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Approximate analytical solutions of scattering states for Klein-Gordon equation with Hulthén potentials for nonzero angular momentum

Research Article

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Abstract: In this paper, using the exponential function transformation approach along with an approximation for the centrifugal potential, the radial Klein-Gordon equation with the vector and scalar Hulthén potential is transformed to a hypergeometric differential equation. The approximate analytical solutions of l-waves scattering states are presented. The normalized wave functions expressed in terms of hypergeometric functions of scattering states on the "$k/2\pi$ scale" and the calculation formula of phase shifts are given. The physical meaning of the approximate analytical solution is discussed.

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Keywords: Hulthén potential • Klein-Gordon equation • scattering states • approximate analytical solution • phase shifts

1. Introduction

The Hulthén potential [1] is one of the important short-range potentials in physics. The potential is given by

$$V(r) = -\frac{V_0}{e^{r/r_0}-1} = -\frac{Ze^2}{r_0}\frac{1}{e^{r/r_0}-1}, \qquad (1)$$

where V_0 is a constant and r_0 is the range of the potentials. If the potential is used for atoms, then $V_0 = Z/r_0$ (with $\hbar = c = e = 1$), where Z is identified as the atomic number. This potential has been applied to a number of areas such as nuclear and particle physics [2], atomic physics [3, 4], molecular physics [5–7], and chemical physics [8], etc. The Hulthén potential behaves like the Coulomb potential near the origin ($r \to 0$), but in the asymptotic region ($r \gg 1$) the Hulthén potential decreases exponentially, so its capacity for bound states is smaller than the Coulomb potential. However, for large values of r_0, the Hulthén potential becomes the Coulomb potential given by $V(r) = -\frac{V_0}{e^{r/r_0}-1} \underset{r_0\gg 1}{\longrightarrow} -\frac{Z}{r}$. Unfortunately, the Hulthén potential can be solved analytically only for the states with zero angular momentum [1]. For the case $l \neq 0$, the Hulthén potential cannot be exactly solved. In the non-relativistic case, for nonzero angular momentum, several techniques were used to obtain approximate solutions, a number of methods have been used to find the bound-state energy eigenvalues numerically [9, 10] and quasi-analytically, such as the variational [9, 11], perturbation

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[12], shifted $1/N$ expansion [13, 14], SUSYQM [15, 16], and AIM [17] methods. In the relativistic case, Dominguez-Adame [18], Chetouani *et al.* [19], and Talukdar *et al.* [20] have given the exact solutions for bound state and the scattering state of s-wave Klein-Gordon equation with vector and scalar Hulthén potentials, respectively. The exact solution for bound state of s-wave Dirac equation with vector and scalar Hulthén potentials has also been discussed [21–23]. Very recently, Chen *et al.* [24] and Soylu *et al.* [25] have presented approximate analytical solutions of bound states for the Klein-Gordon equation and the Dirac equation with vector and scalar Hulthén potentials for nonzero angular momentum, respectively. Haouat *et al.* [26] have discussed approximate analytical solutions of bound states and scattering states for the Klein-Gordon equation and the Dirac equation only with vector Hulthén potentials for the case $l \neq 0$.

The purpose of the current work is to study relativistic characteristics of scattering states for the Hulthén potential in the case $l \neq 0$. In the strong coupling case, the relativistic effects of a moving particle in a potential field should be discussed. In Section 2, we will give the approximate analytical solution of scattering states of l-wave Klein-Gordon equation with vector and scalar Hulthén potentials on the assumption that an effective approximation of $\frac{1}{r^2} \approx \frac{e^{r/r_0}}{r_0^2(e^{r/r_0}-1)^2}$ is used for the centrifugal term in the case of any l-states. In Section 3, we will discuss the physical meaning of the approximate analytical solution.

2. The approximate analytical solutions of scattering states

In spherical coordinates, the Klein-Gordon equation with scalar potential $S(r)$ and vector potential $V(r)$ is written as ($\hbar = c = e = 1$)

$$\left\{-\nabla^2 + [M+S(r)]^2\right\}\Psi(r,\theta,\varphi) = [E-V(r)]^2\Psi(r,\theta,\varphi) \tag{2}$$

Letting $\Psi(r,\theta,\phi) = \frac{u(r)}{r}Y_{lm}(\theta,\varphi)$, the radial equation is represented as

$$\frac{d^2u}{dr^2} + \left\{[E^2-M^2] - 2[MS(r)+EV(r)] + [V^2(r)-S^2(r)] - \frac{l(l+1)}{r^2}\right\}u(r) = 0. \tag{3}$$

Now we consider vector and scalar Hulthén potentials which are written as

$$V(r) = -\frac{V_0}{e^{r/r_0}-1}, \quad S(r) = -\frac{S_0}{e^{r/r_0}-1}, \tag{4}$$

respectively. Substituting Eq. (4) into Eq. (3), the result is

$$\frac{d^2u(r)}{dr^2} + \left\{k^2 + \frac{\beta^2/r_0^2}{e^{r/r_0}-1} - \frac{\nu^2/r_0^2}{(e^{r/r_0}-1)^2} - \frac{l(l+1)}{r^2}\right\}u(r) = 0, \tag{5}$$

where

$$k = (E^2-M^2)^{1/2}, \quad \beta = (2EV_0+2MS_0)^{1/2}r_0, \quad \nu = (S_0^2-V_0^2)^{1/2}r_0. \tag{6}$$

For the scattering states, $E > M, k > 0$. The boundary conditions for Eq. (3) are

$$r \to 0,\ u(r) \to r^{l+1}; \qquad r \to \infty,\ u(r) \to 2\sin(kr - \pi l/2 + \delta_l), \tag{7}$$

The term $\frac{l(l+1)}{r^2}$ in Eq. (3) is known as the centrifugal term. When $l = 0$ (s-wave), Eq. (5) can be exactly solved [18–20], but for the case $l \neq 0$, Eq. (5) cannot be exactly solved. Therefore, we must use an approximation for the centrifugal term similar to the bound states [15–17, 24]. In this approximation, $\frac{1}{r^2} \approx \frac{e^{r/r_0}}{r_0^2(e^{r/r_0}-1)^2}$ is used for the centrifugal term. So, Eq. (5) can be written as

$$\frac{d^2u(x)}{dx^2} + \left\{k^2r_0^2 + \frac{\beta^2}{e^x-1} - \frac{\nu^2}{(e^x-1)^2} - \frac{l(l+1)\,e^x}{(e^x-1)^2}\right\}u(x) = 0, \tag{8}$$

where $x = r/r_0$. If we rewrite equation (8) by using a new variable of the form $z = 1 - e^{-x}$ $(r \in [0, \infty),\ z \in [0, 1])$, we obtain

$$\frac{d^2u(z)}{dz^2} - \frac{1}{1-z}\frac{du(z)}{dz} + \left\{\frac{k^2r_0^2}{(1-z)^2} + \frac{\beta^2}{z(1-z)} - \frac{v^2}{z^2} - \frac{l(l+1)}{z^2(1-z)}\right\} u(z) = 0. \tag{9}$$

Considering the boundary conditions of the scattering states, we take the wave function with the form

$$u(z) = z^{l'+1}(1-z)^{-ik\,r_0} f(z), \tag{10}$$

where

$$l' = \frac{1}{2}\left[\sqrt{4v^2 + (2l+1)^2} - 1\right] = \frac{1}{2}\left[\sqrt{4r_0^2(S_0^2 - V_0^2) + (2l+1)^2} - 1\right]. \tag{11}$$

Substituting Eq. (10) into Eq. (9), we can obtain the following second-order differential equation

$$z(1-z)\frac{d^2f(z)}{dz^2} + [2(l'+1) - (2\,l' + 3 - 2\,i\,kr_0)z]\frac{df(z)}{dz} + [\beta^2 + v^2 - (l'+1)^2 + 2\,i\,kr_0(l'+1)]f(z) = 0\ , \tag{12}$$

which is called the hypergeometric differential equation [27, 28]. Thus, analytical solution as $z = 0 (r \to 0)$is the hypergeometric function

$$f(z) = {}_2F_1(a,\, b\,;\ c\,; z) \tag{13}$$

The parameters are

$$a = l' + 1 + \sqrt{\beta^2 + v^2 - k^2r_0^2} - i\,kr_0, \quad b = l' + 1 - \sqrt{\beta^2 + v^2 - k^2r_0^2} - i\,kr_0, \quad c = 2\,l' + 2. \tag{14}$$

Here the hypergeometric function ${}_2F_1(a,\, b\,;\ c\,; z)$ is a special case of the generalized hypergeometric function

$${}_pF_q(\alpha_1,\, \alpha_2,\, \cdots,\alpha_p;\, \beta_1,\, \beta_2,\, \cdots,\beta_q;\, z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_p)_k}{k!\,(\beta_1)_k(\beta_2)_k\cdots(\beta_q)_k}z^k \tag{15}$$

where the Pochhammer symbol is defined by $(x)_k = \Gamma(x+k)/\Gamma(x)$. Thus, the radial wave function of scattering states is

$$u(r) = N_{k\,l}(1 - e^{-r/r_0})^{l'+1}\, e^{i\,k\,r}\,{}_2F_1(a,\, b;\, c;\, 1 - e^{-r/r_0}) \tag{16}$$

We now study asymptotic form of the above expression for large r, and calculate the normalization constant $N_{k\,l}$ of the radial wave functions and phase shifts. From Eq. (14), we have

$$c - a - b = 2\,i\,kr_0 = (a + b - c)^* \tag{17}$$

$$c - a = l' + 1 - \sqrt{\beta^2 + v^2 - k^2r_0^2} + i\,kr_0 = b^* \tag{18}$$

$$c - b = l' + 1 + \sqrt{\beta^2 + v^2 - k^2r_0^2} + i\,kr_0 = a^* \tag{19}$$

By using the transformation formulas for hypergeometric functions [27, 28]

$$
\begin{aligned}
{}_2F_1(a,\, b;\ \ c\,; z) = {} & \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\, {}_2F_1(a, b; a+b-c+1; 1-z) \\
& + (1-z)^{c-a-b}\, \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\, {}_2F_1(c-a, c-b; c-a-b+1; 1-z)
\end{aligned} \tag{20}
$$

and paying attention to ${}_2F_1(a,\, b;\, c;\, 0) = 1$, we have

$$
\begin{aligned}
{}_2F_1(a,\, b;\ \ c\,; 1-e^{-r/r_0}) = {} & \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\, {}_2F_1(a, b; a+b-c+1;\, e^{-\,r/r_0}) \\
& + (e^{-\,r/r_0})^{c-a-b}\, \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\, {}_2F_1(c-a, c-b; c-a-b+1;\, e^{-\,r/r_0}) \\
& \underset{r\to\infty}{\longrightarrow} \Gamma(c)\left[\frac{\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} + e^{-2\,i\,k\,r}\left(\frac{\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\right)^{*}\right].
\end{aligned} \tag{21}
$$

Letting

$$
\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\, e^{i\,\delta} \tag{22}
$$

then

$$
\left(\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)^{*} = \left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\, e^{-i\,\delta} \tag{23}
$$

where δ is a real number. Eq. (21) then becomes

$$
{}_2F_1(a,\, b;\ \ c\,; 1-e^{-\,r/r_0}) \underset{r\to\infty}{\longrightarrow} \Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\right| e^{-\,i\,k\,r}\left[e^{i\,(k\,r+\delta)} + e^{-\,i\,(k\,r+\delta)}\right] \tag{24}
$$

Substituting Eq. (24) into Eq. (16) leads to

$$
\begin{aligned}
u(r) & \underset{r\to\infty}{\longrightarrow} 2\,N_{k\,l}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \cos(kr+\delta) \\
& \underset{r\to\infty}{\longrightarrow} 2\,N_{k\,l}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right| \sin[kr-\pi\,l/2+(\pi(l+1)/2+\delta)].
\end{aligned} \tag{25}
$$

Comparing Eqs. (7) with (25), we have the phase shifts as

$$
\begin{aligned}
\delta_l & = \pi(l+1)/2 + \arg\Gamma(c-a-b) - \arg\Gamma(c-a) - \arg\Gamma(c-b) \\
& = \pi(l+1)/2 + \arg\Gamma(2ikr_0) - \arg\Gamma\left(l'+1-\sqrt{\beta^2+\nu^2-k^2r_0^2}+ikr_0\right) - \arg\Gamma\left(l'+1+\sqrt{\beta^2+\nu^2-k^2r_0^2}+ikr_0\right)
\end{aligned} \tag{26}
$$

and the normalization constant on the "$k/2\pi$ scale" as

$$
N_{k\,l'} = \frac{1}{\Gamma(c)}\left|\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c-a-b)}\right| = \frac{1}{\Gamma(2l'+2)}\left|\frac{\Gamma(l'+1-\sqrt{\beta^2+\nu^2-k^2r_0^2}+ikr_0)\Gamma(l'+1+\sqrt{\beta^2+\nu^2-k^2r_0^2}+ikr_0)}{\Gamma(2\,i\,kr_0)}\right|. \tag{27}
$$

Where l' and $k,\ \beta,\ \nu$ are given in formulas (11) and (6), respectively. The corresponding normalized wave functions of scattering states as

$$
\begin{aligned}
u(r) = {} & \frac{1}{\Gamma(2l'+2)}\left|\frac{\Gamma(l'+1-\sqrt{\beta^2+\nu^2-k^2r_0^2}+ikr_0)\Gamma(l'+1+\sqrt{\beta^2+\nu^2-k^2r_0^2}+ikr_0)}{\Gamma(2\,i\,kr_0)}\right| (1-e^{-r/r_0})^{l'+1}\, e^{ikr} \\
& \times {}_2F_1(l'+1+\sqrt{\beta^2+\nu^2-k^2r_0^2}-ikr_0,\, l'+1-\sqrt{\beta^2+\nu^2-k^2r_0^2}-ikr_0; 2l'+2; 1-e^{-r/r_0}).
\end{aligned} \tag{28}
$$

3. Discussion

After approximately solving the scattering states of l-wave Klein-Gordon equation with vector and scalar Hulthén potentials, we have several remarks.

(1) When $l = 0$, both the centrifugal term and the approximation centrifugal term are zero. Eqs. (26) and (28) reduce to the exact phase shifts formula and the normalized wave functions on the "$k/2\pi$ scale" for the scattering states of s-wave Klein-Gordon equation with Hulthén potential, respectively.

(2) When $r_0 \gg 1$, the Hulthén potential becomes the Coulomb potential, i. e.

$$V(r) = -\frac{V_0}{e^{r/r_0} - 1} \xrightarrow[r_0 \gg 1]{} -\frac{Z_v}{r}, \quad S(r) = -\frac{S_0}{e^{r/r_0} - 1} \xrightarrow[r_0 \gg 1]{} -\frac{Z_s}{r}. \tag{29}$$

Therefore, $V_0 = Z_v/r_0$, $S_0 = Z_s/r_0$, and Eq. (6) and (11) becomes respectively,

$$k = (M^2 - E^2)^{1/2}, \quad \beta = \sqrt{2r_0(EZ_V + MZ_s)}, \quad \nu = \sqrt{(Z_s^2 - Z_v^2)}, \tag{30}$$

$$l' = \frac{1}{2}\left[\sqrt{4(Z_s^2 - Z_v^2) + (2\,l + 1)^2} - 1\right]. \tag{31}$$

From Eq. (14), (30), and (31), we can obtain

$$\lim_{r_0 \gg 1} a = \lim_{r_0 \gg 1} \left(l' + 1 + \sqrt{\beta^2 + \nu^2 - k^2 r_0^2} - i\,k r_0\right) = l' + 1 - i(EZ_v + MZ_s)/k, \tag{32}$$

and

$$\lim_{r_0 \gg 1} b = \lim_{r_0 \gg 1} \left(l' + 1 - \sqrt{\beta^2 + \nu^2 - k^2 r_0^2} - i\,k r_0\right) = \lim_{r_0 \gg 1} (-i2k\,r_0) \to \infty. \tag{33}$$

And using the relation of hypergeometric function with confluent hypergeometric function [27, 28],

$$\lim_{b \to \infty} {}_2F_1(a, b; c; z/b) = {}_1F_1(a; c; z) \tag{34}$$

we can rewrite the radial wave function (28) as

$$u(r) = A_{k\,l'}(kr)^{l'+1}\, e^{i\,k\,r}\, {}_1F_1(l' + 1 - i(EZ_v + MZ_s)/k\,;\, 2\,l' + 2;\, -2\,i\,k\,r). \tag{35}$$

The above expression is the same as the radial wave function for the scattering states of Klein-Gordon equation with vector and scalar Coulomb potential [29], where the normalization constant is

$$A_{k\,l'} = \frac{2^{l'+1}\, |\Gamma(l' + 1 - i(EZ_v + MZ_s)/k)|\, e^{\pi(EZ_v + MZ_s)/2k}}{\Gamma(2l' + 2)}\ , \tag{36}$$

and corresponding phase shifts are represented as

$$\delta_l = \arg\Gamma(l' + 1 - i(EZ_v + MZ_s)/k\,) + \pi(l - l')/2. \tag{37}$$

(3) In the case that the scalar potential is equal to the vector potential, $S_0 = V_0$, $l' = l$, then Eqs. (26) and (28) reduce respectively to

$$\delta_l = \pi(l+1)/2 + \arg\Gamma(2ikr_0) - \arg\Gamma(l+1-\sqrt{2V_0(E+M)r_0^2 - k^2r_0^2} + ikr_0) - \arg\Gamma(l+1+\sqrt{2V_0(E+M)r_0^2 - k^2r_0^2} + ikr_0), \tag{38}$$

$$u(r) = \frac{1}{(2l+1)!}\left|\frac{\Gamma(l+1-\sqrt{2V_0(E+M)r_0^2-k^2r_0^2}+ikr_0)\Gamma(l+1+\sqrt{2V_0(E+M)r_0^2-k^2r_0^2}+ikr_0)}{\Gamma(2\,i\,kr_0)}\right|(1-e^{-r/r_0})^{l+1}\,e^{ikr} \times {}_2F_1(l+1+\sqrt{2V_0(E+M)r_0^2-k^2r_0^2}-ikr_0, l+1-\sqrt{2V_0(E+M)r_0^2-k^2r_0^2}-ikr_0; 2l+2; 1-e^{-r/r_0}). \tag{39}$$

(4) Finally, we discuss the non-relativistic limit of the phase shifts and the radial wave functions. When $S(r) = V(r)$, Eq. (2) reduces to a Schrödinger-like equation for the potential $2V(r)$. In other words, the non-relativistic limit of scattering states is the Schrödinger equation for the potential $-2V_0/[\exp(r/r_0) - 1]$. By using methods of Ref. [30], we have the non-relativistic representation of the phase shifts and the normalized radial wave functions of scattering states on the "$k/2\pi$ scale" for the potential $-2V_0/[\exp(r/r_0) - 1]$ as (where $\hbar = c = e = 1$)

$$\delta_l = \pi(l+1)/2 + \arg\Gamma(2ikr_0) - \arg\Gamma(l+1+ikr_0-\sqrt{4MV_0r_0^2-k^2r_0^2}) - \arg\Gamma(l+1+ikr_0+\sqrt{4MV_0r_0^2-k^2r_0^2}), \tag{40}$$

$$u(r) = \frac{1}{(2l+1)!}\left|\frac{\Gamma(l+1-\sqrt{4MV_0r_0^2-k^2r_0^2}+ikr_0)\Gamma(l+1+\sqrt{4MV_0r_0^2-k^2r_0^2}+ikr_0)}{\Gamma(2\,i\,kr_0)}\right|(1-e^{-r/r_0})^{l+1}\,e^{ikr} \times {}_2F_1(l+1+\sqrt{4MV_0r_0^2-k^2r_0^2}-ikr_0, l+1-\sqrt{4MV_0r_0^2-k^2r_0^2}-ikr_0; 2l+2; 1-e^{-r/r_0}), \tag{41}$$

where $k = \sqrt{2ME_{non}}$, M and E_{non} are rest mass and non-relativistic energy, respectively.

In the weak coupling condition, $E = M + E_{non}$, $E + M = M + E_{non} + M \approx 2M$, $k = (E^2 - M^2)^{1/2} = \sqrt{(E+M)(E-M)} \approx \sqrt{2ME_{non}}$. Thence Eqs. (38) and (39) reduce to the non-relativistic phase shifts expression (40) and the normalized radial wave functions expression (41) on the "$k/2\pi$ scale", respectively.

4. Conclusions

In this paper, the approximate analytical solution of any l - waves scattering states for the Klein-Gordon equation with vector and scalar Hulthén potential is presented. Using an exponential function transformation, the radial Klein-Gordon equation is transformed into a hypergeometric differential equation on the assumption that an effective approximation of $\frac{1}{r^2} \approx \frac{e^{r/r_0}}{r_0^2(e^{r/r_0}-1)^2}$ is used for the centrifugal term in the case of any l- states. The normalized wave functions expressed in terms of hypergeometric functions of scattering states on the "$k/2\pi$ scale" and the calculation formula of phase shifts are given. When $l = 0$, the result is an exact solution of scattering states of s-wave Klein-Gordon equation with vector and scalar Hulthén potentials. In the case of $r_0 \gg 1$, the result reduces to an exact solution of scattering states of Klein-Gordon equation with vector and scalar Coulomb potentials. Furthermore, we discussed non-relativistic limit of the phase shifts and the radial wave functions.

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