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Kernels of representations of Drinfeld doubles of finite groups

Research Article

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Abstract: A description of the commutator of a normal subcategory of the fusion category of representation $\operatorname{Rep} A$ of a semisimple Hopf algebra A is given. Formulae for the kernels of representations of Drinfeld doubles $D(G)$ of finite groups G are presented. It is shown that all these kernels are normal Hopf subalgebras.

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Introduction

In [1] the notion of a normal fusion subcategory of a fusion category was introduced as a categorification of the notion of a normal subgroup of a group. For the fusion category $\operatorname{Rep} A$ of a semisimple Hopf algebra it follows that a fusion subcategory is normal if and only if it is of the form $\operatorname{Rep}(A/\!/L)$ where L is a normal Hopf subalgebra of A [1, Lemma 2.10].

Also in [7] the authors introduced the notion of commutator of a fusion subcategory of a given fusion category, and proved that if $\mathcal{D} \subset \mathcal{C}$ then its commutator $\mathcal{D}^{co}$ is a fusion subcategory if the Grothendieck ring of $\mathcal{C}$ is commutative. In this paper we show that the commutator $\mathcal{D}^{co}$ is a fusion subcategory for any normal fusion subcategory $\mathcal{D} = \operatorname{Rep}(A/\!/L)$ of the fusion category $\mathcal{C} = \operatorname{Rep} A$. Moreover an explicit formula for $\mathcal{D}^{co}$ is given in Theorem 2.2.

Recently the author introduced in [3] the notion of kernel of representation of a semisimple Hopf algebra, which generalizes the notion of kernel of group representations. It was proven that if the character of the representation is central

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in the dual Hopf algebra then the kernel is a normal Hopf subalgebra. It is not known whether the kernel of an arbitrary representation is in general a normal Hopf subalgebra. Hopf algebras having the property that all kernels of representations are normal are studied in [4]. It is shown that if a Hopf algebra has this property then its dual also has this property.

In this paper we prove that in the case $D(G)$, of a Drinfeld double of a finite group G, all representations have the kernels normal Hopf subalgebras of $D(G)$. In order to do this we give an explicit description of kernels of all irreducible representations of $D(G)$. Note that the structure of all Hopf subalgebras of $D(G)$ can be deduced from [5, Theorem 4.1]. In order to be able to handle the kernels of representations here we give a different description of these Hopf subalgebras and show that this description coincides with the description given in Theorem 4.1 of the above paper.

Fusion subcategories of the category $\operatorname{Rep} D(G)$ of a finite group G were recently studied in [15]. They are parameterized in terms of a pair of commuting subgroups of G and a G-invariant bicharacter defined on their product. The main ingredient used in [15] is the notion of centralizer for a fusion subcategory introduced in [13]. In this paper we will identify from the above mentioned parametrization all normal fusion subcategories of $\operatorname{Rep} D(G)$, i.e. those that are of the form $\operatorname{Rep}(D(G)/\!\!/L)$ where L is a normal Hopf subalgebra of $D(G)$. We show that their associated bicharacters satisfy a stronger condition than that of G-invariance given in [15].

This paper is organized as follows. The first section recalls some basic results on semisimple Hopf algebras and kernels of their representations that are needed in the paper. Kernels and centers of irreducible representations of the dual Hopf algebra $\mathbb{C}G^*$ are computed here.

The second section is concerned with fusion subcategories of $\operatorname{Rep} A$ for a semisimple Hopf algebra A. The notion of a normal fusion subcategory from [1] is also recalled here. Next the formula for the commutator of a normal fusion subcategory of $\operatorname{Rep} A$ is given. Other necessary and sufficient conditions for a simple A-module to be in the commutator of a normal fusion subcategory are stated in Corollary 2.4.

In Section 3 a basis of central characters of $\mathcal{Z}(D(G)) \cap C(D(G)^*)$ is given in Theorem 3.6. It is used in the next section to decide when a Hopf subalgebra of $D(G)$ is a normal Hopf subalgebra.

Section 4 describes the Hopf subalgebras of $D(G)$ following the method from [5]. Also it introduces a new class of Hopf subalgebras of $D(G)$ and shows that this class covers all Hopf subalgebras of $D(G)$. In subsection 4.4 a description of all normal Hopf subalgebras of $D(G)$ is given and it is shown that the kernel of any representation of $D(G)$ is normal.

In the last section the parametrization from [15] of fusion subcategories of $\operatorname{Rep} D(G)$ is recalled. Then using the results of the previous section all normal fusion subcategories of $\operatorname{Rep} D(G)$ are identified.

We work over the algebraically closed field $k = \mathbb{C}$. For a vector space V the dimension $\dim_{\mathbb{C}} V$ is denoted by $|V|$. We use Sweedler's notation $\Delta(x) = \sum x_1 \otimes x_2$ for comultiplication. All the other Hopf notation is that used in [12].

1. Preliminaries

1.1. General conventions

Let A be a semisimple Hopf algebra over $\mathbb{C}$. Then A is finite dimensional and also cosemisimple [10]. If K is a Hopf subalgebra of H then K is also a semisimple and cosemisimple Hopf algebra [12]. For any two subcoalgebras C and D of H we denote by CD the subcoalgebra of H generated as k-vector space by all elements of the type cd with $c \in C$ and $d \in D$.

Let $G_0(A)$ be the Grothendieck group of the category of left A-modules. Then $G_0(A)$ becomes a ring under the tensor product of modules and $C(A) = G_0(A) \otimes_{\mathbb{Z}} k$ is a semisimple subalgebra of A^* [20]. Moreover, $C(A)$ has a vector space basis given by the set $\operatorname{Irr} A$ of irreducible characters of A and $C(A) = \operatorname{Cocom} A^*$, the space of cocommutative elements of A^*. By duality, the character ring of A^* is a semisimple subalgebra of A and $C(A^*) = \operatorname{Cocom} A$. If M is an A-module with character χ then M^* is also an A-module with character $\chi^* = \chi \circ S$. This induces an involution "$\,*\,$": $C(A) \to C(A)$ on $C(A)$. Let $m_A(\chi, \mu)$ be the multiplicity form on $C(A)$. For $d \in \operatorname{Irr} A^*$ denote by C_d the simple subcoalgebra of A whose character as A^*-module equals d [9]. Denote by t_A the integral in A^* with $t_A(1) = |A|$. It is known that t_A is also the regular character of A [12]. Let also Λ_A be the idempotent integral of A. Thus $t_A(\Lambda_A) = 1$.

1.2. The subcoalgebra associated to a comodule

Let W be a right H-comodule. Since H is finite dimensional it follows that W is a left H^*-module via the module structure $f.w = f(w_1)w_0$ where $\rho(w) = w_0 \otimes w_1$ is the given right H-comodule structure of W. Then one can associate to W a subcoalgebra of H denoted by C_W [9]. This is the minimal subcoalgebra of H with the property that $\rho(W) \subset W \otimes C_W$ Moreover, it can be shown that $C_W = (\mathrm{Ann}_{H^*} W)^{\perp}$ and C_W is called the subcoalgebra of H associated to the right H-comodule W.

If W is simple right H-comodule (or equivalently W is an irreducible H^*-module) then the associated subcoalgebra C_W is a co-matrix coalgebra. More precisely, if $\dim W = q$ then $\dim C_W = q^2$ and it has a k-linear basis given by x_{ij} with $1 \le i, j \le q$. The coalgebra structure of C_W is then given by $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$ for all $1 \le i, j \le q$. Moreover the irreducible character $d \in C(H^*)$ of W is given by formula $d = \sum_{i=1}^{q} x_{ii}$. It is easy to check that W is an irreducible H^*-module if and only if C_W is a simple subcoalgebra of H. This establishes a canonical bijection between the set $\mathrm{Irr}\, H^*$ of simple right H^*-comodules and the set of simple subcoalgebras of H. For any irreducible character $d \in \mathrm{Irr}\, H^*$ we denote by C_d be the simple subcoalgebra of H associated to the character d (see also [9]).

1.3. Kernels of characters for semisimple Hopf algebras

Let M be a representation of A which allows the character χ. Define $\ker_A \chi$ as the set of all irreducible characters $d \in \mathrm{Irr}\, A^*$ which act as the scalar $\epsilon(d)$ on M. Then [3, Proposition 1.2] implies that

$$\ker_A \chi = \{d \in \mathrm{Irr}\, A^* : \chi(d) = \epsilon(d)\chi(1)\}.$$

Similarly, let $\mathrm{z}_A \chi$ be the the set of all irreducible characters $d \in \mathrm{Irr}\, A^*$ which act as a scalar $\alpha\epsilon(d)$ on M, where α is a root of unity. Then from the proposition cited above it follows

$$\mathrm{z}_A \chi = \{d \in \mathrm{Irr}\, A^* : |\chi(d)| = \epsilon(d)\chi(1)\}.$$

Clearly, $\ker_A \chi \subset \mathrm{z}_A \chi$. Since the sets $\ker_A \chi$ and $\mathrm{z}_A \chi$ are closed under multiplication and " $*$ " they generate Hopf subalgebras of A denoted by A_χ and $\mathrm{Z}_A \chi$, respectively (see [3]).

Recall that a Hopf subalgebra L of H is called a normal Hopf subalgebra if it is stable under the left and right adjoint action of A on itself. When H is a semisimple Hopf algebra it is equivalent for L to be closed only under the left adjoint action, i.e. $h_1 L S(h_2) \subset L$ for any $h \in H$. Let $L^+ = L \cap \ker \epsilon$ and set $H /\!/ L = H/HL^+$. Since HL^+ is a Hopf ideal of H (see [12]) it follows that $H/\!/L$ is a quotient of Hopf algebra of H. Moreover, $(H/\!/L)^*$ can be regarded as a Hopf subalgebra of H^* via the dual map of the canonical Hopf projection $\pi_L \colon H \to H/\!/L$.

Remark 1.1.

Suppose that K is a Hopf subalgebra of a semisimple Hopf algebra A via $i \colon K \hookrightarrow A$. The restriction functor from A-modules to K-modules induces a map $\mathrm{res} \colon C(A) \to C(K)$. It is easy to see that $\mathrm{res} = i^* \lceil_{C(A)}$, the restriction of the dual map $i^* \colon A^* \to K^*$ to the subalgebra of characters $C(A) \subset A^*$.

Remark 1.2.

Let B be a normal Hopf subalgebra of a semisimple Hopf algebra A and χ a character of A allowing the representation M_χ. Then M_χ is a representation of $A/\!/B$ if and only if $A_\chi \supset B$.

1.4. Examples: kernels and centers in $\mathbb{C}G^*$

In this subsection we apply the previous constructions of $\ker_A \chi$ and $\mathrm{z}_A \chi$ for $A = \mathbb{C}G^*$. Note that in this case one has $\mathrm{Irr}\, A \cong G$.

Let $\chi \in \mathrm{Irr}\, G$. It follows that $h \in \ker_{\mathbb{C}G} \chi$ if and only if $N(h) \subset \ker_{\mathbb{C}G} \chi$ where $N(h)$ is the smallest normal subgroup of G containing the group element h.

Lemma 1.3.
For any $h \in G$ one has $\mathbb{C}G_h^* = \mathbb{C}[G/N(h)]^*$.

Proof. By its definition $\mathbb{C}G_h^*$ is determined by all the characters $\chi \in \operatorname{Irr} G$ such that $\chi(h) = \chi(1)$. These are precisely the characters of $G/N(h)$. □

Lemma 1.4.
For any $h \in G$ one has $Z_{\mathbb{C}G^*}h = \mathbb{C}[G/[G, N(h)]]^*$.

Proof. By its definition Z_h is generated by the set of irreducible characters χ of G with the property that $\chi(h) = \omega\chi(1)$ for some root of unity ω. Since $N(h)$ is generated by the conjugacy class of h it follows that every element of $N(h)$ acts as a scalar on the representation M_χ allowed by χ. Then if $n \in N(h)$ and $g \in G$ it follows that $gng^{-1}n^{-1}$ acts as identity on M_χ. Thus $\chi \in \operatorname{Irr}(G/[G, N(h)])$. Conversely, for any $\chi \in \operatorname{Irr}(G/[G, N(h)])$ one has that left multiplication by any $n \in N(h)$ is a morphism of $\mathbb{C}G$ modules and Schur's lemma implies the conclusion. □

2. On the commutator of a normal fusion subcategory of $\operatorname{Rep} A$

2.1. Fusion categories

In this subsection we recall some basic facts on fusion categories from [6, 7]. As usually, by a fusion category we mean a k-linear semisimple rigid tensor category $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and such that the unit object $1_{\mathcal{C}}$ of $\mathcal{C}$ is simple. We refer the reader to [6] for a general theory of such categories.

Let $\mathcal{C}$ be a fusion subcategory and $\mathcal{O}(\mathcal{C})$ be its set of simple objects considered up to isomorphism. Recall that a fusion subcategory $\mathcal{D}$ of $\mathcal{C}$ is a full abelian replete subcategory of $\mathcal{C}$ such that if $X, Y \in \mathcal{D}$ then $X^* \in \mathcal{D}$ and $X \otimes Y \in \mathcal{D}$. For a set X of objects of $\mathcal{C}$ let $\langle X \rangle$ be the fusion subcategory of $\mathcal{C}$ generated by the set X. Recall that this means that $\langle X \rangle$ is the smallest fusion subcategory of $\mathcal{C}$ whose set of objects is containing the set $X \cup X^*$ where $X^* = \{x^* : x \in X\}$.

2.2. Fusion subcategories of $\operatorname{Rep} A$

Let A be a semisimple Hopf algebra. It is known that $\operatorname{Rep} A$ is a fusion category. Moreover there is a maximal central Hopf subalgebra $K(A)$ of A such that $(\operatorname{Rep} A)_{\mathrm{ad}} = \operatorname{Rep}(A/\!/K(A))$, see [7]. Since $K(A)$ is commutative it follows that $K(A) = (\mathbb{C}U_A)^*$ where U_A is the universal grading group of $\operatorname{Rep} A$. For example if $A = \mathbb{C}G$ then $K(A) = \mathbb{C}\mathcal{Z}(G)$ and $U_A = \widehat{\mathcal{Z}(G)}$, the linear dual group of the center $\mathcal{Z}(G)$ of G.

Let $\mathcal{D}$ be a fusion subcategory of $\operatorname{Rep} A$ and $\mathcal{O}(\mathcal{D})$ be its set of objects. Then $I_{\mathcal{D}} = \bigcap_{V \in \mathcal{O}(\mathcal{D})} \operatorname{Ann}_A V$ is a Hopf ideal in A [18] and $\mathcal{D} = \operatorname{Rep}(A/I_{\mathcal{D}})$. For a fusion category $\mathcal{D} \subset \operatorname{Rep} A$ define its regular character as $r_{\mathcal{D}} = \sum_{X \in \operatorname{Irr}\mathcal{D}} \dim_{\mathbb{C}} X \chi_X$ where $\operatorname{Irr}\mathcal{D}$ is the set of irreducible objects of $\mathcal{D}$ and χ_X is the character of X as A-module. Thus $r_{\mathcal{D}} \in C(A)$.

2.3. Normal fusion subcategory

Recall [1] that a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is called normal if there is a normal tensor functor $F \colon \mathcal{C} \to \mathcal{E}$ such that $\mathcal{D} = \ker F$. By its definition, $\ker F$ is the fusion subcategory of $\mathcal{C}$ generated by the set of all objects $X \in \mathcal{C}$ with $F(X) = (\operatorname{FPdim} X)1_{\mathcal{D}}$. For the definition of the Frobenius–Perron dimension $\operatorname{FPdim} X$ one has to consult [6]. It follows from [1, Lemma 2.10] that a normal fusion subcategory of $\operatorname{Rep} A$ has to be of the type $\operatorname{Rep}(A/\!/L)$ with L a normal Hopf subalgebra of A.

2.4. The commutator subcategory of a normal fusion subcategory

Recall the notion of commutator subcategory from [7]. If $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$ then $\mathcal{D}^{co}$ is the full abelian subcategory of $\mathcal{C}$ generated by objects X such that $X\otimes X^* \in \mathcal{O}(\mathcal{D})$. In the same paper [7] it is proved that $\mathcal{D}^{co}$ is a fusion subcategory if the Grothendieck ring of $\mathcal{C}$ is commutative. In this section we show that $\mathcal{D}^{co}$ is a fusion subcategory for any normal fusion subcategory $\mathcal{D} = \operatorname{Rep}(A/\!/L)$ of $\operatorname{Rep} A$.

Example 2.1.

Let $\mathcal{C} = \operatorname{Rep} G$ be the category of finite dimensional representations of a finite group G and $\mathcal{D} = \operatorname{Rep}(G/N)$ for a normal subgroup N of G. Then it is known that $\mathcal{D}^{co} = \operatorname{Rep}(G/[G,N])$ where $[G,N]$ is the commutator subgroup generated by $gng^{-1}n^{-1}$ with $n\in N$ and $g\in G$ [7].

In this subsection we will describe the category $\operatorname{Rep}(A/\!/L)^{co}$ for any normal Hopf subalgebra L of A. For $a,b\in A$ define $[a,b]=ab-ba$ the usual commutator. Then for L, a Hopf subalgebra of A, let $[A,L]$ be the ideal generated by $[a,l]$ with $a\in A$ and $l \in L$.

Theorem 2.2.

Let L be a normal Hopf subalgebra of A. Then $[A,L]$ is a Hopf ideal of A and

$$\operatorname{Rep}(A/\!/L)^{co} = \operatorname{Rep}(A/[A,L]).$$

Proof. To see that $[A,L]$ is a Hopf ideal note that

$$\Delta([a,l]) = \sum a_1l_1\otimes a_2l_2 - \sum l_1a_1\otimes l_2a_2 = \sum (a_1l_1 - l_1a_1)\otimes a_2l_2 + \sum l_1a_1\otimes (a_2l_2 - l_2a_2).$$

Now consider M an irreducible $A/[A,L]$-module allowing the character $\mathcal{X}\in C(A)$. Since $(la).m = (al).m$ for all $a\in A$, $l\in L$ and $m\in M$ it follows that left multiplication by l on M is a morphism of A-module. Schur's Lemma implies that each $l\in L$ acts by a scalar on M. Thus $\mathcal{X}\downarrow_L^A = \mathcal{X}(1)\psi$ for some linear character ψ of L. Then $(\mathcal{X}\mathcal{X}^*)\downarrow_L^A = \mathcal{X}(1)^2\psi\psi^{-1} = \mathcal{X}(1)^2\epsilon_L$.

Conversely, suppose that $M\otimes M^*$ is a trivial L-module for an irreducible A-module. Let $A = I_M \oplus \operatorname{Ann}_A M$ be the decomposition of A in two-sided ideals where I_M is the minimal ideal in A corresponding to M. It is well known (see [19] for example) that the minimal ideal I_M in A corresponding to M satisfies $I_M \cong M\otimes M^*$ where I_M is regarded as A-module by the adjoint action. Therefore $l_1xSl_2 = \epsilon(l)x$ for all $x\in I_M$ and $l\in L$. Then $lx = (l_1xS(l_2))l_3 = xl$ for all $x\in I_M$ and $l\in L$. Then $(la-al)m = 0$ for all $a\in A$. Indeed if $a\in \operatorname{Ann}_A M$ this is clear since $(la).m = (al).m = 0$. On the other hand if $a\in I_M$ then $al-la=0$ and thus $(al-la).m=0$. This shows that $M\in \operatorname{Rep}(A/[A,L])$. □

Definition 2.3.

Let L be a normal Hopf subalgebra of A. An irreducible character α of L is called A-stable if there is a character $\mathcal{X}\in \operatorname{Rep} A$ such that $\mathcal{X}\downarrow_L^A = \mathcal{X}(1)\alpha/\alpha(1)$. Such a character $\mathcal{X}$ is said to sit over α.

Denote by $G_{st}(L)$ the set of all A-stable linear characters of L. Clearly $G_{st}(L)$ is a finite subgroup of the group $G(L^*)$. We also have the following characterization for simple objects $M\in \mathcal{O}(\operatorname{Rep}(A/\!/L)^{co})$.

Corollary 2.4.

Let M be an irreducible module of A allowing a character $\mathcal{X}$. Then the following are equivalent:

1) *$M\in \mathcal{O}(\operatorname{Rep}(A/\!/L)^{co})$,*
2) *$\mathcal{X}\downarrow_L^A = \mathcal{X}(1)\alpha$ for some A-stable linear character α of L,*
3) *L acts trivially on some tensor power $M^{\otimes n}$ of M.*

Proof. 1) ⇒ 2) Suppose that $M \in \mathcal{O}(\mathrm{Rep}(A/\!/L)^{\mathrm{co}})$ and let $\chi\downarrow_L^A = \sum_{\alpha\in\mathrm{Irr}\,K} m_\alpha \alpha$. Since $m_L(\epsilon_L, \alpha\beta^*) = \delta_{\alpha,\beta}$, counting the multiplicity of ϵ_L in $\chi\downarrow_L^A \chi^*\downarrow_L^A$ implies that $\chi\downarrow_L^A = \chi(1)\alpha$ for an L-linear character α. If n is the order of α in $G(L^*)$ then clearly $\chi^n\downarrow_L^A = \chi(1)^n\epsilon_L$ which shows that L acts trivially on $M^{\otimes n}$.

2) ⇒ 3) If $\chi\downarrow_L^A = \chi(1)\alpha$ then clearly $\chi^n = \chi(1)^n\epsilon_L$ where n is the order of α in $G_{\mathrm{st}}(L)$.

3) ⇒ 2) Suppose that L acts trivially on some tensor power $M^{\otimes n}$ of M. This means that $\chi^n\downarrow_A^L = \chi(1)^n\epsilon_L$. Let

$$\chi\downarrow_L^A = \sum_{\alpha\in\mathrm{Irr}\,L} m_\alpha \alpha$$

for some nonnegative integers m_α and

$$\chi^{n-1}\downarrow_L^A = \sum_{\alpha\in\mathrm{Irr}\,L} n_\alpha \alpha$$

for some nonnegative integers n_α. Recall from [16] that $m_L(\epsilon_L, \alpha\beta) > 0$ if and only if $\alpha = \beta^*$. Since $\chi^n\downarrow_L^A = \chi(1)^n\epsilon_L$ this implies that $\chi\downarrow_L^A = \chi(1)\alpha/\alpha(1)$ and $\chi^{n-1}\downarrow_L^A = \chi(1)^{n-1}\alpha^*/\alpha(1)$ for a fixed character α. It follows by counting the multiplicity of ϵ_L in $\chi^n\downarrow_L^A$ that $\alpha(1) = 1$. Thus α is an A-stable linear character of L.

2) ⇒ 1) One has $\alpha^* = \alpha^{-1}$ and therefore $(\chi\chi^*)\downarrow_L^A = \chi\downarrow_L^A\chi^*\downarrow_L^A = \chi(1)^2\alpha\alpha^* = \chi(1)^2\epsilon_L$. □

3. Kernels of representations of $D(G)$ and its central characters

3.1. The Drinfeld double $D(G)$

Let $D(G)$ be the Drinfeld double of a finite group G over the complex field $\mathbb{C}$. Recall that as a coalgebra $D(G) = \mathbb{C}G^{*\,\mathrm{cop}} \otimes \mathbb{C}G$ and the multiplication is given by

$$(p_x \bowtie g)(p_y \bowtie h) = p_x p_{gyg^{-1}} \bowtie gh = \delta_{x,gyg^{-1}} p_x \bowtie gh.$$

Moreover, the antipode is given by the formula

$$S(p_x \bowtie g) = g^{-1} p_{x^{-1}} = p_{g^{-1}x^{-1}g}\, g^{-1}.$$

A vector space basis for $D(G)$ is given by $\{p_x \bowtie y\}_{x,y\in G}$ where $\{p_x\}_{x\in G}$ is the dual basis of the basis of $\mathbb{C}G$ given by the group elements.

3.2. Irreducible representations of $D(G)$

Let $\mathcal{R}$ be a set of representatives of conjugacy classes of G. Then the irreducible representations of $D(G)$ are parameterized by pairs (a, γ) where $a \in \mathcal{R}$ and $\gamma \in \mathrm{Irr}\, C_G(a)$ is an irreducible character of the centralizer $C_G(a)$ of a in G. Their characters are denoted by $\widehat{(a, \gamma)}$ respectively.

3.3. Some results on group representations

In this subsection we give some results on group representations that are needed in the sequel. Let H be a subgroup of G. Denote by $\mathrm{core}_G H$ the largest normal subgroup of G contained in H. Then $\mathrm{core}_G H = \bigcap_{g\in G} gHg^{-1}$. Let N be a normal subgroup of G. Then it is known that G acts on the set of irreducible characters of N.

Let α be an irreducible character of N. The set of characters of G lying over α is denoted by $(\mathrm{Irr}\, G)|_\alpha$. It is the set of irreducible characters χ of G such that α is a constituent of $\chi\downarrow_N^G$.

Lemma 3.1.
Let N be a normal subgroup of G and $\alpha \in \operatorname{Irr} N$. Then the induced character $\alpha\uparrow_N^G$ vanishes outside N. If α is a G-stable character of N then $\alpha\uparrow_N^G(g) = |G|\alpha(g)/|N|$ for all $g \in N$.

Proof. Denote by M_α the representation allowed by α. Let $G = \bigcup_{i=1}^s x_i N$ be a coset decomposition of G. Then $g(x_i \otimes_N m) = x_j \otimes_N (x_j^{-1} g x_i).m$ where j is chosen such that $g x_i N = x_j N$. Thus if $g \notin N$ then $i \neq j$ and $\alpha\uparrow_N^G(g) = 0$. On the other hand, if $g \in N$ then $\alpha\uparrow_N^G(g) = \sum_{i=1}^s \alpha(x_i^{-1} g x_i)$. But if α is G-stable then $\alpha(x_i^{-1} g x_i) = \alpha(g)$ for all i and the formula follows. $\square$

Let $a \in G$ and denote by $\overline{C}_G(a)$ the core subgroup $\operatorname{core}_G C_G(a)$ of the centralizer $C_G(a)$ of a. For any $a \in G$ let $N(a)$ be the smallest subgroup of G containing a. It is easy to see that $N(a)$ is the subgroup generated by the conjugacy class of a. Also, for any subgroup $N \subset G$ of G let $C_G(N) = \bigcap_{n \in N} C_G(n)$ be the subgroup of the elements of G that commute with each element of N.

Proposition 3.2.
Let $a \in G$ and $N(a)$ be defined as above. Then $N_G(N(a)) = \overline{C}_G(a)$.

Proof. One has $\overline{C}_G(a) = \bigcap_{g \in G} g C_G(a) g^{-1} = \bigcap_{g \in G} C_G(gag^{-1})$. Since $N(a)$ is generated by the conjugacy class of a, $x \in N_G(N(a))$ if and only if x commutes with all conjugates of x. Therefore $x \in N_G(N(a))$ if and only if $x \in \bigcap_{g \in G} C_G(gag^{-1}) = \overline{C}_G(a)$. $\square$

Lemma 3.3.
Let $h \in G$ and $\mathcal{X} \in Z_{\mathbb{C}G^*}(h)$ with $\mathcal{X}(h) = \omega \mathcal{X}(1)$, $\omega \in \mathbb{C}^*$. Then all irreducible characters μ of G which satisfy $\mu(h) = \omega\mu(1)$ are constituents of $\mathcal{X}\epsilon_{N(h)}\uparrow_{N(h)}^G$ where $\epsilon_{N(h)}$ is the trivial character of the normal subgroup $N(h)$ of G.

Proof. If $\mu(h) = \omega\mu(1)$ then $h \in \ker_{\mathbb{C}G} \mu\mathcal{X}^*$. Since $N(h)$ is a normal subgroup it easily follows (see for example [2, Theorem 4.3]) that $\mu\mathcal{X}^*$ has all the constituents inside $\epsilon_{N(h)}\uparrow_{N(h)}^G$. Thus $m_G\big(\mu, \mathcal{X}\epsilon_{N(h)}\uparrow_{N(h)}^G\big) = m_G\big(\mu\mathcal{X}^*, \epsilon_{N(h)}\uparrow_{N(h)}^G\big) > 0$. $\square$

3.4. Central characters in $D(G)$

Lemma 3.4.
Let $c = \sum_{h \in G} \mathcal{X}_h \bowtie h$ be a character of $D(G)^*$ with $\mathcal{X}_h \in C(G)$. Then c is central in $D(G)$ if and only if $\mathcal{X}_{ghg^{-1}} = \mathcal{X}_h$ and $\mathcal{X}_h$ vanishes on $G \setminus C_G(h)$.

Proof. The character c is central if and only it is invariant under the adjoint action of $D(G)$ on itself. Thus c is central if and only if $gcg^{-1} = g$ for all $g \in G$ and $p_x.c = \delta_{x,1} c$ for all $x \in G$. The first condition is equivalent to $\mathcal{X}_{ghg^{-1}} = \mathcal{X}_h$ for all $g \in G$. For the second condition one has

$$p_x.c = \sum_{uv=x} p_v c p_{u^{-1}} = \sum_{h \in G} \sum_{uv=x} p_v (\mathcal{X}_h \bowtie h) p_{u^{-1}} = \sum_{h \in G} \sum_{uv=x} p_v \mathcal{X}_h p_{hu^{-1}h^{-1}} \bowtie h = \sum_{h \in G} \sum_{\{u : uhu^{-1}h^{-1} = x\}} p_{hu^{-1}h^{-1}} \mathcal{X}_h \bowtie h.$$

Suppose that c is central. Then $p_x.c = \delta_{x,1} c$ if and only if

$$\sum_{\{u : uhuh^{-1} = x\}} p_{hu^{-1}h^{-1}} \mathcal{X}_h = \delta_{x,1} \mathcal{X}_h$$

for all $h \in G$. If $x = 1$ this means precisely that $\mathcal{X}_h$ is zero outside $C_G(h)$. The converse is immediate. $\square$

For a conjugacy class $\mathcal{C}$ of G let $p_{\mathcal{C}} = \sum_{x \in \mathcal{C}} p_x$ and $z_{\mathcal{C}} = \sum_{x \in \mathcal{C}} x$. The elements $\{p_{\mathcal{C}}\}_{\mathcal{C}}$ form a basis for the character ring $C(G)$ of G and $\{z_{\mathcal{C}}\}_{\mathcal{C}}$ form a basis for the center $\mathcal{Z}(\mathbb{C}G)$. For a character $\mathcal{X}$ of a group G and a conjugacy class $\mathcal{C}$ of G denote by $\mathcal{X}_{\mathcal{C}}$ the value of $\mathcal{X}$ on the conjugacy class $\mathcal{C}$. Thus $\mathcal{X} = \sum_{\mathcal{C}} \mathcal{X}_{\mathcal{C}} p_{\mathcal{C}}$.

The following remark is straightforward.

Remark 3.5.
Let H be a subgroup of G. Then a character $\chi \in C(G)$ vanishes outside H if and only if it vanishes outside $\mathrm{core}_G H$.

Theorem 3.6.
A basis for $\mathcal{Z}\big(D(G)\big) \cap C\big(D(G)^*\big)$ is given by the elements $p_{\mathcal{D}} \bowtie z_{\mathcal{C}}$ where $\mathcal{D}$ and $\mathcal{C}$ run through all conjugacy classes of G that centralize each other element-wise.

Proof. Let $a \in \mathcal{C}$. Then $\overline{C}_G(a) = \bigcap_{g \in G} g\, C_G(a) g^{-1} = \bigcap_{x \in \mathcal{C}} C_G(x)$. Thus any element of a conjugacy class $\mathcal{D}$ centralize each element of another conjugacy class $\mathcal{C}$ if and only if $\mathcal{D}$ is contained in $\overline{C}_G(a)$. In this case the above remark implies $p_{\mathcal{D}}$ vanishes outside the centralizer of each element of $\mathcal{C}$. The previous lemma implies that $p_{\mathcal{D}} \bowtie z_{\mathcal{C}}$ is central in $D(G)$. The same lemma also implies that any central character is a linear combination of such characters. □

4. Hopf subalgebras of Drinfeld doubles $D(G)$

If N is a normal subgroup of G let $G_{\mathrm{st}}(N)$ be the group of linear characters of N that are stable under the conjugation action of G on the subgroup N. It is not difficult to check the following result. Its proof follows from Lemma 3.1.

Lemma 4.1.
Let N be a normal subgroup of G and $x, x' \in G_{\mathrm{st}}(N)$. Then

$$x \uparrow_N^G x' \uparrow_N^G = \frac{|G|}{|N|}\, (x x')\uparrow_N^G .$$

If x is a character of N then as in subsection 1.2 denote by $C_{x\uparrow_N^G}$ the subcoalgebra of $\mathbb{C}G^*$ associated to the character $x\uparrow_N^G$. On the other hand if N is a normal subgroup of G let $\pi_N \colon G \to G/N$ be the natural group projection and for any $g \in G$ denote by $C_{\pi_N^{-1}(\overline{g})}$ the vector subspace of kG with a basis given by the elements $\{gn : n \in N\}$.

Lemma 4.2.
Let N be a normal subgroup of G and $x \in G_{\mathrm{st}}(N)$. Then

$$C_{x\uparrow_N^G} = \big\{ f \in k^G : n \rightharpoonup f = x(n) f \big\}.$$

Proof. Note that

$$\big\{ f \in k^G : n \rightharpoonup f = x(n) f \big\} = \big(\langle an - x(n) a : a \in G \rangle \big)^{\perp}.$$

By a straightforward computation it can be checked that the annihilator $\mathrm{Ann}_{kG}(x\uparrow_N^G)$ is the linear subspace of kG generated by the elements $\{an - x(n)a : a \in G\}$. This implies the conclusion of the lemma. □

4.1. Hopf subalgebras of $D(G)$

In this subsection we describe all Hopf subalgebras of $D(G)$. Let N and M be normal subgroups of G, $\mathcal{X} \subset G_{\mathrm{st}}(N)$ and $\psi \colon \mathcal{X} \to G/M$ be a monomorphism of groups. Let $D(N, M, \mathcal{X}, \psi)$ be the subcoalgebra of $D(G)$ given by

$$D(N, M, \mathcal{X}, \psi) = \bigoplus_{x \in \mathcal{X}} C_{x\uparrow_N^G} \bowtie C_{\pi_M{}^{-1}(\psi(x))}.$$

In the next theorem it will be shown that $D(N, M, \mathcal{X}, \psi)$ is a Hopf subalgebra of $D(G)$.

Proposition 4.3.
With the above notation $D(N,M,\mathcal{X},\psi)$ is a Hopf subalgebra of $D(G)$ for any given datum as above.

Proof. In order to show that a subcoalgebra of a semisimple Hopf algebra is a Hopf subalgebra is enough to show that the set of irreducible characters associated to the respective subcoalgebra is closed under multiplication and duality "$*$" (see [17, Theorem 6]). Since $D(G)$ is finite dimensional it is enough to show that this set is closed under multiplication only. Since x is a G-stable character by Frobenius reciprocity it follows that $x\uparrow_N^G = \sum_{X\in\mathcal{A}_x} X(1)X$ and therefore $\dim_{\mathbb{C}} C_{x\uparrow_N^G} = |G|/|N|$. Then $\dim_{\mathbb{C}} D(N,M,\mathcal{X},\psi) = |\mathcal{X}|\cdot|G|\cdot|M|/|N|$. Let $\psi_0(x)\in G$ such that $\psi(x)=\psi_0(x)M$ for all $x\in M$. Since ψ is a group morphism it follows that $(\psi_0(x)\Lambda_M)(\psi_0(x')\Lambda_M)=\psi_0(xx')\Lambda_M$ for all $x,x'\in M$. Here $\Lambda_M=(1/|M|)\sum_{m\in M} m$. Let

$$\Lambda_{D(N,M,\mathcal{X},\psi)} = \frac{|N|}{|\mathcal{X}|\cdot|G|\cdot|M|}\sum_{x\in\mathcal{X}} x\uparrow_N^G \bowtie \psi_0(x)\Lambda_M.$$

One has

$$\Lambda^2_{D(N,M,\mathcal{X},\psi)} = \left(\frac{|N|}{|\mathcal{X}|\cdot|G|\cdot|M|}\right)^2 \sum_{x,x'\in\mathcal{X}} x\uparrow_N^G x'\uparrow_N^G \bowtie \psi_0(x)\Lambda_M\psi_0(x')\Lambda_M = \frac{|N|}{|\mathcal{X}|\cdot|G|\cdot|M|}\sum_{x,x'\in\mathcal{X}} (xx')\uparrow_N^G \bowtie \psi_0(xx')\Lambda_M$$

by the above lemma. This shows that $D(N,M,\mathcal{X},\psi)$ is a Hopf subalgebra of $D(G)$. □

4.2. Definition of the Hopf subalgebras $C(M,H,\lambda)$

Define the linear harpoon operators on $\mathbb{C}^G$ by $L_a(f)=a\rightharpoonup f$ and $R_a(f)=f\leftharpoonup a$ for all $a\in G$. Let $M,H\le G$ be subgroups of G with M a normal subgroup of G. Let also $\lambda\colon M\times H\to k^*$ be an invariant twisted bicharacter on $M\times H$, i.e. a function satisfying the following three properties:

$$\lambda(mn,h)=\lambda(m,h)\lambda(n,h), \tag{1}$$
$$\lambda(m,hl)=\lambda(m,h)\lambda(hmh^{-1},l), \tag{2}$$
$$\lambda(a^{-1}ma,h)=\lambda(m,h) \tag{3}$$

for all $a\in G$, $m,n\in M$ and $h,l\in H$. Define the following vector subspace of $D(G)$:

$$C(M,H,\lambda)=\bigoplus_{h\in H} C_\lambda(h)\# h$$

where $C_\lambda(h)=\{f\in\mathbb{C}^G : m\rightharpoonup f=\lambda(m,h)f,\ m\in M\}$. Note that $D(G)$ fits into a cocentral exact sequence of Hopf algebras

$$\mathbb{C}\to\mathbb{C}G^*\to D(G)\to\mathbb{C}G\to\mathbb{C}$$

and the results from [5] therefore can be applied. Then [5, Theorem 4.1] shows that any Hopf subalgebra of $D(G)$ is of the type $C(M,H,\lambda)$.

4.3. The correspondence between the two Hopf subalgebras

Proposition 4.4.
With the above notations one has:

(a) *$D(N,M,\mathcal{X},\psi)=C(N,H,\lambda_{\mathcal{X},\psi})$, where $H=\langle\psi(\mathcal{X})M\rangle$ and $\lambda_{\mathcal{X},\psi}\colon N\times H\to k^*$ is given by $\lambda_{\mathcal{X},\psi}(n,\psi_0(x)m)=x(n)$ on the generators $\psi(x)m$ of the subgroup H of G.*

(b) *$C(M,H,\lambda)=D(M,\ker_H\lambda,\mathcal{X},\psi)$ where $\ker_H\lambda=\{h\in H : \lambda(\cdot,h)=1\}$ and $\mathcal{X}=\{\lambda(\cdot,h) : h\in H\}$. Moreover, $\psi\colon\mathcal{X}\to G/\ker_H\lambda$ is defined by $\psi(\lambda(\cdot,h))=\overline{h}$.*

Proof. (a) As above suppose that $\psi(x) = \psi_0(x)M$ for a chosen element $\psi_0(x) \in G$. Then it is easy to see that $\lambda_{\mathcal{X},\psi}$ is well defined and satisfies equations (1) and (2). The rest of the proof follows from Lemma 4.2.

(b) This part is also straightforward. Note that the compatibility condition (3) implies that $\mathcal{X} \subset G_{\mathrm{st}}(N)$. $\square$

[5, Theorem 4.1] together with the previous proposition imply the following.

Corollary 4.5.
Any Hopf subalgebra of $D(G)$ is of the type $D(N, M, \mathcal{X}, \psi)$ for a given datum as above.

4.4. Kernels of representations of $D(G)$ and normal Hopf subalgebras of $D(G)$

Theorem 4.6.
A Hopf subalgebra $D(N, M, \mathcal{X}, \psi)$ of $D(G)$ is normal if and only $\psi(\mathcal{X}) \subset C_G(N)/M \cap \mathcal{Z}(G/M)$ and $[N, M] = 1$.

Proof. In order to see when $D(N, M, \mathcal{X}, \psi)$ is a normal Hopf subalgebra of $D(G)$ it is enough to decide when its integral $\Lambda_{D(N,M,\mathcal{X},\psi)}$ is central in $D(G)$ (see [11]). In order to do this one needs to verify the conditions from Lemma 3.4 for centrality of a character of $D(G)^*$.

First note that $\psi_0(x)m \neq \psi_0(y)m'$ for $x \neq y$ and for any $m, m' \in M$ since ψ is a monomorphism. The equality of characters $X_{ghg^{-1}} = X_h$ from Lemma 3.4 is satisfied if and only if $g\psi_0(x)Mg^{-1} = \psi_0(x)M$ for all $g \in G$. This is equivalent to the fact that $\psi(x) \in \mathcal{Z}(G/M)$. On the other hand, for the second condition note that by Lemma 3.1, $x\!\uparrow_N^G$ is zero outside N and does not vanish on any element of N. Thus the second condition of Lemma 3.4 is equivalent to $N \subset C_G(\psi_0(x)m)$ for all $m \in M$. For $x = 1$ this implies that $[N, M] = 1$ and then for an arbitrary $x \in \mathcal{X}$ it follows that $\psi(x) \in C_G(N)/M$. $\square$

4.5. Kernels of irreducible characters of $D(G)$

Since $D(G) = \mathbb{C}G^{*\,\mathrm{cop}} \bowtie \mathbb{C}G$ as vector spaces, the dual Hopf algebra $D(G)^*$ can be identified with $\mathbb{C}G^* \otimes \mathbb{C}G^{\mathrm{op}}$ via $\langle f \otimes l, p_x \bowtie g\rangle = \langle f, g\rangle\langle p_x, l\rangle$ for any $g, x, l \in G$ and $f \in \mathbb{C}G^*$.

For a subgroup H of G denote by $(G/H)_l$ the set of representatives for the left cosets of H in G.

Lemma 4.7.
For an irreducible representation (a, γ) of $D(G)$ its character $\widehat{(a, \gamma)}$ is given by

$$\widehat{(a,\gamma)} = \sum_{g \in (G/C_G(a))_l} \sum_{z \in C_G(a)} \gamma(z) p_{gag^{-1}} \otimes gzg^{-1}.$$

Proof. It is enough to show that

$$\widehat{(a,\gamma)}(p_x \bowtie l) = \gamma(g^{-1}lg) \tag{4}$$

if there is $g \in G$ such that $x = gag^{-1}$ and $g^{-1}lg \in C_G(a)$. Also if there is no such $g \in G$ then $\widehat{(a,\gamma)}(p_x \bowtie l) = 0$.

The representation corresponding to (a, γ) is given by $\mathbb{C}G \otimes_{\mathbb{C}\,C_G(a)} M_\gamma$ where M_γ is the module allowing the character γ. The action of $\mathbb{C}G^*$ is given by $p_x\big(g \otimes_{\mathbb{C}\,C_G(a)} m\big) = \delta_{x,gag^{-1}}\big(g \otimes_{\mathbb{C}\,C_G(a)} m\big)$ and the action of $\mathbb{C}G$ is the action of induced module. Using Lemma 3.1 a straightforward computation implies formula (4). $\square$

Proposition 4.8.
Let M be a subgroup of G, $\gamma \in \operatorname{Irr} M$ an irreducible character of M and $a \in G$. Then the set $\mathcal{S}$ of pairs $(X, l) \in \mathrm{Z}_{\mathbb{C}G^*} a \times \operatorname{core}_G M$ such that $X(a)/X(1) = (\gamma(glg^{-1})/\gamma(1))^{-1}$ for all $g \in G$ is of the form

$$\mathcal{S} = \bigsqcup_{i=0}^{s} \big[(\operatorname{Irr} G)|_{f_0^i} \times l_0^i \operatorname{core}_G(\ker_M \gamma) \big]$$

for some G-stable character $f_0 \in G_{\mathrm{st}}(N(a))$ of $N(a)$ and some $l_0 \in Z_{\mathbb{C}[C_G(a)]}\gamma$. Moreover f_0 has order s and $l_0^s \in \operatorname{core}_G(\ker_M\gamma)$.

Proof. Define the following set of scalars:

$$H = \left\{\omega \in \mathbb{C} \,:\, \text{there is } (\chi, l) \in \mathcal{S} \text{ with } \omega = \frac{\chi(a)}{\chi(1)}\right\}.$$

It can be easily checked that H is a subgroup of $\mathbb{C}^*$. Since $\mathbb{C}$ is algebraically closed and H is finite it follows that H is a cyclic group. Therefore $H = \{1, \omega, \dots, \omega^{s-1}\}$ for some root of unity ω of order s. Let χ_0 and l_0 such that $(\chi_0, l_0) \in \mathcal{S}$ and $\chi_0(a)/\chi_0(1) = \omega = (\gamma(gl_0g^{-1})/\gamma(1))^{-1}$ for all $g \in G$. Then $\chi_0{\downarrow}^G_{N(a)} = \chi_0(1)f_0$ for some $f_0 \in G_{\mathrm{st}}(N(a))$ of order s. Also note that $g_0^s \in \operatorname{core}_G(\ker_M\gamma)$. Lemma 3.3 implies the conclusion of the proposition. □

Corollary 4.9.

With the notation of the previous proposition, the Hopf subalgebra of $D(G)$ generated by the characters $\chi \bowtie l$, with $(\chi, l) \in \mathcal{S}$, is of the form $D\big(N(a), \operatorname{core}_G(\ker_M\gamma), \mathcal{X}, \psi\big)$ where $\mathcal{X} = \{f_0^i\}_{i=1}^s$ and $\psi\colon \mathcal{X} \to G/\operatorname{core}_G(\ker_M\gamma)$ is given by $\psi(f_0^i) = \overline{l_0^i}$.

Proof. Note that $\mathcal{S}$ is closed under multiplication, i.e. if $(\chi, l), (\chi', l') \in \mathcal{S}$ then $(\mu, ll') \in \mathcal{S}$ for any μ, an irreducible constituent of $\chi\chi'$. Then using [17, Theorem 6] it follows that this set generates a Hopf subalgebra of $D(G)$ which clearly should coincide with $D\big(N(a), \operatorname{core}_G(\ker_M\gamma), \mathcal{X}, \psi\big)$. □

Remark 4.10.

Note that since $\gamma(gl_0g^{-1})/\gamma(1) = \gamma(l_0)/\gamma(1)$ it follows that $gl_0g^{-1} \in l_0 \operatorname{core}_G(\ker_M\gamma)$ which means that $\psi(f_0) \in \mathcal{Z}\big(G/\operatorname{core}_G(\ker_M\gamma)\big)$ and therefore $\psi(\mathcal{X}) \subset \mathcal{Z}\big(G/\operatorname{core}_G(\ker_M\gamma)\big)$.

Theorem 4.11.

Let $a \in \mathcal{R}$ and $\gamma \in \operatorname{Irr} C_G(a)$. Then the Hopf subalgebra $D_{(a,\gamma)} = D(G)_{\widehat{(a,\gamma)}}$ is a normal Hopf subalgebra of $D(G)$. Moreover, with the above notation one has

$$D_{(a,\gamma)} = D\big(N(a), \operatorname{core}_G(\ker_{C_G(a)}\gamma), \langle f_0\rangle, \psi\big)$$

for some G-stable linear character f_0 of $N(a)$ and some group monomorphism $\psi\colon \langle f_0\rangle \to \mathcal{Z}\big(G/\operatorname{core}_G(\ker_{C_G(a)}\gamma)\big) \cap C_G(N(a))/\ker_{C_G(a)}\gamma$.

Proof. First we will describe $\ker \widehat{(a,\gamma)}$. An irreducible character of $D(G)^*$ is given by $\chi \bowtie l$ with $\chi \in \operatorname{Irr} G$ and $l \in G$. It follows that $\chi \bowtie l \in \ker_{D(G)} \widehat{(a,\gamma)}$ if and only if

$$\widehat{(a,\gamma)}(\chi \bowtie l) = \chi(1)\gamma(1)\,\frac{|G|}{|C_G(a)|}\,.$$

It follows from Lemma 4.7 that

$$\widehat{(a,\gamma)}(\chi \bowtie l) = \chi(a)\left(\sum_{\{g \in (G/C_G(a))_l : l \in gC_G(a)g^{-1}\}} \gamma(g^{-1}lg)\right).$$

Note that the above the sum has at most $|G|/|C_G(a)|$ terms and the absolute value for each term satisfies $|\chi(a)|\cdot|\gamma(g^{-1}lg)| \le \chi(1)\gamma(1)$. Then we deduce that the above equality is satisfied if and only if there is a root of unity $\omega \in \mathbb{C}^*$ such that $\chi(a) = \omega\chi(1)$ and $l \in \operatorname{core}_G(Z_{C_G(a)}\gamma)$ with the property that $\gamma(g^{-1}lg) = \omega^{-1}\gamma(1)$ for all $g \in G$.

Thus the set $\mathcal{S} = \ker_{D(G)}\widehat{(a,\gamma)}$ satisfies the hypothesis of Proposition 4.8. Then as in Corollary 4.9 it follows that there is f_0 a G-stable linear character of $N(a)$ and $l_0 \in G$ such that

$$\ker_{D(G)}\widehat{(a,\gamma)} = \bigsqcup_{i=0}^{s}\big[(\operatorname{Irr} G)|_{f_0^i} \times l_0^i \operatorname{core}_G(\ker_{C_G(a)}\gamma)\big].$$

Thus one can take $\mathcal{X}$ to be the group generated by f_0 and define ψ by sending f_0^i to the class of g_0^i modulo $\operatorname{core}_G(\ker_{C_G(a)}\gamma)$, for all $1 \leq i \leq s-1$. It can be easily checked that the map ψ satisfies the additional hypothesis from Theorem 4.6. Note that $[N(a), \overline{C}_G(a)] = 1$ by Proposition 3.2. Thus $\psi(\mathcal{X}) \subset C_G(N(a))/\operatorname{core}_G(\ker_{C_G(a)}\gamma)$. The other condition $\psi(\mathcal{X}) \subset \mathcal{Z}\big(G/\operatorname{core}_G(\ker_{C_G(a)}\gamma)\big)$ of Theorem 4.6 follows from Remark 4.10. □

The description of the kernels from the previous theorem implies the following corollary.

Corollary 4.12.
With the above notation one has $D_{(1,\gamma)} = \mathbb{C}G^* \bowtie \mathbb{C}(\ker_G\gamma)$.

A proof similar to the proof of Theorem 4.11 gives

Proposition 4.13.
If (a,γ) is a representation of $D(G)$ then

$$Z_{D(G)}\widehat{(a,\gamma)} = \mathbb{C}[G/[G,N(a)]]^* \bowtie \mathbb{C}(\operatorname{core}_G(Z_{C_G(a)}\gamma)).$$

Note that Lemma 1.4 is also needed in order to compute the center $Z_{\mathbb{C}G^*}a$.

5. Normal fusion subcategories of Rep $D(G)$

5.1. Fusion subcategories of Rep $D(G)$

In this subsection we recall the parametrization of fusion subcategories of Rep $D(G)$ given in [15].

Let $\mathcal{D}$ be a fusion subcategory of $\mathcal{C}$. Then, following [14], the fusion subcategroy $\mathcal{D}$ is completely determined by two canonical normal subgroups $K_{\mathcal{D}}$ and $H_{\mathcal{D}}$ of G and a G-invariant bicharacter $B_{\mathcal{D}}\colon K_{\mathcal{D}} \times H_{\mathcal{D}} \to \mathbb{C}^*$. The subgroups $K_{\mathcal{D}}$ and $H_{\mathcal{D}}$ are defined as follows:

$$K_{\mathcal{D}} = \big\{gag^{-1} : g \in G \text{ and } (a,\gamma) \in \mathcal{D} \text{ for some } \gamma\big\}$$

and $H_{\mathcal{D}}$ is the normal subgroup of G such that $\mathcal{D} \cap \operatorname{Rep} G = \operatorname{Rep}(G/H_{\mathcal{D}})$. Note that $K_{\mathcal{D}}$ is the fusion subcategory of $\mathbb{C}G^*$ determined by restricting all simple objects of $\mathcal{D}$ to $\mathbb{C}G^*$.

The bicharacter $B_{\mathcal{D}}\colon K_{\mathcal{D}} \times H_{\mathcal{D}} \to \mathbb{C}^*$ is defined by

$$B_{\mathcal{D}}(g^{-1}ag, h) = \frac{\gamma(ghg^{-1})}{\gamma(1)}$$

if $(a,\gamma) \in \mathcal{D}$. This is well defined and does not depend on γ [15]. Recall, see again [15], that a bicharacter is called G-invariant if and only if $B(xkx^{-1}, xhx^{-1}) = B(k,h)$ for all $x \in G$, $k \in K$ and $h \in H$.

Conversely, any two normal subgroups K and H of G that centralize each other element-wise together with a G-invariant bicharacter $B\colon K \times H \to \mathbb{C}^*$ give rise to a fusion category denoted by $S(K,H,B)$ in [15]. It is defined as the full abelian subcategory of Rep $D(G)$ generated by the objects (a,γ) such that $a \in K \cap \mathcal{R}$ and $\gamma \in \operatorname{Irr} C_G(a)$ such that $\gamma(h) = B(a,h)\gamma(1)$ for all $h \in H$.

5.2. Normal fusion subcategories of $\operatorname{Rep} D(G)$

In this subsection we will identify all the normal fusion subcategories $S(K,H,B)$ of $\operatorname{Rep} D(G)$.

Remark 5.1.
Let B be a normal Hopf subalgebra of a semisimple Hopf algebra A and $a \in A$. Then $\overline{a}(A/\!/B) \neq 0$ if and only if $a\Lambda_B \neq 0$. Indeed it is easy to verify that the map $A/\!/B \to A\Lambda_B$ given by $a + AB^+ \mapsto a\Lambda_B$ is an isomorphism of A-modules (see also [8, Lemma 3.2]).

Theorem 5.2.
Let $D(N,M,\mathcal{X},\psi)$ be a normal Hopf subalgebra of $D(G)$ as in Theorem 4.6. Then the normal fusion subcategory $\operatorname{Rep}(D(G)/\!/D(N,M,\mathcal{X},\psi))$ can be identified with $\mathcal{S}(K,H,B)$ where $K=N$, $H=\langle \psi_0(x), M : x\in\mathcal{X}\rangle$ and $B\colon K\times H \to \mathbb{C}^*$ is given by $B(n,\psi_0(x)m) = x(n)^{-1}$.

Proof. Let $\mathcal{D} = \operatorname{Rep}(D(G)/\!/D(N,M,\mathcal{X},\psi))$. For the subgroup $H_{\mathcal{D}}$ one has to look at $\operatorname{Rep}(D(G)/\!/D(N,M,\mathcal{X},\psi)) \cap \operatorname{Rep}(\mathbb{C}G)$. Thus

$$H_{\mathcal{D}} = \bigcap_{\{X\in \operatorname{Irr} G : (1,X)\in\mathcal{D}\}} \ker_G X.$$

But $(1,X)\in\mathcal{D}$ if and only if $D_{(1,X)} \supset D(N,M,\mathcal{X},\psi)$. Note that $D_{(1,X)} = \mathbb{C}G^* \bowtie \ker_G X$ by Corollary 4.12. Then it follows that $D_{(1,X)} \supset D(N,M,\mathcal{X},\psi)$ if and only if $\ker_G X \supset \langle \psi(x), m : x\in\mathcal{X},\ m\in M\rangle$. Thus $H_{\mathcal{D}} = \langle \psi(x), m : x\in\mathcal{X},\ m\in M\rangle$.

The subgroup $K_{\mathcal{D}}$ is generated by all $x\in G$ such that the relation $p_x(D(G)/\!/D(N,M,\mathcal{X},\psi)) \neq 0$ holds. The above remark implies that $K_{\mathcal{D}}$ is given by $x\in G$ such that $p_x\Lambda_{D(N,M,\mathcal{X},\psi)} \neq 0$. Formula for $\Lambda_{D(N,M,\mathcal{X},\psi)}$ from Theorem 4.5 shows that $x\in K_{\mathcal{D}}$ if and only if $(f^i\!\uparrow_N^G)(x)\neq 0$ for some i and some $f\in\mathcal{X}$. Lemma 3.1 shows that $K_{\mathcal{D}} \subset N$. Since f is a linear character it follows that $K_{\mathcal{D}} = N$.

In order to describe the bicharacter B suppose now that we have $(a,\gamma)\in \operatorname{Rep}(D(G)/\!/D(N,M,\mathcal{X},\psi))$. Then $D_{(a,\gamma)} \supset D(N,M,\mathcal{X},\psi)$. But using again Lemma 3.1 one has

$$B(gag^{-1},h) = \frac{\gamma(g^{-1}hg)}{\gamma(1)} = \left(\frac{x\!\uparrow_N^G(a)}{|G|/|N|}\right)^{-1} = x(a)^{-1}. \qquad \square$$

For any bicharacter $B\colon K\times H\to k^*$ define

$$K^\perp = \{h\in H : B(a,h) = 1,\ a\in K\}.$$

Using the previous theorem we can give the following criteria.

Theorem 5.3.
A fusion subcategory $S(K,H,B)$ is a normal fusion subcategory of $\operatorname{Rep} D(G)$ if and only if

$$B(gag^{-1},h) = B(a,h) = B(a,ghg^{-1}) \tag{5}$$

for all $a,g,h\in G$. In these conditions

$$S(K,H,B) = \operatorname{Rep}(D(G)/\!/D(K,K^\perp,\mathcal{X},\psi)) \tag{6}$$

where $\mathcal{X} = \{B(\cdot,h) : h\in H\}$ and $\psi\colon \mathcal{X}\to \mathcal{Z}(G/K^\perp)\cap C_G(K)/K^\perp$ is given by $\psi(x) = \overline{h}$ for any $h\in H$ with $x = B(\cdot,h)$.

Proof. Suppose that $S(K,H,B)$ is a normal fusion subcategory. Then, using the previous theorem, (6) follows for some data $(N,M,\mathcal{X},\psi)$. Thus $B(gag^{-1},\psi_0(x)m) = x(gag^{-1}) = x(a) = B(a,\psi_0(x)m)$. Conversely, suppose that (5) is satisfied. Then clearly $\mathcal{X}$ is a group of G-stable linear characters of K. Moreover it is straightforward to check that ψ takes values inside $\mathcal{Z}(G/K^\perp)\cap C_G(K)/K^\perp$. The rest of the theorem follows from Theorem 5.2. $\square$

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