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Weighted inequalities for some integral operators with rough kernels

Research Article

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Abstract: In this paper we study integral operators with kernels

$$K(x,y) = k_1(x-A_1y)\cdots k_m(x-A_my),$$

$k_i(x) = \Omega_i(x)/|x|^{n/q_i}$ where $\Omega_i\colon \mathbb{R}^n \to \mathbb{R}$ are homogeneous functions of degree zero, satisfying a size and a Dini condition, A_i are certain invertible matrices, and $n/q_1+\cdots+n/q_m = n-\alpha$, $0\le\alpha<n$. We obtain the appropriate weighted L^p-L^q estimate, the weighted BMO and weak type estimates for certain weights in $A(p,q)$. We also give a Coifman type estimate for these operators.

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1. Introduction

Let $0\le\alpha<n$, $1<m\in\mathbb{N}$. For $1\le i\le m$, let $1<q_i<\infty$ be such that $n/q_1+\cdots+n/q_m = n-\alpha$. We denote by $\Sigma=\Sigma_{n-1}$ the unit sphere in $\mathbb{R}^n$. Let $\Omega_i\in L^1(\Sigma)$. If $x\neq 0$, we write $x' = x/|x|$. We extend this function to $\mathbb{R}^n\setminus\{0\}$ as $\Omega_i(x)=\Omega_i(x')$. Let

$$k_i(x) = \frac{\Omega_i(x)}{|x|^{n/q_i}}. \tag{1}$$

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In this paper we study the integral operator

$$T_\alpha f(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \tag{2}$$

with $K(x,y) = k_1(x - A_1 y) \cdots k_m(x - A_m y)$, where A_i, are certain invertible matrices and $f \in L^\infty_{\text{loc}}(\mathbb{R}^n)$.

In the case $A_i = a_i I$, $a_i \in \mathbb{R}$, Godoy and Urciuolo in [6] obtain the $L^p(\mathbb{R}^n, dx)$–$L^q(\mathbb{R}^n, dx)$ boundedness of this operator for $0 \le \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. In the case that Ω_i are smooth functions, in [12], Rocha and Urciuolo consider the operator T_α for matrices $A_1, \dots, A_m$ satisfying the following hypothesis:

$$A_i \text{ is invertible and } A_i - A_j \text{ is invertible for } i \ne j,\ 1 \le i,j \le m. \tag{H}$$

They obtain that T_α is a bounded operator from $H^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$, for $0 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$.

For $0 \le \alpha < n$ and $1 \le s < \infty$ we define

$$M_{\alpha,s} f(x) = \sup_B |B|^{\alpha/n} \left(\frac{1}{|B|} \int_B |f(x)|^s\, dx \right)^{1/s},$$

where the supremum is taken along all balls B such that x belongs to B. We observe that $M = M_{0,1}$, where M is the classical Hardy–Littlewood maximal operator, also for $0 < \alpha < n$, $M_\alpha = M_{\alpha,1}$ is the classical fractional maximal operator. It is well known [9] that if w is a weight (i.e. w is a non negative function and $w \in L^1_{\text{loc}}(\mathbb{R}^n, dx)$) then M_α is a bounded operator from $L^p(\mathbb{R}^n, w^p)$ into $L^q(\mathbb{R}^n, w^q)$, for $1 < p < n/\alpha$ and $1/q = /p - \alpha/n$, if and only if

$$\sup_B \left[\left(\frac{1}{|B|} \int_B w^q \right)^{1/q} \left(\frac{1}{|B|} \int_B w^{-p'} \right)^{1/p'} \right] < \infty, \tag{3}$$

where $1/p + 1/p' = 1$. The class of weights that satisfy (3) is called $A(p,q)$.

Throughout this paper we understand that for $p = \infty$, $\left(\int_E |f|^p \right)^{1/p}$ stands for $\|f \chi_E\|_\infty$, for any measurable set E. With this in mind we define the class $A(p,q)$ still by (3) for all $1 \le p \le \infty$ and $1 \le q \le \infty$. If A_p, $p \ge 1$, denotes the classical Muckenhoupt class of weights, we note that $w \in A(p,q)$ if and only if $w^q \in A_{1+q/p'}$, and as a particular case $w \in A(p,p)$ is equivalent to $w^p \in A_p$. We recall that $A_\infty = \bigcup_{p \ge 1} A_p$. Also, the statement $w \in A(\infty,\infty)$ is equivalent to $w^{-1} \in A_1$.

In [10, 11] we consider $\Omega_i \equiv 1$ and weights satisfying the following condition: *There exists* $c > 0$ *such that*

$$w(A_i x) \le c\, w(x), \tag{4}$$

for *a.e.* $x \in \mathbb{R}^n$, $1 \le i \le m$.

We note that if w is a power weight then w satisfies (4). Observe that there are other weights that satisfy this condition. For example, consider

$$w(x) = \begin{cases} -\ln|x| & \text{if} \quad |x| \le e^{-1}, \\ 1 & \text{if} \quad |x| > e^{-1}. \end{cases}$$

In [7], it is shown that $w \in A_1$ and it is easy to check that for any $a \in \mathbb{R} \setminus \{0\}$ there exists C_a such that $w(ax) \le C_a w(x)$, for a.e. $x \in \mathbb{R}$. In [11] we obtain weighted estimates for this kind of operator and certain weights satisfying (4), precisely as for the classical fractional integral operator I_α with $0 < \alpha < n$, or the singular integral operator with $\alpha = 0$, we prove the $L^p(\mathbb{R}^n, w^p)$–$L^q(\mathbb{R}^n, w^q)$ boundedness of T_α for weights $w \in A(p,q)$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $0 \le \alpha < n$.

Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n, dx)$, we define the sharp maximal function by

$$M^\# f(x) = \sup_{B \ni x} \frac{1}{|B|} \int \left| f(y) - \frac{1}{|Q|} \int_B |f| \right| dy,$$

and the space

$$\mathrm{BMO} = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx) : M^\# f \in L^\infty(\mathbb{R}^n, dx) \right\},$$

the norm in this space is $\|f\|_* = \|M^\# f\|_\infty$. There is also a weighted version of BMO, denoted by BMO(w), that is described by the semi norm

$$|||f|||_w = \sup_B \|w\chi_B\|_\infty \left(\frac{1}{|B|} \int_B \left| f(x) - \frac{1}{|B|} \int_B f \right| dx \right).$$

It is easy to check that $|||f||| \simeq \|wM^\# f\|_\infty$. In [11] we also obtain the weighted weak type $(1, n/(n-\alpha))$ estimate for $w \in A(1, n/(n-\alpha))$ and w satisfying (4). We also prove that if $w \in A(n/\alpha, \infty)$ and w satisfies (4) then

$$|||T_\alpha f|||_w \le C \left(\int (|f|w)^{n/\alpha} \right)^{\alpha/n}. \tag{5}$$

The key argument to obtain the above stated results was the Coifman type estimate (see [11, Theorem 2.1])

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} |M_\alpha f(x)|^p\, w(x)\, dx,$$

$f \in L^\infty_c(\mathbb{R}^n, dx)$, $p > 0$ and $w \in A_\infty$ satisfying (4).

For integral operators with rough kernels of the form

$$T_{\Omega,\alpha} f(x) = \int \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy,$$

in [3, 8, 13] the authors obtain weighted estimates for $T_{\Omega,0}$ for certain functions Ω homogeneous of degree zero and $\Omega \in L^p(S^{n-1})$ for some $p > 1$. In [2] the authors prove the corresponding weighted results for $\alpha > 0$. Also in [1] the authors obtain a Coifman type inequality for general fractional integrals operators with kernels satisfying a Hörmander condition given by a Young function. In Section 2 we describe this condition.

In this paper we consider the operator T_α defined in (2) where, for $1 \le i \le m$, k_i is given by (1) and the matrices A_i satisfy the hypothesis (H). For $1 \le p \le \infty$ and $\Omega_i \in L^1(\Sigma)$, we define the L^p-modulus of continuity as

$$\varpi_{i,p}(t) = \sup_{|y| \le t} \left\| \Omega_i(\cdot + y) - \Omega_i(\cdot) \right\|_{p,\Sigma}.$$

We will make the following hypotheses about the functions Ω_i, $1 \le i \le m$:

$$\text{there exists } p_i > q_i \text{ such that } \Omega_i \in L^{p_i}(\Sigma), \tag{H$_1$}$$

$$\int_0^1 \varpi_{i,p_i}(t) \frac{dt}{t} < \infty. \tag{H$_2$}$$

In Section 2 we obtain a pointwise estimate that relates $(M^\# |T_\alpha f|^\delta(x))^{1/\delta}$, for $0 < \delta < 1$, with a fractional maximal function of an appropriate power of f. This estimate is the fundamental key to obtain weighted inequalities for the operator T_α. These inequalities are developed in Section 3. We give first a Coifman type estimate for these operators that allows us to get the adequate weighted L^p-L^q estimate for certain weights in $A(p, q)$. The results that we obtain in Theorems 3.3 and 3.4 are the analogs of [2, Theorems 1 and 2]. We also get the corresponding weighted BMO and weak type estimates.

Throughout this paper c and C will denote positive constants, not the same at each occurrence.

2. Pointwise estimate

We denote by $|x| \sim R$ the set $\{x \in \mathbb{R}^n : R < |x| \leq 2R\}$ and for $1 \leq r \leq \infty$,

$$\|f\|_{r,|x|\sim R} = \left(\frac{1}{|B(0,2R)|} \int_{B(0,2R)} |f|^r \chi_{|x|\sim R} \right)^{1/r}.$$

In [1] the authors introduce the following definition.

Definition 2.1.
Given $0 \leq \alpha < n$ and $1 \leq r \leq \infty$, we say that $k \in H_{r,\alpha}$ if there exist $c \geq 1$ and $C > 0$ such that for all $y \in \mathbb{R}^n$ and $R > c|y|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} \left\| k(\cdot - y) - k(\cdot) \right\|_{r,|x|\sim 2^m R} \leq C.$$

In Proposition 4.2 of the mentioned paper they prove that that if k_i is as in (1) and Ω_i satisfies (H_2) then $k_i \in H_{n/q_i',p_i}$.

Theorem 2.2.
Let $0 \leq \alpha < n$ and let T_α be the integral operator defined by (2). We suppose that for $1 \leq i \leq m$, the matrices A_i and the functions Ω_i satisfy hypotheses (H), (H_1) and (H_2). If $s \geq 1$ is defined by $1/p_1 + \cdots + 1/p_m + 1/s = 1$, then there exists $C > 0$ such that for $0 < \delta \leq 1$ and $f \in L_c^\infty(\mathbb{R}^n, dx)$,

$$\left(M^\# |T_\alpha f|^\delta (x) \right)^{1/\delta} \leq C \sum_{i=1}^{m} M_{\alpha,s} f(A_i^{-1} x).$$

Proof. Let $f \in L_c^\infty(\mathbb{R}^n, dx)$, $f \geq 0$ and $0 < \delta \leq 1$. As in [6] it can be proved that T_α is a bounded operator from $L^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$, for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, so $T_\alpha(f) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ and $M_\delta^\#(T_\alpha f)(x)$ is well defined for all $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and let $B = B(x_B, R)$ be a ball that contains x, centered at x_B with radius R, and $T_\alpha f(x_B) < \infty$. We write $\widetilde{B} = B(x_B, 4R)$, and for $1 \leq i \leq m$ we also set $\widetilde{B}_i = A_i^{-1} \widetilde{B}$. Let $f_1 = f \chi_{\bigcup_{1 \leq i \leq m} \widetilde{B}_i}$ and let $f_2 = f - f_1$.

We choose $a = T_\alpha f_2(x_B)$. By Jensen's inequality and from the inequality

$$\left| t^\delta - s^\delta \right|^{1/\delta} \leq |t - s|,$$

which holds for any positive t, s, we get

$$\begin{aligned} \left(\frac{1}{|B|} \int_B \left| (T_\alpha f)^\delta (y) - a^\delta \right| dy \right)^{1/\delta} &\leq \left(\frac{1}{|B|} \int_B |T_\alpha f(y) - a| \, dy \right) \\ &\leq \left(\frac{1}{|B|} \int_B |T_\alpha f_1(y)| \, dy \right) + \left(\frac{1}{|B|} \int_B |T_\alpha f_2(y) - a| \, dy \right) = \mathrm{I} + \mathrm{II}. \end{aligned}$$

We consider first the case $0 < \alpha < n$.

$$\mathrm{I} = \frac{1}{|B|} \int_B |T_\alpha f_1(y)| \, dy \leq \frac{1}{|B|} \int_B \sum_{i=1}^{m} \int_{\widetilde{B}_i} |K(y,z)| f(z) \, dz \, dy = \sum_{i=1}^{m} \frac{1}{|B|} \int_{\widetilde{B}_i} f(z) \int_B |K(y,z)| \, dy \, dz.$$

If $z \in \widetilde{B}_i$ let

$$\mathcal{C}^l = \left\{ y \in B : |y - A_l z| \leq |y - A_r z|, \ 1 \leq r \leq m \right\},$$

then

$$\int_B |K(y,z)|\,dy \le \int_{\mathcal{C}^l} |K(y,z)|\,dy + \cdots + \int_{\mathcal{C}^m} |K(y,z)|\,dy.$$

For $1 \le l \le m$ and $j \in \mathbb{N}$, let

$$\mathcal{C}_j^l = \left\{ y \in B : |y - A_l z| \le |y - A_r z|,\ 1 \le r \le m,\ |y - A_l z| \sim 2^{-j-1}R \right\}.$$

We observe that if $y \in B$ then $|y - A_l z| \le 5R < 8R$. By Hölder's inequality,

$$\int_{\mathcal{C}^l} |K(y,z)|\,dy \le \sum_{j=-3}^{\infty} \int_{\mathcal{C}_j^l} |K(y,z)|\,dy \le C \sum_{j=-3}^{\infty} \left[\left\| k_1(\cdot - A_1 z)\chi_{\mathcal{C}_j^l} \right\|_{p_1} \cdots \left\| k_m(\cdot - A_m z)\chi_{\mathcal{C}_j^l} \right\|_{p_m} (2^{-j}R)^{n/s} \right]. \tag{6}$$

If $p_l < \infty$, then

$$\begin{aligned} \left\| k_l(\cdot - A_l z)\chi_{\mathcal{C}_j^l} \right\|_{p_l} &= \left(\int_{2^{-j-1}R \le |u| \le 2^{-j}R} \left(\frac{|\Omega_l(u)|}{|u|^{n/q_l}}\,du \right)^{p_l} \right)^{1/p_l} \\ &\le C 2^{jn/q_l} R^{-n/q_l} \left(\int_{2^{-j-1}R \le |u| \le 2^{-j}R} |\Omega_l(u)|^{p_l}\,du \right)^{1/p_l} \le C 2^{jn/q_l} R^{-n/q_l} 2^{-jn/p_l} R^{n/p_l} \|\Omega_l\|_{p_l}, \end{aligned} \tag{7}$$

where the last inequality follows since Ω_l is homogeneous of degree zero. We observe that if $p_l = \infty$ we also have

$$\left\| k_l(\cdot - A_l z)\chi_{\mathcal{C}_j^l} \right\|_{\infty} \le C 2^{jn/q_l} R^{-n/q_l} \|\Omega_l\|_{\infty}.$$

For $1 \le r \le m$, $r \ne l$, we observe that if $y \in \mathcal{C}_j^l$ then $|y - A_r z| \ge |y - A_l z| > 2^{-j-1}R$. So if $p_r < \infty$, then

$$\begin{aligned} \left\| k_r(\cdot - A_r z)\chi_{\mathcal{C}_j^l} \right\|_{p_r} &\le \left(\sum_{k \ge 0} \int_{\{2^{-j+k-1}R \le |u| \le 2^{-j+k}R\}} \left(\frac{|\Omega_r(u)|}{|u|^{n/q_r}} \right)^{p_r} \right)^{1/p_r} \\ &\le C \sum_{k \ge 0} 2^{(j-k)n/q_r} R^{-n/q_r} 2^{(-j+k)n/p_r} R^{n/p_r} \|\Omega_r\|_{p_r} \\ &\le C 2^{jn/q_ir} R^{-n/q_r} 2^{-jn/p_r} R^{n/p_r} \|\Omega_r\|_{p_r} \sum_{k \ge 0} 2^{k(n/p_r - n/q_r)} \\ &\le C 2^{jn/q_r} R^{-n/q_r} 2^{-jn/p_r} R^{n/p_r} \|\Omega_r\|_{p_r}, \end{aligned} \tag{8}$$

the last inequality follows since $p_r > q_r$. Again, if $p_r = \infty$ we get

$$\left\| k_r(\cdot - A_r z)\chi_{\mathcal{C}_j^l} \right\|_{\infty} \le C 2^{jn/q_r} R^{-n/q_r} \|\Omega_r\|_{\infty}.$$

Then from (6), (7) and (8) we obtain

$$\begin{aligned} \int_{\mathcal{C}^l} |K(y,z)|\,dy &\le C \sum_{j=-3}^{\infty} 2^{jn/q_1} R^{-n/q_1} 2^{-jn/p_1} R^{n/p_1} \|\Omega_1\|_{p_1} \cdots 2^{jn/q_m} R^{-n/q_m} 2^{-jn/p_m} R^{n/p_m} \|\Omega_m\|_{p_m} (2^{-j}R)^{n/s} \\ &\le C R^{\alpha} \|\Omega_1\|_{p_1} \cdots \|\Omega_m\|_{p_m}. \end{aligned}$$

So,

$$\mathrm{I} \le C \sum_{i=1}^{m} \frac{R^{\alpha}}{|B|} \int_{\tilde{B}_i} f(z)\,dz \le C \sum_{i=1}^{m} M_{\alpha} f(A_i^{-1}x) \le C \sum_{i=1}^{m} M_{\alpha,s} f(A_i^{-1}x).$$

On the other hand,

$$\mathrm{II} = \frac{1}{|B|}\int_B \left|T_\alpha f_2 y - T_\alpha f_2 x_B\right| dy \leq \frac{1}{|B|}\int_B \int_{\left(\bigcup_{i=1}^m \widetilde{B}_i\right)^c} |K(y,z) - K(x_B,z)| f(z)\, dz\, dy$$
$$\leq \sum_{l=1}^m \frac{1}{|B|}\int_B \int_{\mathcal{Z}_l} |K(y,z) - K(x_B,z)| f(z)\, dz\, dy,$$

where

$$\mathcal{Z}^l = \left(\bigcup_{i=1}^m \widetilde{B}_i\right)^c \cap \left\{z : |x_B - A_l z| \leq |x_B - A_r z|,\ 1 \leq r \leq m\right\}.$$

We estimate now $|K(y,z) - K(x_B,z)|$ for $y \in B$ and $z \in \mathcal{Z}^l$. It is easy to check that

$$|K(y,z) - K(x_B,z)| \leq \sum_{i=1}^m \left[\prod_{r=1}^i |k_{r-1}(x_B - A_{r-1}z)| \left|k_i(y - A_i z) - k_i(x_B - A_i z)\right| \prod_{r=i}^m \left|k_{r+1}(y - A_{r+1}z)\right|\right], \tag{9}$$

where we define $k_0 = k_{m+1} \equiv 1$.

For simplicity we estimate the first summand of (9), the other summands follow in analogous way. For $j \in \mathbb{N}$, let $\mathcal{D}_j^l = \{z \in \mathcal{Z}^l : |x_B - A_l z| \sim 2^{j+1}R\}$. We use Hölder's inequality to get

$$\int_{\mathcal{Z}^l} |k_1(y - A_1 z) - k_1(x_B - A_1 z)| \prod_{r=2}^m |k_r(y - A_r z)| f(z)\, dz$$
$$= \sum_{j=1}^\infty \int_{\mathcal{D}_j^l} |k_1(y - A_1 z) - k_1(x_B - A_1 z)| \prod_{r=2}^m |k_r(y - A_r z)| f(z)\, dz$$
$$\leq \sum_{j=1}^\infty \left\|\left(k_1(y - A_1\cdot) - k_1(x_B - A_1\cdot)\right)\chi_{\mathcal{D}_j^l}\right\|_{p_1} \prod_{r=2}^m \left\|k_r(y - A_r\cdot)\chi_{\mathcal{D}_j^l}\right\|_{p_r} \left\|f\chi_{\mathcal{D}_j^l}\right\|_s.$$

Now, if $p_l < \infty$,

$$\begin{aligned}
\left\|k_l(y - A_l\cdot)\chi_{\mathcal{D}_j^l}\right\|_{p_l} &= \left(\int_{\mathcal{D}_j^l} \frac{|\Omega_l(y - A_l z)|^{p_l}}{|y - A_l z|^{np_l/q_l}}\, dz\right)^{1/p_l} \\
&\leq C(R2^j)^{-n/q_l}\left(\int_{\{2^j R < |y - A_l z| \leq 2^{j+3}R\}} |\Omega_l(y - A_l z)|^{p_l}\, dz\right)^{1/p_l} \\
&\leq C(2^j R)^{-n/q_l + n/p_l}\left(\int_{\{1<|u|\leq 8\}} |\Omega_l(u)|^{p_l} du\right)^{1/p_l} \\
&\leq C(2^j R)^{-n/q_l + n/p_l}\|\Omega_l\|_{p_l},
\end{aligned} \tag{10}$$

where the first inequality follows since $|x_B - A_l z|/2 \leq |y - A_l z| \leq 2|x_B - A_l z|$. If $p_l = \infty$ we also get

$$\left\|k_l(y - A_l\cdot)\chi_{\mathcal{D}_j^l}\right\|_\infty \leq C(2^j R)^{-n/q_l}\|\Omega_l\|_\infty.$$

For $r \neq l$, we observe that if $z \in \mathcal{D}_l^j$ then $|x_B - A_r z| \geq |x_B - A_l z| \geq 2^{j+1}R$, so we decompose $\mathcal{D}_j^l = \bigcup_{k \geq j}\left(\mathcal{D}_j^l\right)_{k,r}$ where

$$\left(\mathcal{D}_j^l\right)_{k,r} = \left\{z \in \mathcal{D}_j^l : |x_B - A_r z| \sim 2^{k+1}R\right\}.$$

If $p_r < \infty$,

$$\begin{aligned}
\left\|k_r(y - A_r\cdot)\chi_{\mathcal{D}_j^l}\right\|_{p_r} &= \sum_{k=j+1}^\infty \left(\int_{(\mathcal{D}_j^l)_{k,r}} |k_r(y - A_r z)|^{p_r}\, dz\right)^{1/p_r} \\
&\leq C\|\Omega_r\|_{p_r} \sum_{k=j+1}^\infty (2^k R)^{-n/q_r + n/p_r} \leq C\|\Omega_r\|_{p_r}(2^j R)^{-n/q_r + n/p_r},
\end{aligned} \tag{11}$$

where the geometric sums converge since $p_r > q_r$. If $p_r = \infty$,

$$\left\|k_r(y - A_r\cdot)\chi_{\mathcal{D}_j^l}\right\|_\infty = \sum_{k=j+1}^{\infty}\left\|k_r(y - A_r\cdot)\chi_{(\mathcal{D}_j^l)_{k,r}}\right\|_\infty \le C\|\Omega_r\|_\infty(2^jR)^{-n/q_r}.$$

Now for $l = 1$,

$$\left\|\left(k_1(y - A_1\cdot) - k_1(x_B - A_1\cdot)\right)\chi_{\mathcal{D}_j^1}\right\|_{p_1} \le C\left\|\left(k_1(y - x_B + \cdot) - k_1(\cdot)\right)\chi_{|x|\sim 2^{j+1}R}\right\|_{p_1}. \tag{12}$$

Since $n/p_2 + \cdots + n/p_m - (n/q_2 + \cdots + n/q_m) = \alpha - n/s - n/p_1 + n/q_1$, then (10), (11) and (12) imply

$$\begin{aligned}
&\int_{\mathcal{Z}^1}\left|k_1(y - A_1z) - k_1(x_B - A_1z)\right|\prod_{r=2}^{m}|k_r(y - A_rz)|f(z)\,dz\\
&\quad\le C\sum_{j=1}^{\infty}(2^jR)^{n/q_1-n/p_1}\left\|\left(k_1(y - x_B + \cdot) - k_1(\cdot)\right)\chi_{|x|\sim 2^{j+1}R}\right\|_{p_1}(2^jR)^{\alpha}\left(\frac{1}{(2^jR)^n}\int_{\mathcal{D}_j^1}f^s(z)\,dz\right)^{1/s}\\
&\quad\le CM_{\alpha,s}f(A_1^{-1}x)\sum_{j=1}^{\infty}(2^jR)^{n/q_1-n/p_1}\left\|\left(k_1(y - x_B + \cdot) - k_1(\cdot)\right)\chi_{|x|\sim 2^{j+1}R}\right\|_{p_1} \le CM_{\alpha,s}f(A_1^{-1}x),
\end{aligned}$$

where the last inequality follows since $k_1 \in H_{n/q_1',p_1}$. For $l \ne 1$ we observe that

$$\begin{aligned}
&\left\|\left(k_1(y - A_1\cdot) - k_1(x_B - A_1\cdot)\right)\chi_{\mathcal{D}_j^l}\right\|_{p_1} \le \sum_{k=j+1}^{\infty}\left\|\left(k_1(y - A_1\cdot) - k_1(x_B - A_1\cdot)\right)\chi_{(\mathcal{D}_j^l)_{k,1}}\right\|_{p_1}\\
&\quad\le C\sum_{k=j+1}^{\infty}(2^kR)^{n/p_1-n/q_1}(2^kR)^{n/q_1-n/p_1}\left\|\left(k_1(y - x_B + \cdot) - k_1(\cdot)\right)\chi_{|x|\sim 2^{k+1}R}\right\|_{p_1} \le C(2^jR)^{n/p_1-n/q_1},
\end{aligned}$$

where the last inequality follows since $p_1 > q_1$ and since $k_1 \in H_{n/q_1',p_1}$. So as in the case $l = 1$ we obtain

$$\int_{\mathcal{Z}^l}\left|k_1(y - A_1z) - k_1(x_B - A_1z)\right|\prod_{r=2}^{m}|k_r(y - A_rz)|f(z)\,dz \le CM_{\alpha,s}f(A_l^{-1}x).$$

Then

$$\mathrm{II} \le C\sum_{i=1}^{m}M_{\alpha,s}f(A_i^{-1}x).$$

Now we start with the case $\alpha = 0$.

If $p_i = \infty$ for all $1 \le i \le m$, we decompose

$$\left(\frac{1}{|B|}\int_B\left|(T_0f)^\delta(y) - a^\delta\right|dy\right)^{1/\delta} \le \left(\frac{C}{|B|}\int_B(T_0f_1)^\delta(y)\,dy\right)^{1/\delta} + \left(\frac{C}{|B|}\int_B\left|(T_0f_2)^\delta(y) - a^\delta\right|dy\right)^{1/\delta} = \mathrm{I} + \mathrm{II}.$$

To estimate I we observe that

$$|T_0f(x)| \le C\int|x - A_1y|^{-n/q_1}\cdots|x - A_my|^{-n/q_m}f(y)\,dy = CTf(x). \tag{13}$$

In [11] we obtain that the operator T is of weak-type $(1,1)$ with respect to the Lebesgue measure. Thus taking $0 < \delta < 1$ and using Kolmogorov's inequality (see [7, Exercise 2.1.5, p. 91]) we get

$$\mathrm{I} \le \frac{C}{|B|}\int_{\mathbb{R}^n}f_1(y)\,dy \le \sum_{j=1}^{m}\frac{C}{|B|}\int_{\tilde{B}_j}f(y)\,dy \le C\sum_{j=1}^{m}Mf(A_j^{-1}x).$$

To estimate II, we first use Jensen's inequality and then proceed just as in the case $0 < \alpha < n$ to get

$$\mathrm{II} \le C \sum_{j=1}^{m} Mf(A_j^{-1}x),$$

and so the theorem follows in this case.

If $p_i < \infty$ for some $1 \le i \le m$, by Jensen's inequality,

$$\begin{aligned}\left(\frac{1}{|B|}\int_B \left|(T_0 f)^\delta(y) - a^\delta\right| dy\right)^{1/\delta} &\le \left(\frac{1}{|B|}\int_B |T_0 f(y) - a|\, dy\right)\\ &\le \left(\frac{1}{|B|}\int_B |T_0 f_1(y)|\, dy\right) + \left(\frac{1}{|B|}\int_B |T_0 f_2(y) - a|\, dy\right) = \mathrm{I} + \mathrm{II}.\end{aligned}$$

As in [6] it can be proved that T_0 is bounded on $L^p(\mathbb{R}^n, dx)$ for $1 < p < \infty$. So, by Hölder's inequality,

$$\mathrm{I} \le \left(\frac{1}{|B|}\int_B |T_0 f_1(y)|^p\, dy\right)^{1/p} \le C\left(\frac{1}{|B|}\int_{\mathbb{R}^n} |f_1(y)|^p\, dy\right)^{1/p} \le C \sum_{j=1}^{m} M_{0,p} f(A_j^{-1}x).$$

As before, to estimate II we proceed as in the case $0 < \alpha < n$ to get

$$\mathrm{II} \le C \sum_{j=1}^{m} M_{0,s} f(A_j^{-1}x).$$

If we chose $p = s$ the theorem follows in this case. □

3. Weighted estimates

Our next aim is to obtain weighted L^p-L^q estimates for the operator T_α and certain classes of weights. The fundamental tool to get these results is the following theorem about a Coifman type inequality.

Theorem 3.1.
Let assumptions of Theorem 2.2 on $\alpha, T_\alpha, A_i, \Omega_i$ and s hold. Let $0 < p < \infty$ and $w \in A_\infty$ satisfy (4). Then there exists $C > 0$ such that for $f \in L_c^\infty(\mathbb{R}^n, dx)$

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} |M_{\alpha,s} f(x)|^p\, w(x)\, dx,$$

always holds if the left hand side is finite.

Proof. Let $w \in A_\infty$, then there exists $r > 1$ such that $w \in A_r$. For $0 < p < \infty$ we take $0 < \delta < 1$, such that $1 < r < p/\delta$, thus $w \in A_{p/\delta}$. If $\|T_\alpha f\|_{p,w} < \infty$ then also $\|(T_\alpha f)^\delta\|_{p/\delta,w} < \infty$. Under these conditions we can apply [5, Theorem 2.20, p. 410], and from Theorem 2.2 we get

$$\begin{aligned}\int_{\mathbb{R}^n} |T_\alpha f(x)|^p\, w(x)\, dx &\le \int_{\mathbb{R}^n} (M(T_\alpha f)^\delta(x))^{p/\delta} w(x)\, dx \le C \int_{\mathbb{R}^n} (M_\delta^\#(T_\alpha f)(x))^p\, w(x)\, dx\\ &\le C \int_{\mathbb{R}^n} \left(\sum_{i=1}^{m} M_{\alpha,s} f(A_i^{-1}x)\right)^p w(x)\, dx \le C \sum_{i=1}^{m} \int_{\mathbb{R}^n} (M_{\alpha,s} f)^p(x)\, w(A_i x)\, dx \le C \int_{\mathbb{R}^n} (M_{\alpha,s} f(x))^p\, w(x)\, dx,\end{aligned}$$

where the last inequality follows since w satisfies (4). □

Lemma 3.2.

Let assumptions of Theorem 2.2 on $\alpha, T_\alpha, A_i, \Omega_i$ and s hold. Suppose $w^s \in A(p/s, q/s)$ with $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $f \in L^\infty_c(\mathbb{R}^n, dx)$ then $T_\alpha(f) \in L^q(\mathbb{R}^n, w^q)$.

Proof. The proof follows similar lines as the proof of [11, Lemma 2.2]. Since $w^s \in A(p/s, q/s)$ then $w^q \in A_r$ with $r = 1 + q/s \cdot 1/(p/s)' = q/n \cdot (n/s - \alpha)$.

Let $\mathcal{M}_j = \max\{|A_j| : |y| = 1\}$ and let $\mathcal{M} = \max_{1\le j\le m}\{\mathcal{M}_j\}$. Suppose $\operatorname{supp} f \subseteq B(0, R)$. If $|x| > 2\mathcal{M}R$ and $y \in \operatorname{supp} f$, then for $1 \le i \le m$,

$$|x - A_i y| \ge |x| - |A_i y| = |x| - |y|\left|A_i \frac{y}{|y|}\right| \ge |x| - R\mathcal{M} \ge \frac{|x|}{2},$$

so by Hölder's inequality,

$$|T_\alpha f(x)| = \left|\int k_1(x - A_1 y) \cdots k_m(x - A_m y) f(y)\, dy\right| \le \left\|k_1(x - A_1\,\cdot)\chi_{\{|x-A_1\cdot|\ge|x|/2\}}\right\|_{p_1} \cdots \left\|k_m(x - A_m\,\cdot)\chi_{\{|x-A_m\cdot|\ge|x|/2\}}\right\|_{p_m} \|f\|_s.$$

Now,

$$\begin{aligned}\left\|k_i(x - A_i\,\cdot)\chi_{\{|x-A_i\cdot|\ge|x|/2\}}\right\|_{p_i} &= \sum_{k\in\mathbb{N}}\left\|k_i(x - A_i\,\cdot)\chi_{\{|x-A_i\cdot|\sim 2^{k-2}|x|\}}\right\|_{p_i}\\ &\le C\sum_{k\in\mathbb{N}} 2^k |x|^{-n/q_i}\left\|\Omega_i\chi_{\{|\cdot|\sim 2^{k-2}|x|\}}\right\|_{p_i} \le \sum_{k\in\mathbb{N}} 2^k |x|^{-n/q_i + n/p_i}\|\Omega_i\|_{p_i} = C|x|^{-n/q_i+n/p_i}\|\Omega_i\|_{p_i}.\end{aligned}$$

So,

$$|T_\alpha f(x)| \le C|x|^{\sum_{i=1}^m -n/q_i + n/p_i}\|\Omega_1\|_{p_1}\cdots\|\Omega_m\|_{p_m}\|f\|_s = C|x|^{\alpha - n/s}\|f\|_s.$$

Thus

$$\begin{aligned}\int_{|x|>2\mathcal{M}R} |T_\alpha f(x)|^q w^q(x)\, dx &= \sum_{k\in\mathbb{N}} \int_{|x|\sim 2^k\mathcal{M}R} |T_\alpha f(x)|^q w^q(x)\, dx\\ &\le C\sum_{k\in\mathbb{N}} \int_{|x|\sim 2^k\mathcal{M}R} |x|^{(\alpha-n/s)q} w^q(x)\, dx \le C\sum_{k\in\mathbb{N}} (2^k\mathcal{M}R)^{(\alpha-n/s)q} w^q\big(B(0, 2^{k+1}\mathcal{M}R)\big).\end{aligned}$$

Since $w^q \in A_r$, there exists $\tilde{r} < r = q/n\cdot(n/s - \alpha)$ such that $w^q \in A_{\tilde{r}}$ so $w^q\big(B(0, 2^{k+1}\mathcal{M}R)\big) \le C2^{kn\tilde{r}}$ (see [5, Lemma 2.2]) so the last sum is finite. To study

$$\int_{|x|\le 2\mathcal{M}R} |T_\alpha f(x)|^q w^q(x)\, dx,$$

we recall that in [6] the authors obtain the boundedness of T_α from $L^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$ for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, and so it is left to continue the proof as in [11]. □

We are now ready to prove the weighted boundedness result.

Theorem 3.3.

Let assumptions of Theorem 2.2 on $\alpha, T_\alpha, A_i, \Omega_i$ and s hold. Suppose w satisfies (4) and $w^s \in A(p/s, q/s)$ with $s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then there exits $C > 0$ such that for $f \in L^\infty_c(\mathbb{R}^n, dx)$,

$$\left(\int_{\mathbb{R}^n} |T_\alpha f(x)|^q w^q(x)\, dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\, dx\right)^{1/p}.$$

Proof. Since $w^s \in A(p/s, q/s)$ for $1/q = 1/p - \alpha/n$ then $w^q \in A_r \subset A_\infty$, with $r = q/n \cdot (n/s - \alpha)$. By Lemma 3.2 we have that $T_\alpha f \in L^q(\mathbb{R}^n, w^q)$. Moreover we recall that $w^s \in A(p/s, q/s)$ implies that $M_{\alpha s}$ is bounded from $L^{p/s}(\mathbb{R}^n, w^{p/s})$ into $L^{q/s}(\mathbb{R}^n, w^{q/s})$, so we apply Theorem 3.1 to obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |T_\alpha f(x)|^q w^q(x)\,dx\right)^{1/q} &\le C\left(\int_{\mathbb{R}^n} (M_{\alpha,s} f(x))^q w^q(x)\,dx\right)^{1/q}\\ &= C\left(\int_{\mathbb{R}^n} (M_{\alpha s}|f(x)|^s)^{q/s} w^q(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\,dx\right)^{1/p}. \qquad \square\end{aligned}$$

By a standard duality argument we obtain the following theorem.

Theorem 3.4.

Let assumptions of Theorem 2.2 on $\alpha, T_\alpha, A_i, \Omega_i$ and s hold. Suppose w satisfies $w^{-1}(A_i^{-1}x) \le C w^{-1}(x)$ for all $1 \le i \le m$ and $w^{-s} \in A(q'/s, p'/s)$ with $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $q < s'$. Then there exits $C > 0$ such that for $f \in L_c^\infty(\mathbb{R}^n, dx)$,

$$\left(\int_{\mathbb{R}^n} |T_\alpha f(x)|^q w^q(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\,dx\right)^{1/p}.$$

Proof. We observe that the adjoint T_α^* of the operator T_α is the integral operator with kernel

$$\widetilde{K}(x,y) = \widetilde{k_1}(x - A_1^{-1}y)\cdots\widetilde{k_m}(x - A_m^{-1}y),$$

where for $1 \le i \le m$

$$\widetilde{k_i}(x) = \frac{\widetilde{\Omega}_i(x)}{|A_i x|^{n/qi}} = \frac{\overline{\Omega}_i(-A_i x)}{|A_i x|^{n/qi}}.$$

It is easy to check that $\widetilde{\Omega}_i$ satisfies (H_1) and (H_2) and also that $\widetilde{k_i} \in H_{n/q_i', p_i}$ for all $1 \le i \le m$. We take g with $\|g\|_{q', w^{-q'}} \le 1$, thus

$$\int_{\mathbb{R}^n} T_\alpha f(x) g(x)\,dx = \int_{\mathbb{R}^n} f(x) T_\alpha^* g(x)\,dx.$$

Hence

$$\|T_\alpha f\|_{q,w^q} = \sup_g \left|\int_{\mathbb{R}^n} f(x) T_\alpha^* g(x)\,dx\right| \le \|f\|_{p,w^p} \sup_g \|T_\alpha^* g\|_{p', w^{-p'}}.$$

Since $1/q = 1/p - \alpha/n$ and $1 < p < q < s'$ then $1/p' = 1/q' - \alpha/n$ and $s < q' < n/\alpha$, so as in Theorem 3.3 we obtain

$$\|T_\alpha^* g\|_{p', w^{-p'}} \le C\|g\|_{q', w^{-q'}} \le C, \qquad \text{and so} \qquad \|T_\alpha f\|_{q,w^q} \le C\|f\|_{p,w^p}. \qquad \square$$

We now obtain an estimate of the type (5) for the operator T_α and for certain weights in the class $A(n/\alpha, \infty)$.

Theorem 3.5.

Let assumptions of Theorem 2.2 on $\alpha, T_\alpha, A_i, \Omega_i$ and s hold. Suppose $w^s \in A(n/\alpha s, \infty)$ and satisfies (4), then there exits $C > 0$ such that for $f \in L_c^\infty(\mathbb{R}^n, dx)$,

$$|||T_\alpha f|||_w \le C\left(\int (|f(x)| w(x))^{n/\alpha}\,dx\right)^{\alpha/n}.$$

Proof. We observe that if

$$w^s \in A\left(\frac{n}{\alpha s},\infty\right) \qquad \text{then} \qquad \|wM_{\alpha,s}f\|_\infty \le C\|fw\|_{n/\alpha}. \tag{14}$$

Indeed, by Hölder's inequality we get

$$\frac{1}{|B|^{1-\alpha s/n}}\int_B |f(x)|^s\,dx \le \frac{1}{|B|^{1-\alpha s/n}}\left(\int_B |f(x)|^{n/\alpha}w^{n/\alpha}(x)\,dx\right)^{\alpha s/n}\left(\int_B w^{-s(n/\alpha s)'}(x)\,dx\right)^{1/(n/\alpha s)'}.$$

Then, for $x\in B$, since $w^s\in A(n/(\alpha s),\infty)$ we get

$$\begin{aligned} w(x)\left(\frac{1}{|B|^{1-\alpha s/n}}\int_B |f(x)|^s\,dx\right)^{1/s} &\le \left(\int_B |f(x)|^{n/\alpha}w^{n/\alpha}(x)\,dx\right)^{\alpha/n}\|w^s\chi_B\|_\infty^{1/s}\left(\frac{1}{|B|}\int_B w^{-s(n/\alpha s)'}(x)\,dx\right)^{1/(n/\alpha s)'s}\\ &\le C\left(\int_{\mathbb{R}^n}|f(x)|^{n/\alpha}w^{n/\alpha}(x)\,dx\right)^{\alpha/n}, \end{aligned}$$

thus $w(x)M_{\alpha,s}f(x)\le C\|fw\|_{n/\alpha}$, and (14) follows. Now, using Theorem 2.2 and (14), we get

$$\begin{aligned} \|\|T_\alpha f\|\|_w \simeq \|wM^\# T_\alpha f\|_\infty &\le C\sum_{i=1}^m \left\|wM_{\alpha,s}f(A_i^{-1}\cdot)\right\|_\infty \le C\sum_{i=1}^m\left(\int |f(A_i^{-1}x)w(x)|^{n/\alpha}\,dx\right)^{\alpha/n}\\ &\le C\sum_{i=1}^m\left(\int |f(x)w(A_ix)|^{n/\alpha}\,dx\right)^{\alpha/n} \le C\left(\int |f(x)w(x)|^{n/\alpha}\,dx\right)^{\alpha/n}, \end{aligned}$$

where the last inequality follows since w satisfies hypothesis (4). □

Finally we prove that T_α satisfies a weighted weak type $(1, n/(n-\alpha))$ estimate for certain weights in $A(1, n/(n-\alpha))$.

Theorem 3.6.

Let the assumptions of Theorem 2.2 on $\alpha, T_\alpha, A_i, \Omega_i$ and s hold. Suppose $w^s\in A(1,n/(n-\alpha s))$ and satisfies (4), then there exists $C>0$ such that for $f\in L_c^\infty(\mathbb{R}^n,dx)$,

$$\sup_{\lambda>0}\lambda\left(w^{sn/(n-\alpha s)}\{x:|T_\alpha f(x)|>\lambda\}\right)^{(n-\alpha s)/sn}\le C\left(\int |f(x)|^s w^s(x)\,dx\right)^{1/s}.$$

Proof. Given $w\in A_\infty$, there exists $\beta>0$ and $C>0$ such that

$$w\left\{x: Mf(x)>2\lambda,\ M^\# f(x)\le\gamma\lambda\right\}\le C\gamma^\beta w\{x:Mf(x)>\lambda\},$$

for any $\gamma>0$ (see [4, p. 146]). For $q\ge 1$, as in [11, Theorem 3.2], we obtain that

$$\sup_{\lambda>0}\lambda^q w\{x:Mf(x)>\lambda\}\le C\sup_{\lambda>0}\lambda^q w\{x:M^\# f(x)>\gamma\lambda\},$$

for some $\gamma>0$. We consider first the case $s>1$. If $w^s\in A(1,n/(n-\alpha s))$ then $w^{sn/(n-\alpha s)}\in A_\infty$. So for $q=sn/(n-\alpha s)$, we obtain

$$\begin{aligned} \sup_{\lambda>0}\lambda\left(w^{sn/(n-\alpha s)}\{x:|T_\alpha f|(x)>\lambda\}\right)^{(n-\alpha s)/(sn)} &\le C\sup_{\lambda>0}\lambda\left(w^{sn/(n-\alpha s)}\{x:MT_\alpha f(x)>\lambda\}\right)^{(n-\alpha s)/sn}\\ &\le C\sup_{\lambda>0}\lambda\left(w^{sn/(n-\alpha s)}\{x:M^\# T_\alpha f(x)>\gamma\lambda\}\right)^{(n-\alpha s)/sn}\\ &\le C\sup_{\lambda>0}\lambda\left(w^{sn/(n-\alpha s)}\left\{x:\sum_{i=1}^m M_{\alpha,s}f(A_i^{-1}x)>C\gamma\lambda\right\}\right)^{(n-\alpha s)/sn}, \end{aligned}$$

where the last inequality follows from Theorem 2.2, with $\delta = 1$. Since w satisfies (4), it is easy to check that

$$w^{sn/(n-\alpha s)}\{x : M_{\alpha,s}f(A_i^{-1}x) > \lambda\} \leq C_i w^{sn/(n-\alpha s)}\{x : M_{\alpha,s}f(x) > \lambda\},$$

so

$$\sup_{\lambda>0} \lambda\big(w^{sn/(n-\alpha s)}\{x : |T_\alpha f|(x) > \lambda\}\big)^{(n-\alpha s)/sn} \leq C \sup_{\lambda>0} \lambda\big(w^{sn/(n-\alpha s)}\{x : M_{\alpha,s}f(x) > \lambda\}\big)^{(n-\alpha s)/sn}$$
$$\leq C \sup_{\lambda>0} \lambda\big(w^{sn/(n-\alpha s)}\{x : M_{\alpha s}|f|^s(x) > \lambda^s\}\big)^{(n-\alpha s)/sn} \leq C\left(\int |f(x)|^s w^s(x)\,dx\right)^{1/s},$$

where the last inequality follows since $w^s \in A(1, n/(n-\alpha s))$, and since $M_{\alpha s}$ is of weak type $(1, n/(n-\alpha s))$. If $s = 1$, T_α is bounded by the operator T defined in (13) so we proceed as in the proof of [11, Theorem 3.2]. □

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