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Skeletally Dugundji spaces

Research Article

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Abstract: We introduce and investigate the class of skeletally Dugundji spaces as a skeletal analogue of Dugundji space. Our main result states that the following conditions are equivalent for a given space X: (i) X is skeletally Dugundji; (ii) every compactification of X is co-absolute to a Dugundji space; (iii) every C^*-embedding of the absolute $p(X)$ in another space is strongly π-regular; (iv) X has a multiplicative lattice in the sense of Shchepin [Shchepin E.V., Topology of limit spaces with uncountable inverse spectra, Uspekhi Mat. Nauk, 1976, 31(5), 191–226 (in Russian)] consisting of skeletal maps.

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1. Introduction

In this paper we introduce a class of skeletally Dugundji spaces as a skeletal analogue of Dugundji spaces [12]. The paper can be considered as a continuation of [8–10, 21], where it was shown that I-favorable spaces [4] coincide with the class of skeletally generated spaces [21]. The last class is a skeletal counterpart of κ-metrizable compacta [17].

Recall that a map $f: X \to Y$ is called *skeletal* [11] (resp., *semi-open*) if the set $\mathrm{Int}_Y \mathrm{cl}_Y f(U)$ (resp., $\mathrm{Int}_Y f(U)$) is non-empty, for any $U \in \mathcal{T}_X$. Obviously, every semi-open map is skeletal, and both notions are equivalent for closed maps. Our definition of skeletally Dugundji spaces is similar to the spectral characterization of Dugundji spaces obtained by

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Haydon [6]. We say that a space X is *skeletally Dugundji* if there exists an inverse system $S = \{X_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$ with surjective skeletal bonding maps, where τ is identified with the first ordinal $\omega(\tau)$ of cardinality τ, satisfying the following conditions: (i) X_0 is a separable metric space and all maps $p_\alpha^{\alpha+1}$ have metrizable kernels (i.e., there exists a separable metric space M_α such that $X_{\alpha+1}$ is embedded in $X_\alpha \times M_\alpha$ and $p_\alpha^{\alpha+1}$ coincides with the restriction $\pi \restriction X_{\alpha+1}$ of the projection $\pi\colon X_\alpha \times M_\alpha \to X_\alpha$); (ii) for any limit ordinal $\gamma < \tau$ the space X_γ is a (dense) subset of $\varprojlim \{X_\alpha, p_\alpha^\beta : \alpha < \beta < \gamma\}$; (iii) X is embedded in $\varprojlim S$ such that $p_\alpha(X) = X_\alpha$ for each α, where $p_\alpha\colon \varprojlim S \to X_\alpha$ is the α-th limit projection; (iv) for every bounded continuous real-valued function f on $\varprojlim S$ there exists $\alpha \in A$ such that $p_\alpha \prec f$ (the last relation means that there exists a continuous function g on X_α with $f = g \circ p_\alpha$). If the inverse system S and X satisfy conditions (ii) and (iii), X is said to be the *almost limit* of S, notation $X = \text{a-}\varprojlim S$. We also say that an inverse system S is factorizing if it satisfies condition (iv).

In this paper we provide different characterizations of skeletally Dugundji spaces. One of our starting points is the result of Shapiro [15] that a compactum X is co-absolute to a Dugundji space (i.e., the absolute of X is the absolute of a Dugundji space) if and only if X is the limit space of a continuous inverse system $S = \{X_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$ satisfying all conditions from the definition of skeletally Dugundji spaces, except that the maps $p_\alpha^{\alpha+1}$ do not necessarily have metrizable kernels but have countable π-weight (see Section 3 for this notion). Let us note that necessity of the above result was announced in [15, Theorem 3] without a proof. We establish in Section 2 that any space co-absolute to a space with a multiplicative in the sense of Shchepin [16] lattice of open maps has a multiplicative lattice of skeletal maps. This extends the necessity of Shapiro's result, mentioned above. Some properties of skeletally Dugundji spaces are provided in Section 3. The following result is the main theorem from this section.

Theorem 3.3.

For any space X the following are equivalent:

(i) *X is skeletally Dugundji;*

(ii) *every compactification of X is co-absolute to a Dugundji space;*

(iii) *every C^*-embedding of the absolute $p(X)$ in another space is strongly π-regular;*

(iv) *X has a multiplicative lattice of skeletal maps.*

Here, we say that a subspace X of a space Y is strongly π-regularly embedded in Y if there exists a function $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_Y$ between the topologies of X and Y such that:

(e1) $\mathrm{e}(\varnothing) = \varnothing$ and $\mathrm{e}(U) \cap X$ is a dense subset of U;

(e2) $\mathrm{e}(U \cap V) = \mathrm{e}(U) \cap \mathrm{e}(V)$ for any $U, V \in \mathcal{T}_X$.

Such a function was considered in [14] under the name π-regular operator. It follows from Corollary 3.2 that if every embedding of a compactum X in another space is strongly π-regular, then X is skeletally Dugundji. A positive answer to the following question would provide an external characterization of skeletally Dugundji spaces similar to the characterization of Dugundji spaces given by Shirokov in [18].

Question 1.1.

Is any embedding of a skeletally Dugundji compactum in a Tychonoff cube strongly π-regular?

Another question arises from Corollary 3.6 that if X is a skeletally Dugundji space, then the absolute $p(\beta X)$ of βX is a retract of any extremally disconnected space in which $p(\beta X)$ is embedded.

Question 1.2.

Let X be a compact space such that its absolute $p(X)$ is a retract of any extremally disconnected space containing $p(X)$. Is X skeletally Dugundji?

When X is 0-dimensional, this question is equivalent to the open problem following [7, Observation 5.3.10]. According to [13], Question 1.2 has a positive answer if the weight of X is $\leq \omega_1$. All spaces in this paper are Tychonoff and the single-valued maps are continuous.

2. Spaces co-absolute with AE(0)-spaces

Haydon [6] established that the class of compact absolute extensors for zero-dimensional spaces (br., AE(0)-spaces) coincides with the class of Dugundji spaces and any compactum X belongs to that class iff X can be represented as the limit space of a continuous inverse system $S = \{X_\alpha, p^\beta_\alpha : \alpha < \beta < w(X)\}$ with open bonding maps such that each map $p^{\alpha+1}_\alpha$ has a metrizable kernel, see [6]. Dugundji spaces can be also characterized as the compact spaces X possessing a multiplicative lattice in the sense of Shchepin [16] consisting of open maps. This means that there exists a family Ψ of open maps with domain X such that:

(L1) For any map $f\colon X \to f(X)$ there exists $\phi \in \Psi$ with $\phi \prec f$ and $w(\phi(X)) \le w(f(X))$.

(L2) Ψ is multiplicative, i.e., if $\{\phi_\alpha : \alpha \in A\} \subset \Psi$, then the diagonal product $\triangle\{\phi_\alpha : \alpha \in A\}$ belongs to Ψ.

Let us note that a general definition of AE(0) in the class of all Tychonoff spaces was introduced by Chigogidze [3]. By [20, Theorem 1], for every C-embedding of an AE(0)-space X in $\mathbb{R}^A$ there exists an upper semi-continuous compact-valued map $r\colon \mathbb{R}^A \to X$ with $r(x) = \{x\}$ for all $x \in X$. Then, following the terminology of [20, Lemmas 3–6], all restrictions $\pi_B{\restriction}X$, where $B \subset A$ is r-admissible, are open and form a multiplicative lattice. Therefore, every AE(0)-space possesses a multiplicative lattice of open maps.

Proposition 2.1.
Let X be C-embedded in $\mathbb{R}^\Gamma$ for some Γ and X has a multiplicative lattice Ψ of quotient maps. Then the family $\mathcal{A} = \{B \subset \Gamma : \phi_B \in \Psi\}$, where $\phi_B = \pi_B{\restriction}X\colon X \to \pi_B(X)$ is the restriction of the projection $\pi_B\colon \mathbb{R}^\Gamma \to \mathbb{R}^B$, has the following properties:

(i) *$\mathcal{A}$ is additive, i.e., the union of any elements from $\mathcal{A}$ is also from $\mathcal{A}$.*

(ii) *Every $C \subset \Gamma$ is contained in some $B \in \mathcal{A}$ with $|B| \le |C| \cdot \aleph_0$.*

Proof. Let Ψ be a multiplicative lattice on X consisting of quotient maps. Suppose $B = \bigcup B_\alpha$ with $B_\alpha \in \mathcal{A}$ for all α. Then $\phi_B = \triangle\phi_{B_\alpha}$ because $\triangle\phi_{B_\alpha} \in \Psi$ is quotient. So, $B \in \mathcal{A}$. Assume $C \subset \Gamma$ is infinite of cardinality $|C| = \tau$. We construct by induction an increasing sequence $\{B(k)\} \subset \Gamma$ and a sequence $\{\phi_k\} \subset \Psi$ such that $B(1) = C$, $|B(k)| = \tau$, $w(\phi_k(X)) \le \tau$ and $\phi_{B(k+1)} \prec \phi_k \prec \phi_{B(k)}$ for all k. Suppose the construction is done up to level k for some $k \ge 1$. We consider each $\phi_k(X)$ as a subspace of $\mathbb{R}^\tau$. Since X is C-embedded in $\mathbb{R}^\Gamma$, there exists a map $g_k\colon \mathbb{R}^\Gamma \to \mathbb{R}^\tau$ extending ϕ_k. Then g_k depends on τ many coordinates of $\mathbb{R}^\Gamma$, so we can find a set $B(k+1) \subset \Gamma$ of cardinality τ containing $B(k)$ such that $\pi_{B(k+1)} \prec g_k$. Consequently, $\phi_{B(k+1)} \prec \phi_k$. Next, by condition (L1), there exists $\phi_{k+1} \in \Psi$ with $\phi_{k+1} \prec \phi_{B(k+1)}$ and $w(\phi_{k+1}(X)) \le \tau$. This completes the construction. Finally, let $B = \bigcup_{k=1}^\infty B(k)$ and $\phi = \triangle_{k=1}^\infty \phi_k$. Obviously, $|B| = \tau$ and $\phi_B = \phi \in \Psi$. Hence, $C \subset B \in \mathcal{A}$. □

Shapiro [15, Theorem 3] stated without a proof that if a compactum X is co-absolute to a Dugundji space, then X is the limit space of a continuous inverse system $S = \{X_\alpha, p^\beta_\alpha : \alpha < \beta < w(X)\}$ such that X_0 is a point, the bonding maps are skeletal and each $p^{\alpha+1}_\alpha$ has a countable π-weight (see Section 3 for this notion). The next theorem is a generalization of Shapiro's result (recall that any Dugundji space is an AE(0), and hence has a multiplicative lattice of open maps).

Theorem 2.2.
Let X be a space with a multiplicative lattice of open maps. Then every space co-absolute with X has a multiplicative lattice of semi-open maps.

Proof. Suppose Y is co-absolute with X and Z is their common absolute. So, there exist irreducible perfect maps $\theta_X\colon Z \to X$ and $\theta_Y\colon Z \to Y$. Consider the set-valued maps $r_X\colon X \to Y$ and $r_Y\colon Y \to X$ defined by $r_X(x) = \theta_Y\big(\theta_X^{-1}(x)\big)$, $x \in X$, $r_Y(y) = \theta_X\big(\theta_Y^{-1}(y)\big)$, $y \in Y$. Since θ_X is irreducible, for every open $V \subset Y$ the set $r^\#_X(V) = \{x \in X : r_X(x) \subset V\}$ is non-empty and open in X, and $r^\#_X(V) = \theta^\#_X\big(\theta_Y^{-1}(V)\big)$, where $\theta^\#_X(W) = \big\{x \in X : \theta_X^{-1}(x) \subset W\big\}$, $W \subset Z$. Hence, r_X is upper semi-continuous and compact valued. Similarly, r_Y is also upper semi-continuous and compact valued.

The next claim follows from the fact that both θ_X and θ_Y are closed irreducible maps, and Z is extremally disconnected.

Claim 1. *For every open $V \subset Y$ the set $\theta_X^{\#}\big(\overline{\theta_Y^{-1}(V)}\big)$ is regularly open and $\overline{r_X^{\#}(V)} = \overline{r_X^{-1}(V)} = \overline{\theta_X^{\#}\big(\overline{\theta_Y^{-1}(V)}\big)}$.*

Consider the disjoint union $X \oplus Y$ of X and Y as a C-embedded subspace in $\mathbb{R}^\Gamma$ for some Γ. For every $B \subset \Gamma$ let $\phi_B = \pi_B{\restriction}X$, $p_B = \pi_B{\restriction}Y$, $X_B = \phi_B(X)$ and $Y_B = p_B(Y)$. If $B \subset C \subset \Gamma$, there exist natural maps $\phi_B^C\colon X_C \to X_B$ and $p_B^C\colon Y_C \to Y_B$. Let Ψ be a multiplicative lattice for X consisting of open maps. Since X is also C-embedded in $\mathbb{R}^\Gamma$, the family $\mathcal{A} = \{B \subset \Gamma : \phi_B \in \Psi\}$ satisfies conditions (i)–(ii) from Proposition 2.1.

Claim 2. *Let $C \in \mathcal{A}$ be a set of cardinality τ and $\{V_\alpha\}$ an open family in Y_C of the same cardinality. Then there exist $B \in \mathcal{A}$ containing C and a family $\{G_\alpha\}$ of open subsets of $\phi_B(X)$ such that $|B| = \tau$ and $r_X^{\#}\big(p_B^{-1}(V_\alpha)\big)$ is a dense subset of $\phi_B^{-1}(G_\alpha)$ for each α.*

Because every disjoint open family in X is at most countable [19], any family of open subsets of X contains a dense countable subfamily. For every α choose a cover γ_α of $r_X^{\#}\big(p_B^{-1}(V_\alpha)\big)$ consisting of open subsets of X of the form $U \cap X$, where U is a standard open set in $\mathbb{R}^\Gamma$. Then there exists a dense countable subfamily ω_α of γ_α. Since each element of ω_α depends on finitely many coordinates, by Proposition 2.1, we can find a set $B \in \mathcal{A}$ containing C of cardinality $|B| = \tau$ such that $\phi_B(W)$ is open in $\phi_B(X)$ and $\phi_B^{-1}(\phi_B(W)) = W$ for every $W \in \omega_\alpha$ and every α. Then each $O_\alpha = \phi_B(W_\alpha)$ is open in $\phi_B(X)$ and $\phi_B^{-1}(O_\alpha) = W_\alpha$, where $W_\alpha = \bigcup\{W : W \in \omega_\alpha\}$. Because ϕ_B is open, we have $\phi_B^{-1}(\overline{O_\alpha}) = \overline{W_\alpha} = \overline{r_X^{\#}\big(p_B^{-1}(V_\alpha)\big)}$ and $\phi_B\big(r_X^{\#}\big(p_B^{-1}(V_\alpha)\big)\big) \subset \operatorname{Int}\overline{O_\alpha} = G_\alpha$. Hence, $r_X^{\#}\big(p_B^{-1}(V_\alpha)\big) \subset \phi_B^{-1}(G_\alpha) \subset \overline{r_X^{\#}\big(p_B^{-1}(V_\alpha)\big)}$ for every α. This completes the proof of Claim 2.

For any $B \subset \Gamma$ let Ω_B and Λ_B be bases for X_B and Y_B, respectively, having cardinality $\le |B|$ such that Ω_B consists of open sets of the form $U \cap X_B$ and Λ_B consists of all finite unions of sets of the form $V \cap Y_B$, where U and V are standard open sets in $\mathbb{R}^B$. Denote by Σ the family of all $B \subset \Gamma$ satisfying the following conditions:

(a) $B \in \mathcal{A}$;

(b) for every $V \in \Lambda_B$ there exists an open set $G_V \subset \phi_B(X)$ such that $r_X^{\#}\big(p_B^{-1}(V)\big)$ is dense in $\phi_B^{-1}(G_V)$;

(c) for any $U \in \Omega_B$ there exists an open set $V_U \subset Y_B$ with $p_B^{-1}(V_U) \subset r_Y^{\#}\big(\phi_B^{-1}(U)\big)$.

Claim 3. *Σ is additive.*

First, let us show that Σ is finitely additive. It suffices to prove that if $B(1), B(2)$ belong to Σ, then so does $B = B(1) \cup B(2)$. Because $\mathcal{A}$ is additive, we need to check that B satisfies conditions (b) and (c).

Suppose $V \subset Y_B$ is open and $V = V_1 \cup V_2 \cup \dots \cup V_m$, where each V_i is of the form $W_i \cap Y_B$ with W_i being a standard open subset of $\mathbb{R}^B$. So, for every i we have $W_i = \prod\{W_i(\alpha) : \alpha \in B\}$ such that all $W_i(\alpha)$, $\alpha \in B$, are open intervals and the set $k(V_i) = \{\alpha : W_i(\alpha) \ne \mathbb{R}\}$ is finite. Then the family $\{V_1, \dots, V_m\}$ is the union of the following families: $\gamma_j = \{V_i : k(V_i) \subset B(j)\}$, $j = 1, 2$, and $\gamma_{1,2} = \{V_i : k(V_i) \setminus B(j) \ne \varnothing,\ j = 1, 2\}$. Let $O_j^* = \bigcup\big\{p_{B(j)}^B(V_i) : V_i \in \gamma_j\big\}$ and $O_j = \bigcup\{V_i : V_i \in \gamma_j\}$, $j = 1, 2$. Then, according to (b), $r_X^{\#}\big(p_{B(j)}^{-1}(O_j^*)\big)$ is a dense subset of $\phi_{B(j)}^{-1}(G_j^*)$ for some open $G_j^* \subset X_{B(j)}$, $j = 1, 2$. Since $p_{B(j)}^{-1}(O_j^*) = p_B^{-1}(O_j)$, for every $j = 1, 2$,

(d) $r_X^{\#}\big(p_B^{-1}(O_j)\big)$ is dense in $\phi_B^{-1}(G_j)$ with $G_j = \big(\phi_{B(j)}^B\big)^{-1}(G_j^*)$.

If $V_i \in \gamma_{1,2}$, then $V_i = V_i(1) \cap V_i(2)$ with $V_i(j) \in \gamma_j$, $j = 1, 2$. Hence, for each j there exists an open set $G_{V_i(j)} \subset X_{B(j)}$ such that $\phi_{B(j)}^{-1}(G_{V_i(j)})$ contains $r_X^{\#}\big(p_{B(j)}^{-1}(V_i(j)^*\big)$ as a dense subset, where $V_i(j)^* = p_{B(j)}^B(V_i(j))$. Because $p_B^{-1}(V_i) = p_{B(1)}^{-1}(V_i(1)^*) \cap p_{B(2)}^{-1}(V_i(2)^*)$, the set $r_X^{\#}\big(p_B^{-1}(V_i)\big)$, being the intersection of the open sets $r_X^{\#}\big(p_{B(1)}^{-1}(V_i(1)^*)\big)$ and $r_X^{\#}\big(p_{B(2)}^{-1}(V_i(2)^*)\big)$, is dense in $\phi_B^{-1}(G_{V_i})$, where $G_{V_i} = \big(\phi_{B(1)}^B\big)^{-1}(G_{V_i(1)}) \cap \big(\phi_{B(2)}^B\big)^{-1}(G_{V_i(2)})$. Therefore, we have finitely many open sets $G_W \subset X_B$ such that

(e) $r_X^{\#}\big(p_B^{-1}(W)\big)$ is dense in $\phi_B^{-1}(G_W)$, $W \in \gamma_{1,2}$.

Obviously, $V = \bigcup\{W : W \in \gamma_{1,2}\} \cup O_1 \cup O_2$, and let $G(V) = \bigcup\{G_W : W \in \gamma_{1,2}\} \cup G_1 \cup G_2$. It follows from (d) and (e) that the set

$$L = \bigcup\big\{r_X^{\#}\big(p_B^{-1}(W)\big) : W \in \gamma_{1,2}\big\} \cup r_X^{\#}\big(p_B^{-1}(O_1)\big) \cup r_X^{\#}\big(p_B^{-1}(O_2)\big)$$

is dense in $\phi_B^{-1}(G(V))$. On the other hand, by Claim 1,

$$\overline{L} = \bigcup\big\{\overline{r_X^{-1}\big(p_B^{-1}(W)\big)} : W \in \gamma_{1,2}\big\} \cup \overline{r_X^{-1}\big(p_B^{-1}(O_1)\big)} \cup \overline{r_X^{-1}\big(p_B^{-1}(O_2)\big)}.$$

Therefore, $\overline{L} = \overline{r_X^{-1}(p_B^{-1}(V))} = \overline{\phi_B^{-1}(G(V))}$. Let $G_V = \operatorname{Int}\overline{G(V)}$. Because ϕ_B is an open map, $\phi_B^{-1}(G_V) = \operatorname{Int}\overline{r_X^{-1}\big(p_B^{-1}(V)\big)}$. Finally, since $r_X^{\#}\big(p_B^{-1}(V)\big)$ is open in X and it is dense in $\overline{r_X^{-1}\big(p_B^{-1}(V)\big)}$ (see Claim 1), $r_X^{\#}\big(p_B^{-1}(V)\big)$ is a dense subset of $\phi_B^{-1}(G_V)$. Thus, B satisfies condition (b).

To show that B satisfies condition (c), let $U \in \Omega_B$. Then $U = \phi^B_{B(1)}(U(1)) \cap \phi^B_{B(2)}(U(2))$ with $U(i) \in \Omega_{B(i)}$, $i = 1, 2$. So, there are open sets $V_{U(i)} \subset Y_{B(i)}$ such that $p^{-1}_{B(i)}(V_{U(i)}) \subset r_Y^{\#}\big(\phi^{-1}_{B(i)}(U(i))\big)$, $i = 1, 2$. Let $V_U = \big(p^B_{B(1)}\big)^{-1}(V_{U(1)}) \cap \big(p^B_{B(2)}\big)^{-1}(V_{U(2)})$. Then, $p_B^{-1}(V_U) = p^{-1}_{B(1)}(V_{U(1)}) \cap p^{-1}_{B(2)}(V_{U(2)})$ and $\phi_B^{-1}(U) = \phi^{-1}_{B(1)}(U(1)) \cap \phi^{-1}_{B(2)}(U(2))$. Consequently, $p_B^{-1}(V_U) \subset r_Y^{\#}\big(\phi_B^{-1}(U)\big)$. Hence, $B \in \Sigma$.

Suppose now that $B = \bigcup B_\alpha$ is the union of infinitely many $B_\alpha \in \Sigma$, and let $V \in \Lambda_B$. Then there exists a set $C \subset B$ which is a union of finitely many B_{α_i}, $i = 1, \dots, k$, such that $(p^B_C)^{-1}(p^B_C(V)) = V$ and $V^* = p^B_C(V) \in \Lambda_C$. Since $C \in \Sigma$, there exists an open set G^*_V in X_C with $r_X^{\#}\big(p_C^{-1}(V^*)\big)$ being a dense subset of $\phi_C^{-1}(G^*_V)$. Obviously, $p_C^{-1}(V^*) = p_B^{-1}(V)$ and $\phi_C^{-1}(G^*_V) = \phi_B^{-1}(G_V)$, where $G_V = (\phi^B_C)^{-1}(G^*_V)$. Consequently, $r_X^{\#}\big(p_B^{-1}(V)\big)$ is a dense subset of $\phi_B^{-1}(G_V)$. Hence, B satisfies (b). Similarly, one can show that B also satisfies condition (c). Therefore, $B \in \Sigma$. This complete the proof of Claim 3.

***Claim** 4. Every infinite $C \in \mathcal{A}$ of cardinality τ is contained in a set $B \in \Sigma$ with $|B| = \tau$.*

We construct by induction sets $B(k) \in \mathcal{A}$, open subsets $\{G_V : V \in \Lambda_{B(2k)}\}$ of $\phi_{B(2k+1)}(X)$ and open subsets $\{V_U : U \in \Omega_{B(2k+1)}\}$ of $Y_{B(2k+2)}$ such that for every $k \geq 0$ we have:

(i) $B(0) = C$, $|B(k)| = \tau$ and $B(k) \subset B(k+1)$;

(ii) $r_X^{\#}\big(p^{-1}_{B(2k)}(V)\big)$ is a dense subset of $\phi^{-1}_{B(k+1)}(G_V)$ for every $V \in \Lambda_{B(2k)}$;

(iii) $p^{-1}_{B(2k+2)}(V_U) \subset r_Y^{\#}\big(\phi^{-1}_{B(2k+1)}(U)\big)$ for every $U \in \Omega_{B(2k+1)}$.

Suppose the construction is done up to level $2k$. Since the family $\big\{p^{-1}_{B(2k)}(V) : V \in \Lambda_{B(2k)}\big\}$ is of cardinality $\leq \tau$, by Claim 2, there exist a set $B(2k+1) \in \mathcal{A}$ of cardinality τ containing $B(2k)$ and open sets $\{G_V : V \in \Lambda_{B(2k)}\}$ in $\phi_{B(k+1)}(X)$ satisfying (ii). Because each $r_Y^{\#}\big(\phi^{-1}_{B(2k+1)}(U)\big)$, $U \in \Omega_{B(2k+1)}$, is open in Y, we can find a set $B(2k+2) \in \mathcal{A}$ containing $B(2k+1)$ such that $|B(k+2)| = \tau$ and $p_{B(2k+2)}(Y)$ contains an open family $\{V_U : U \in \Omega_{B(2k+1)}\}$ satisfying (iii). This completes the inductive step. Obviously, $B = \bigcup_{k=1}^{\infty} B(k) \in \mathcal{A}$ and $|B| = \tau$. Repeating the arguments from the proof of Claim 3, one can show that B satisfies conditions (b) and (c). So, $B \in \Sigma$, which completes the proof of Claim 4.

***Claim** 5. For any $B \in \Sigma$ the map $p_B \colon Y \to Y_B$ is semi-open.*

First, let us show that if $B \in \Sigma$ and $\phi_B(x_1) = \phi_B(x_2)$ for some $x_1, x_2 \in X$, then $p_B(r_X(x_1)) = p_B(r_X(x_2))$. Indeed, suppose $p_B(r_X(x_2)) \subset V$, where $V \subset Y_B$ is open. Since $p_B(r_X(x_2))$ is compact and Λ_B is finitely additive, we can assume that $V \in \Lambda_B$. Then $x_2 \in r_X^{\#}\big(p_B^{-1}(V)\big)$. By condition (b), $r_X^{\#}\big(p_B^{-1}(V)\big)$ is a dense subset of $\phi_B^{-1}(G_V)$ for some open set G_V in X_B, so $x_1 \in \phi_B^{-1}(x_2) \subset \phi_B^{-1}(G_V)$. On the other hand, according to Claim 1, we have $\theta_X^{\#}\big(\overline{\theta_Y^{-1}\big(p_B^{-1}(V)\big)}\big) = \operatorname{Int}\overline{r_X^{\#}\big(p_B^{-1}(V)\big)}$. Hence, $\theta_X^{\#}\big(\overline{\theta_Y^{-1}\big(p_B^{-1}(V)\big)}\big)$ contains the set $\phi_B^{-1}(G_V)$. Thus, $r_X(x_1) \subset \theta_Y\big(\overline{\theta_Y^{-1}\big(p_B^{-1}(V)\big)}\big) = \overline{p_B^{-1}(V)}$. Finally, we obtain $p_B(r_X(x_1)) \subset \overline{V}$, which implies $p_B(r_X(x_1)) \subset p_B(r_X(x_2))$. Similarly, $p_B(r_X(x_2)) \subset p_B(r_X(x_1))$.

To show that p_B is semi-open, let $W \subset Y$ be open. Then $\phi_B\big(r_X^{\#}(W)\big)$ is open in $\phi_B(X)$. According to (c), there exist $U \in \Omega_B$ and open $V_U \subset Y_B$ with $U \subset \phi_B\big(r_X^{\#}(W)\big)$ and $p_B^{-1}(V_U) \subset r_Y^{\#}\big(\phi_B^{-1}(U)\big)$. The last inclusion implies $V_U \subset p_B\big(r_Y^{\#}\big(\phi_B^{-1}(U)\big)\big)$. It is easily seen that $r_Y^{\#}\big(\phi_B^{-1}(U)\big) \subset r_X\big(\phi_B^{-1}(U)\big)$. So, we have the inclusions

$$V_U \subset p_B\big(r_Y^{\#}\big(\phi_B^{-1}(U)\big)\big) \subset p_B\big(r_X\big(\phi_B^{-1}(U)\big)\big).$$

Therefore, it suffices to show that $p_B\big(r_X\big(\phi_B^{-1}(U)\big)\big)$ is contained in $p_B(W)$. To this end, let $x \in \phi_B^{-1}(U)$. Then there exists $y \in r_X^{\#}(W)$ with $\phi_B(x) = \phi_B(y)$. Hence, $p_B(r_X(x)) = p_B(r_X(y)) \subset p_B(W)$. Thus, $p_B\big(r_X\big(\phi_B^{-1}(U)\big)\big) \subset p_B(W)$, which completes the proof of Claim 5.

We can show now that Y has a multiplicative lattice of semi-open maps. Indeed, let $\Psi_Y = \{p_B : B \in \Sigma\}$. According to the last claim, Ψ_Y consists of semi-open maps. If $\{p_{B(\alpha)}\} \subset \Psi_Y$, then it is easily seen that $\triangle p_{B(\alpha)} = p_B$, where B is the union of all $B(\alpha)$. Hence, $\triangle p_{B(\alpha)} \in \Psi_Y$. Finally, let $g \colon Y \to g(Y)$ be an arbitrary map with $w(g(Y)) = \tau$. Considering $g(Y)$ as a subspace of $\mathbb{R}^\tau$, we can extend g to a map $h \colon \mathbb{R}^\Gamma \to \mathbb{R}^\tau$ (recall that Y is C-embedded in $\mathbb{R}^\Gamma$). Then there exists a set $B \in \Sigma$ of cardinality τ such that $\pi_B \prec h$. Consequently, $p_B \prec g$ and $w(p_B(Y)) \leq \tau$. □

Corollary 2.3.
Every space Y co-absolute with a space possessing a multiplicative lattice of open maps is skeletally Dugundji. In particular, if Y is co-absolute to an AE(0)-space, then Y has a multiplicative lattice of semi-open maps and Y is skeletally Dugundji.

Proof. We are going to show that $Y = \text{a-}\varprojlim S$, where $S = \{Y_\alpha, p_\alpha^\beta : \alpha < \beta < \tau\}$ is a factorizing inverse system with skeletal maps such that Y_0 is a separable metric space and $p_\alpha^{\alpha+1}$ has a metrizable kernel for each α. Let $\mathbb{R}^\Gamma$ and Σ be as in the proof of Theorem 2.2. There is nothing to prove if Y is second countable. So, let $w(Y) > \aleph_0$ and assume that Γ is the set of all ordinals $\alpha < \omega(\tau)$. For every $\alpha \in \Gamma$ there exists a countable set $A(\alpha) \in \Sigma$ containing α. Define $B(0) = A(0)$, $B(\alpha) = \bigcup\{A(\beta) : \beta < \alpha\}$ if α is a limit ordinal, and $B(\alpha) = \bigcup\{A(\beta) : \beta \le \alpha\}$ if α is isolated. Then the inverse system $S = \left\{Y_{B(\alpha)}, p_{B(\alpha)}^{B(\beta)} : \alpha < \beta < \omega(\tau)\right\}$ consists of skeletal maps and $p_{B(\alpha)}^{B(\alpha+1)}$ has a metrizable kernel for all α. Moreover, Y is a dense subset of $\varprojlim S$ with $Y = \text{a-}\varprojlim S$.

The second part of the corollary follows from the fact that each AE(0)-space has a multiplicative lattice of open maps. □

Corollary 2.4.
Let Y be a Lindelöf p-space co-absolute with a space possessing a multiplicative lattice of open maps. Then Y has a multiplicative lattice of perfect skeletal maps.

Proof. Suppose Y is co-absolute with a space X possessing a multiplicative lattice of open maps. Recall that a Lindelöf p-space [1] is a space admitting a perfect map onto a separable metric space, and that this property is invariant under images and preimages of perfect maps. So, X is also a a Lindelöf p-space. Then X and Y can be embedded as closed subsets of a product of the form $M \times \mathbb{I}^\Gamma$ for some Γ, where M is a second countable space. Considering in Proposition 2.1 such a product instead of $\mathbb{R}^\Gamma$, we obtain a family $\mathcal{A} = \{B \subset \Gamma : \phi_B \in \Psi\}$, where $\phi_B = \pi_B{\restriction}X \colon X \to \pi_B(X)$ is the restriction of the projection $\pi_B \colon M \times \mathbb{I}^\Gamma \to M \times \mathbb{I}^B$. Hence, all ϕ_B, $B \in \mathcal{A}$ are perfect and open maps. Similarly, replacing $\mathbb{R}^\Gamma$ in Theorem 2.2 with $M \times \mathbb{I}^\Gamma$, we obtain the family $\Sigma \subset \mathcal{A}$ and that the perfect skeletal maps $p_B = \pi_B{\restriction}Y \colon Y \to \pi_B(Y)$, $B \in \Sigma$, form a multiplicative lattice. □

3. Skeletally Dugundji spaces

The next proposition provides examples of skeletally Dugundji spaces.

Proposition 3.1.
A space X is skeletally Dugundji if it satisfies one of the following conditions:

(i) *X has a multiplicative lattice of skeletal maps;*

(ii) *X is strongly π-regularly C^*-embedded subset of a space with a multiplicative lattice of open maps.*

Proof. Suppose X has a multiplicative lattice Ψ of skeletal maps. For every continuous function $f \in C(X)$ fix a map $\phi_f \in \Psi$ such that $\phi_f \prec f$ and $\phi_f(X)$ is second countable. We assume that $C(X) = \{f_\alpha : \alpha < \omega(\tau)\}$ for some cardinal τ. Let $\varphi_0 = \phi_{f_0}$, φ_α is the diagonal product $\triangle_{\beta<\alpha}\phi_{f_\beta}$ if α is a limit ordinal, and $\varphi_\alpha = \triangle_{\beta\le\alpha}\phi_{f_\beta}$ if α is isolated. Define $X_\alpha = \varphi_\alpha(X)$ and $\varphi_\beta^\alpha : X_\alpha \to X_\beta$, $\beta < \alpha$, to be the natural projection. Since all $\varphi_\alpha \in \Psi$, the inverse system $S = \{X_\alpha, \varphi_\beta^\alpha : \beta < \alpha < \omega(\tau)\}$ consists of skeletal maps and X is the almost limit of S. It follows from the definition of the maps φ_α that for every $f \in C(X)$ there exists α with $\varphi_\alpha \prec f$. So, S is factorizing. This proves (i).

Let M be a space with a multiplicative lattice Φ of open maps and X be strongly π-regularly C^*-embedded in M. Let $\mathrm{e} \colon \mathcal{T}_X \to \mathcal{T}_M$ be a strongly π-regular operator and $M_\phi = \phi(M)$ for any $\phi \in \Phi$. We say that a map $\phi \in \Phi$ is e-admissible if

$$\phi^{-1}\big(\phi(\overline{\mathrm{e}(\phi^{-1}(U) \cap X)})\big) = \overline{\mathrm{e}(\phi^{-1}(U) \cap X)}, \qquad U \in \mathcal{B}_\phi, \tag{1}$$

where $\mathcal{B}_\phi$ is an open base for M_ϕ of cardinality $w(M_\phi)$. We are going to show that the family $\Phi_X = \{\phi{\restriction}X : \phi \text{ is e-admissible}\}$ is a multiplicative lattice on X consisting of skeletal maps. Our arguments follow the proof of [21, Proposition 3.7]. Let $X_\varphi = \varphi(X)$, $\varphi \in \Phi_X$.

Claim 6. *Any $\varphi \in \Phi_X$ is skeletal.*

Let $U \subset X$ be open in X and $\varphi = \phi{\restriction}X$ for some e-admissible $\phi \in \Phi$. Because ϕ is open, it suffices to show that $\phi(\mathrm{e}(U)) \cap X_\varphi \subset \overline{\varphi(U)}^{X_\varphi}$. Suppose there exists a point $z \in \phi(\mathrm{e}(U)) \cap X_\varphi \setminus \overline{\varphi(U)}^{X_\varphi}$ and take $V \in \mathcal{B}_\phi$ containing z such that $V \cap \overline{\varphi(U)} = \varnothing$ (here $\overline{\varphi(U)}$ is the closure in M_ϕ). Since ϕ is e-admissible, $\phi^{-1}\big(\phi(\overline{\mathrm{e}(U_1)})\big) = \overline{\mathrm{e}(U_1)}$, where $U_1 = \phi^{-1}(V) \cap X$. Obviously, $U_1 \cap U = \varnothing$ and $\varphi(U_1) = V \cap X_\varphi$. Because $\mathrm{e}(U_1) \cap X$ is dense in U_1, we have $\overline{\phi(\mathrm{e}(U_1) \cap X)} = \overline{\varphi(U_1)} = \overline{V \cap X_\varphi}$. Since $\phi(\overline{\mathrm{e}(U_1)})$ is closed in M_ϕ (recall that ϕ, being open, is a quotient map), $z \in \phi(\overline{\mathrm{e}(U_1)}) \cap \phi(\mathrm{e}(U))$ which implies $\overline{\mathrm{e}(U_1)} \cap \mathrm{e}(U) \neq \varnothing$. So, $\mathrm{e}(U_1) \cap \mathrm{e}(U) \neq \varnothing$, and consequently, $U \cap U_1 \neq \varnothing$. This contradiction completes the proof of the claim.

Claim 7. *The diagonal product of any family $\{\phi_\alpha : \alpha \in A\}$ of* e*-admissible maps is* e*-admissible.*

For every $\alpha \in A$ fix a base $\mathcal{B}_{\phi_\alpha}$ for M_{ϕ_α} satisfying condition (1). Let $\phi_0 = \triangle_{\alpha\in A}\phi_\alpha$ and $V \in \mathcal{B}_{\phi_0}$, where $\mathcal{B}_{\phi_0}$ is the standard open base of M_{ϕ_0} generated by all $\mathcal{B}_{\phi_\alpha}$. Then $\phi_0^{-1}(V) = \bigcap_{i=1}^k \phi_{\alpha(i)}^{-1}(V_i)$ for some $V_i \in \mathcal{B}_{\phi_{\alpha(i)}}$. The equality $\mathrm{e}\big(\phi^{-1}(V) \cap X\big) = \bigcap_{i=1}^k \mathrm{e}\big(\phi_{\alpha(i)}^{-1}(V_i) \cap X\big)$ together with the facts that ϕ_{α_0} is open and $\phi_{\alpha_0} \prec \phi_{\alpha_i}$ for each i, implies that $\phi_{\alpha_0}^{-1}\big(\phi_{\alpha_0}\big(\overline{\mathrm{e}\big(\phi_{\alpha_0}^{-1}(V) \cap X\big)}\big)\big) = \overline{\mathrm{e}\big(\phi_{\alpha_0}^{-1}(V) \cap X\big)}$. Hence, ϕ_{α_0} is e-admissible.

Claim 8. *For every map $f\colon X \to f(X)$ there exists $\varphi \in \Phi_X$ such that $\varphi \prec f$ and $w(X_\varphi) \leq w(f(X))$.*

We embed $f(X)$ in $\mathbb{I}^\tau$, where $\tau = w(f(X))$. Since X is C^*-embedded in M, f can be extended to a map $\overline{f}\colon M \to \mathbb{I}^\tau$. Next, there exists $\phi_0 \in \Phi$ with $\phi_0 \prec \overline{f}$ and $w(M_{\phi_0}) \leq \tau$. We construct by induction a sequence $\{\phi_n\}_{n\geq 0} \subset \Phi$, such that for every $n \geq 0$ we have:

- $\phi_{n+1} \prec \phi_n$;
- $w(M_{\phi_n}) \leq \tau$;
- ϕ_{n+1} satisfies condition (1) for all $U \in \mathcal{B}_n$ with $\mathcal{B}_n$ being a base for M_{ϕ_n} of cardinality $\leq \tau$.

Suppose ϕ_n is already constructed. For each $U \in \mathcal{B}_n$ there exists $\phi_U \in \Phi$ such that $\phi_U^{-1}\big(\phi_U\big(\overline{\mathrm{e}\big(\phi_U^{-1}(U) \cap X\big)}\big)\big) = \overline{\mathrm{e}\big(\phi_U^{-1}(U) \cap X\big)}$ and $w(M_{\phi_U})$ is countable (see [21, Lemma 3.6] for a similar statement). Obviously, $\phi_{n+1} = \triangle\{\phi_U : U \in \mathcal{B}_n\}$ satisfies the above conditions. This completes the construction. It is easily seen that the diagonal product ϕ of all ϕ_n is e-admissible. Hence, $\varphi = \phi{\restriction}X$ is as required.

Therefore, $\Phi_X = \{\phi{\restriction}X : \phi \text{ is e-admissible}\}$ is indeed a multiplicative lattice on X consisting of skeletal maps. Finally, according to (i), X is skeletally Dugundji. □

Corollary 3.2.

Let M be a product of separable metric spaces. Then every strongly π-regularly C^*-embedded subset of M is skeletally Dugundji.

A map $f\colon X \to Y$ is said to have a π-weight $\leq \tau$ [15], notation $\pi w(f) \leq \tau$, if there exists a family $\mathcal{B}$ of functionally open sets in X with $|\mathcal{B}| \leq \tau$ such that $\gamma = \{U \cap f^{-1}(V) : U \in \mathcal{B}, V \in \mathcal{T}_Y\}$ is a π-base for X (i.e., every open subset of X contains some $W \in \gamma$). Everywhere below $p(X)$ denotes the absolute of X and $\theta_X\colon p(X) \to X$ is the canonical irreducible map.

Theorem 3.3.

For any space X the following are equivalent:

(i) *X is skeletally Dugundji;*

(ii) *every compactification of X is co-absolute to a Dugundji space;*

(iii) *X has a compactification co-absolute to a Dugundji space;*

(iv) *every C^*-embedding of the absolute $p(X)$ in another space is strongly π-regular;*

(v) *X has a multiplicative lattice of skeletal maps.*

Proof. (i)$\Rightarrow$(ii). Since every compactification of X is an irreducible image of βX, it suffices to show that βX is co-absolute with a Dugundji space. Let $S = \{X_\alpha, p^\gamma_\alpha : \alpha < \gamma < \tau\}$ be a factorizing inverse system with skeletal maps such that $X = \text{a-}\varprojlim S$, X_0 is second countable and $p^{\alpha+1}_\alpha$ has a metrizing kernel for all α. Take a second countable compactification Y_0 of X_0 and consider the inverse system $\widetilde{S} = \{Y_\alpha, q^\gamma_\alpha : \alpha < \gamma < \tau\}$, where $Y_\alpha = \beta X_\alpha$, $q^\gamma_\alpha = \beta(p^\gamma_\alpha)$ for $\alpha > 0$, and $q^\gamma_0 \colon \beta X_\gamma \to Y_0$ is the extension of p^γ_0. Then $\pi w(q^{\alpha+1}_\alpha) \le \aleph_0$ for each α because $p^{\alpha+1}_\alpha$ has a metrizable kernel. Moreover, all projections q^γ_α are skeletal. Since S is factorizing, βX is the limit space of $\widetilde{S}$. Therefore, we can apply Shapiro's result [15, Theorem 2] to conclude that βX is co-absolute to a Dugundji space.

(ii)$\Leftrightarrow$(iii). If there exists a compactification $c(X)$ of X which is co-absolute to a Dugundji space, then so is βX as an irreducible preimage of X. Hence, every compactification of X has this property.

(ii)$\Rightarrow$(iv). Suppose βX is co-absolute to an AE(0)-space Y and let $p(X)$ be C^*-embedded in a space Z. Then $\overline{p(X)}^{\beta Z}$ is homeomorphic to $\beta p(X)$. Since $\beta p(X) = p(\beta X)$, $\beta p(X)$ is the absolute of Y. Consider the canonical irreducible maps $\theta_{\beta Z} \colon p(\beta Z) \to \beta Z$ and $\theta_Y \colon \beta p(X) \to Y$. Because $Y \in \text{AE}(0)$, the restriction $\varphi = \theta_Y \circ \theta_{\beta Z} \restriction H \colon H \to Y$ has a continuous extension $\phi \colon p(\beta Z) \to Y$, where $H = \theta^{-1}_{\beta Z}(\beta p(X))$. Finally, for every open $U \subset \beta p(X)$ define $\mathrm{e}(U) = \theta^{\#}_{\beta Z}\big(\phi^{-1}\big(\theta^{\#}_Y(U)\big)\big)$. It is easily seen that $\mathrm{e} \colon \mathcal{T}_{\beta p(X)} \to \mathcal{T}_{\beta Z}$ is a strongly π-regular operator. This implies that $p(X)$ is strongly π-regularly embedded in Z.

(iv)$\Rightarrow$(v). Obviously, item (iv) implies that every embedding of $p(\beta X)$ in any Tychonoff cube $\mathbb{I}^\tau$ is strongly π-regular. So, by Corollary 3.2, $p(\beta X)$ is the limit space of a continuous inverse system $\{Z_\alpha, q^\gamma_\alpha : \alpha < \gamma < \tau\}$ with skeletal projections such that Z_0 is a metric compactum and each $q^{\alpha+1}_\alpha$ has a metrizable kernel. Then, according to [15, Theorem 2], $p(\beta X)$ is co-absolute to a Dugundji space Y. Because $p(\beta X)$ is extremally disconnected, it is the absolute of Y. Since Y has a multiplicative lattice of open maps (as an AE(0)-space), βX has a multiplicative lattice Ψ of skeletal maps (by Theorem 2.2). Finally, observe that $\Psi_X = \{\varphi \restriction X : \varphi \in \Psi\}$ is a multiplicative lattice on X consisting of skeletal maps.

(v)$\Rightarrow$(i). This implication follows directly from Proposition 3.1. □

Recall that X is skeletally generated [21] if there exists a factorizing inverse system $S = \{X_\alpha, p^\beta_\alpha : A\}$ consisting of separable metric spaces X_α and surjective skeletal bonding maps such that: (i) the index set A is σ-complete (every countable chain in A has a supremum in A); (ii) for every countable chain $\{\alpha_n : n \ge 1\} \subset A$ with $\beta = \sup\{\alpha_n : n \ge 1\}$ the space X_β is a (dense) subset of $\varprojlim \{X_{\alpha_n}, p^{\alpha_{n+1}}_{\alpha_n}\}$; (iii) X is embedded in $\varprojlim S$ such that $p_\alpha(X) = X_\alpha$ for each α.

Corollary 3.4.
Every skeletally Dugundji space is skeletally generated.

Proof. Let Ψ be a multiplicative lattice for X consisting of skeletal maps and $\Psi_0 = \{\phi \in \Psi : w(\phi(X)) \le \aleph_0\}$. Define $X_\phi = \phi(X)$ and equip Ψ_0 with the partial order $\phi_1 \prec \phi_2$ iff there exists a map $\phi^2_1 \colon X_{\phi_1} \to X_{\phi_1}$ such that $\phi^2_1 \circ \phi_2 = \phi_1$. It is easily seen that the inverse system $S = \{X_\phi, \phi^\beta_\alpha : \Psi_0\}$ satisfies all conditions from the definition of skeletally generated spaces. □

Corollary 3.5.
Any perfect image of a normal space possessing a multiplicative lattice of open maps is skeletally Dugundji. In particular, any dyadic compactum is skeletally Dugundji.

Proof. Suppose M is a normal space with a multiplicative lattice of open maps and let $f \colon M \to X$ be a perfect surjection. Then there exists a closed subset Z of M such that the restriction $g = f \restriction Z \colon Z \to X$ is irreducible. One can see that $\mathrm{e}(U) = f^{-1}(g^{\#}(U))$, $U \in \mathcal{T}_Z$, defines a strongly π-regular operator $\mathrm{e} \colon \mathcal{T}_Z \to \mathcal{T}_M$. Hence, by Proposition 3.1, Z is a skeletally Dugundji space. Next, Theorem 3.3 yields that βZ is co-absolute to a Dugundji space. Finally, since $\beta g \colon \beta Z \to \beta X$ is irreducible, $p(\beta X) = p(\beta Z)$. Therefore, X is skeletally Dugundji. □

Corollary 3.6.
Let X be a skeletally Dugundji space. Then for every C^*-embedding of $p(X)$ in any extremally disconnected space Z there exists a retraction from βZ onto $p(\beta X)$.

Proof. By Theorem 3.3, there exists a strongly π-regular operator $\mathrm{e}\colon \mathcal{T}_{p(\beta X)} \to \mathcal{T}_{\beta Z}$. Then the operator $\overline{\mathrm{e}}\colon \mathcal{T}_{p(\beta X)} \to \mathcal{T}_{\beta Z}$, $\overline{\mathrm{e}}(U) = \overline{\mathrm{e}(U)}$ has the following properties: $\overline{\mathrm{e}}(U \cap V) = \overline{\mathrm{e}}(U) \cap \overline{\mathrm{e}}(V)$ and $\overline{\mathrm{e}}(U) \cap p(X) = \overline{U}$ for any open sets $U, V \subset p(\beta X)$. Next, define the set-valued map $\Phi\colon \beta Z \to 2^{p(\beta X)}$, $\Phi(z) = \bigcap\{\overline{U} : z \in \overline{\mathrm{e}}(U),\ U \in \mathcal{T}_{p(\beta X)}\}$ if $z \in \bigcup\{\overline{\mathrm{e}}(U) : U \in \mathcal{T}_{p(\beta X)}\}$, and $\Phi(z) = p(\beta X)$ otherwise. It is easily seen that Φ is an upper semi-continuous map with compact non-empty values and $\Phi(z) = z$ for $z \in p(\beta X)$. Finally, according to [5], Φ has a continuous selection $r\colon \beta Z \to p(\beta X)$. Obviously, r is a retraction. □

A combination of Corollary 3.5 and Corollary 3.6 implies a generalization of the following result [2]: the absolute of any dyadic compactum is a retract of any extremally disconnected space in which it is embedded.

Corollary 3.7.
Any product of skeletally Dugundji spaces is skeletally Dugundji.

Proof. Suppose $X = \prod_{\alpha\in A} X_\alpha$ with each X_α being skeletally Dugundji. We can suppose that all X_α are compact spaces. Then, according to Theorem 3.3, for each α there exists a strong π-regular operator $\mathrm{e}\colon \mathcal{T}_{p(X_\alpha)} \to \mathcal{T}_{\mathbb{I}^{A(\alpha)}}$ for some $A(\alpha)$. Denote by $\mathcal{B}$ the family of all standard open sets $U = U_{\alpha_1} \times \cdots \times U_{\alpha_k} \times \{p(X_\alpha) : \alpha \neq \alpha_i,\ i = 1, \dots, k\}$ in $Z = \prod_{\alpha\in A} p(X_\alpha)$, and let $\mathrm{e}'(U) = \mathrm{e}_{\alpha_1}(U_{\alpha_1}) \times \cdots \times \mathrm{e}_{\alpha_k}(U_{\alpha_k}) \times \{\mathbb{I}^{A(\alpha)} : \alpha \neq \alpha_i,\ i = 1, \dots, k\}$. Finally, $\mathrm{e}\colon \mathcal{T}_Z \to \mathcal{T}_Y$, $\mathrm{e}(W) = \bigcup\{\mathrm{e}'(U) : U \subset W,\ U \in \mathcal{B}\}$, where $Y = \prod_{\alpha\in A} \mathbb{I}^{A(\alpha)}$, is a strong π-regular operator. Hence, by Proposition 3.1 (ii), Z is skeletally Dugundji. Since the map $\theta = \prod_{\alpha\in A} \theta_\alpha\colon Z \to X$ is irreducible, X and Z are co-absolute. Therefore, X is skeletally Dugundji. □

For any map $f\colon X \to Y$ let $p(f)$ denote the absolute of f. It is well known that $p(f)$ is a map from $p(X) \to p(Y)$ such that $\theta_Y \circ p(f) = \theta_X \circ f$, where $p(X)$ and $p(Y)$ are the absolutes of X and Y, respectively, and $\theta_X\colon p(X) \to X$, $\theta_Y\colon p(Y) \to Y$ are the corresponding canonical maps. In general, a map f can have many absolutes, but it has a unique one if f is skeletal.

The next lemma was established in [15, Lemma 1] in the case X and Y are compact, but the same arguments provide a more general version.

Lemma 3.8.
Let $f\colon X \to Y$ be a perfect skeletal map of a countable π-weight with X and Y being 0-dimensional Lindelöf p-spaces and $Y \in \mathrm{AE}(0)$. Then there exist a space Z and perfect maps $g\colon X \to Z$, $h\colon Z \to Y$ such that: $f = h \circ g$; g is irreducible; h is an open map having a metrizable kernel; $\dim Z = 0$.

Theorem 3.9.
The following conditions are equivalent for any Čech-complete Lindelöf p-space X:

(i) *X is the limit space of a continuous inverse system $S = \{X_\alpha, \varphi^\gamma_\alpha : \alpha < \gamma < \tau\}$ with perfect skeletal bonding maps such that X_0 is a complete separable metric space and each $\varphi^{\alpha+1}_\alpha$ has a metrizable kernel;*

(ii) *X is co-absolute with an* $\mathrm{AE}(0)$*-space;*

(iii) *X has a multiplicative lattice of perfect skeletal maps.*

Proof. (i) ⇒ (ii). Following the arguments from the proof of [15, Theorem 2], we construct an inverse system $E = \{Y_\alpha, q^\gamma_\alpha : \alpha < \gamma < \tau\}$ consisting of 0-dimensional AE(0)-spaces Y_α and open perfect bonding maps such that all $q^{\alpha+1}_\alpha$ have metrizable kernels and Y_α are co-absolute to X_α. Assume we already have constructed Y_α. So, we have perfect irreducible maps $\theta_{X_\alpha}\colon p(X_\alpha) \to X_\alpha$ and $\theta_{Y_\alpha}\colon p(X_\alpha) \to Y_\alpha$. Since $\varphi^{\alpha+1}_\alpha$ is a perfect skeletal map with a metrizable kernel, its absolute $p(\varphi^{\alpha+1}_\alpha)\colon p(X_{\alpha+1}) \to p(X_\alpha)$ is a perfect open map of countable π-weight. Then $\theta_{Y_\alpha} \circ p(\varphi^{\alpha+1}_\alpha)\colon p(X_{\alpha+1}) \to Y_\alpha$ is a perfect skeletal map of countable π-weight. Moreover, Y_α and $p(X_{\alpha+1})$ are both Čech-complete Lindelöf p-spaces

(recall that the last property is invariant under images and preimages of perfect maps). Hence, by Lemma 3.8, there exist a 0-dimensional space $Y_{\alpha+1}$ and perfect maps $q_\alpha^{\alpha+1}\colon Y_{\alpha+1} \to Y_\alpha$, $\phi\colon p(X_{\alpha+1}) \to Y_{\alpha+1}$ such that $q_\alpha^{\alpha+1}$ is open with a metrizable kernel and ϕ is irreducible. Since $p(X_{\alpha+1})$ is extremally disconnected and ϕ is irreducible, $p(X_\alpha)$ is the absolute of $Y_{\alpha+1}$ and ϕ coincides with the canonical projection $\theta_{Y_{\alpha+1}}$. Because $Y_\alpha \in \mathrm{AE}(0)$ and $q_\alpha^{\alpha+1}$ is open and has a metrizable kernel, $Y_{\alpha+1} \in \mathrm{AE}(0)$.

If $\alpha < \tau$ is a limit ordinal and the construction was done for all $\beta < \alpha$, we take Y_α to be the limit of the inverse system $\{Y_\beta, q_\beta^\gamma : \beta < \gamma < \alpha\}$. Using again that all $Y_\beta \in \mathrm{AE}(0)$ and $q_\beta^{\beta+1}$ are open perfect maps with metrizable kernels, we obtain that Y_α is a 0-dimensional AE(0)-space which is co-absolute with X_α. This completes the construction. Finally, observe that the space $Y = \varprojlim E$ is co-absolute to X and $Y \in \mathrm{AE}(0)$.

(ii) $\Rightarrow$ (iii). Since X is co-absolute to an AE(0)-space Y, Y is also a Čech-complete Lindelöf p-space. Consequently, it has a multiplicative lattice of open perfect maps. Then, by Corollary 2.4, X has a multiplicative lattice of skeletal perfect maps.

(iii) $\Rightarrow$ (i). This implication follows from the arguments used in the proof of Proposition 3.1 (i). □

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