find that the metric D_G is not jointly convex. However, numerical experiment shows that the square of D_G is jointly convex.
Now we discuss some physical interpretations of the G-metric and PG-metric. For the PG-metric, we know that it is really the doubled 1-partitioned trace distance, and we can estimate it using a positive operator valued measure (POVM). A POVM is a set of positive operators M_m satisfying $\sum_m M_m = I$, where I is the identity operator. For two density operators, the traces $\mathrm{Tr}(\mathrm{M_m}\rho) := \mathrm{p_m}$ and $\mathrm{Tr}(\mathrm{M_m}\sigma) := \mathrm{q_m}$ are the probabilities of obtaining a measurement outcome labeled by m [1, 30]. For partitioned trace distances, the following holds [30]:

$$D_k(\rho,\sigma) = \max\{D_k^{\downarrow}(p_m,q_m) : \mathrm{Tr}(\mathrm{M_m}) \leqslant 1\}. \tag{19}$$

Formula (19) gives a nice interpretation of the PG-metric. For the G-metric, we know that super-fidelity $G(\rho,\sigma)$ can be measured with the help of a single setup, namely the one that measures observable V [11]:

$$G(\rho,\sigma) = \mathrm{Tr}(\mathrm{V}\rho\otimes\sigma) + \sqrt{1-\mathrm{Tr}(\mathrm{V}\rho\otimes\rho)}\sqrt{1-\mathrm{Tr}(\mathrm{V}\sigma\otimes\sigma)}.$$

So we get the following representation of G-metric:

$$\begin{aligned} D_G(\rho,\sigma) &= \max|G(\rho,\tau) - G(\sigma,\tau)| \\ &= \max\Big|\mathrm{Tr}(\mathrm{V_1}\rho\otimes\tau) \\ &\quad + \sqrt{1-\mathrm{Tr}(\mathrm{V_1}\rho\otimes\rho)}\sqrt{1-\mathrm{Tr}(\mathrm{V_1}\tau\otimes\tau)} \\ &\quad - \mathrm{Tr}(\mathrm{V_2}\sigma\otimes\tau) \\ &\quad - \sqrt{1-\mathrm{Tr}(\mathrm{V_2}\sigma\otimes\sigma)}\sqrt{1-\mathrm{Tr}(\mathrm{V_2}\tau\otimes\tau)}\Big|. \end{aligned}$$

It is worthwhile to notice that super-fidelity can be obtained from directly measurable quantities, i.e. probabilities. Given three density operators ρ_1,ρ_2,ρ_3, we can introduce the probabilities of the projection onto the antisymmetric subspace of $H \otimes H$ (see [11] for details):

$$p_{ij} := \mathrm{Tr}(\mathrm{P}^{-}\rho_{\mathrm{i}} \otimes \rho_{\mathrm{j}}), \quad \mathrm{i}, \mathrm{j} = 1, 2, 3. \tag{20}$$

Then, the super-fidelity has a nice form [11]:

$$G(\rho_1, \rho_2) = 1 - 2(p_{12} - \sqrt{p_{11}p_{22}}). \tag{21}$$

The probabilities p_{ij} can be experimentally measured [31, 32].

Now, we can get another representation of the G-metric:

$$D_G(\rho_1, \rho_2) = \max 2\left|p_{13} - p_{23} - \sqrt{p_{11}p_{33}} + \sqrt{p_{22}p_{33}}\right|. \tag{22}$$

We will try to give the geometric interpretation of $D_G(\rho, \sigma)$. From [21], we know that the G-metric has a clear geometric interpretation as follows:

$$D_G(\rho, \sigma) = \frac{2}{1+r} \max_v \left| \left(\frac{\cosh\left(\frac{\phi_{w_1}}{2}\right)}{\cosh(\phi_{u_1})} \frac{\cosh\left(\frac{\phi_{w_1}}{2}\right)}{\cosh(\phi_v)} - \frac{\cosh\left(\frac{\phi_{w_2}}{2}\right)}{\cosh(\phi_{u_2})} \frac{\cosh\left(\frac{\phi_{w_2}}{2}\right)}{\cosh(\phi_v)} \right) \right|,$$

where $r = \frac{1}{N-1}, \phi_{u_1}, \phi_{u_2}, \phi_v$ are the hyperbolic triangles for states ρ, σ, τ, w_1 is the Einstein sum of u_1 and v, w_2 is the Einstein sum of u_2 and v, see Figure 1.

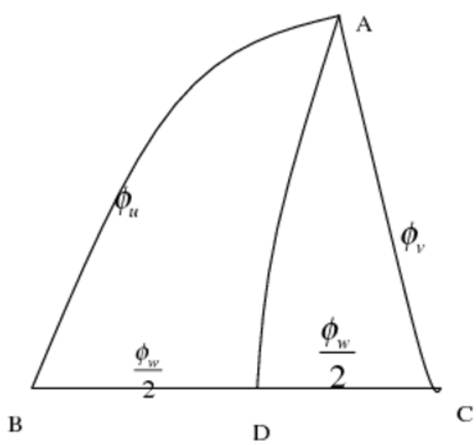

Figure 1. Geometric interpretation of D_G.

We know that $D_G(\rho, \sigma) \geqslant D_{PG}(\rho, \sigma)$, since $D_{PG}(\rho, \sigma)$ is in fact the spectral metric, so we get the following lower bound of the G-metric: $D_G(\rho, \sigma) \geqslant D_s(\rho, \sigma)$.

Now if we get the lower bound and the upper bound of the G-metric, we will see that, from the two bounds we can give a good approximation of the G-metric.

Numerical simulations have been carried out. Thousands of pairs of random quantum states ρ and σ have been obtained by Mathematica. Every $D_G(\rho, \sigma)$ has been obtained through choosing 10^4 random quantum states τ, so the result should be very close to the real value of $D_G(\rho, \sigma)$. Numerical experiment shows that $D_G(\rho, \sigma)$, $D_s(\rho, \sigma)$, and our upper bound

$$\sqrt{\frac{2 \times (N-1)}{N}} \times \sqrt{1 - G(\rho, \sigma)}$$

are very close, so $D_G(\rho, \sigma)$ can be approximately computed through the lower bound $D_s(\rho, \sigma)$ and upper bound of $D_G(\rho, \sigma)$.

The $D_G(\rho, \sigma)$, upper bound of $D_G(\rho, \sigma)$ and $D_s(\rho, \sigma)$ are listed in Figure 2. $D_G(\rho, \sigma)$ are represented by '$\cdot$', $D_s(\rho, \sigma)$ are represented by 'o', and upper bounds of $D_G(\rho, \sigma)$ are denoted by '$\triangle$'.

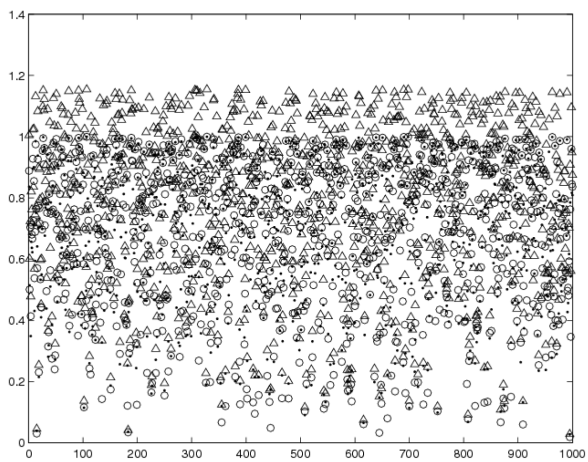

Figure 2. $D_G(\rho, \sigma)$, upper bound of $D_G(\rho, \sigma)$ and $D_s(\rho, \sigma)$.

4. Conclusion

In summary, we have introduced a new way to define metrics of quantum states from super-fidelity. We find that, for the qubit case, our metric D_G coincides with the metric $C(\rho, \sigma)$ introduced in [13]. We proved that the metric D_{PG} is contractive under quantum operation, while the metric

D_G does not behave monotonically under quantum operation. Also, we rigorously proved that D_{PG} is jointly convex, and numerically proved that the square of D_G is jointly convex. The new metric may be used to tasks in quantum information theory, such as the geometrical entanglement measure [26], finding the bound of entanglement measure [14, 27, 28], characterizing the quantum phase transitions [29]. All these show that the metric D_G is worthwhile studying.

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