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The geometry of the space of Cauchy data of nonlinear PDEs

Research Article

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Abstract: First-order jet bundles can be put at the foundations of the modern geometric approach to nonlinear PDEs, since higher-order jet bundles can be seen as constrained iterated jet bundles. The definition of first-order jet bundles can be given in many equivalent ways – for instance, by means of Grassmann bundles. In this paper we generalize it by means of flag bundles, and develop the corresponding theory for higher-oder and infinite-order jet bundles. We show that this is a natural geometric framework for the space of Cauchy data for nonlinear PDEs. As an example, we derive a general notion of transversality conditions in the Calculus of Variations.

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1. Introduction

Algebraic Geometry entered adulthood when its intellectual energies, traditionally committed to find concrete solutions of algebraic equations (i.e., *points* of an arithmetic space), began to wonder about the structure of the equations themselves (i.e., *ideals* in rings over arithmetic fields). The theory of nonlinear PDEs underwent a similar development, though highly ramified and dependent on the intermittent and diversified impulses coming from natural sciences, and it is still inappropriate to speak about "the" theory of nonlinear PDEs, for none of the proposed frameworks was enthusiastically embraced by the mainstream. The reader may find relevant historical information, as well as an exhaustive list of references in the 2010 review [7].

This paper is committed to the perspective that (smooth) solutions of a (regular enough) system of nonlinear PDEs in n independent variables (henceforth called *equation*, for short) are to be interpreted as the maximal integral submanifolds

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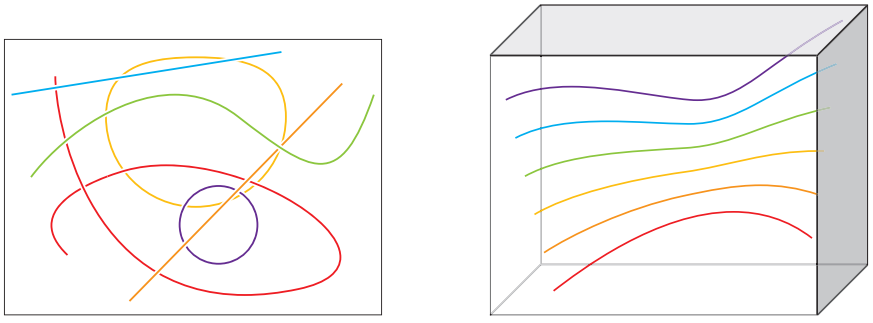
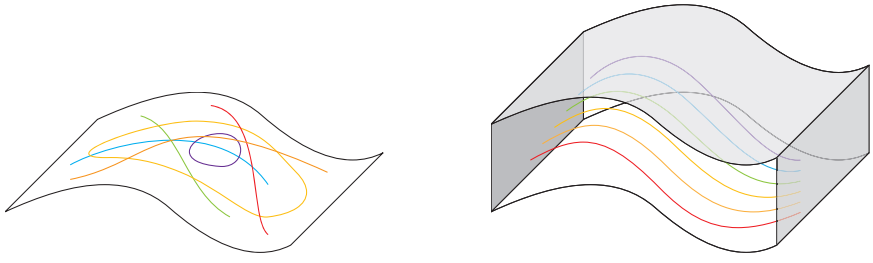
(henceforth called *leaves*) of an n-dimensional involutive distribution on a pro-finite manifold, and adheres to the philosophy that relevant invariants of the equation are encoded by an appropriate cohomological theory, possibly twisted with nonlocal coefficients, called *characteristic* or *leafwise* cohomology (of the equation). In such a framework, the space of leaves itself, which is (as a rule) quite bad-behaved, can be put aside, and the focus diverted to the characteristic cohomologies of the foliation. We shall use the word "secondary" (following [15]) as a synonymous of "leafwise" in the pro-finite context[1] and we adopt the same framework and terminology which can be found, e.g., in the introductory section of [17] (for example, a *secondary point* is just a leaf, a *secondary manifold* is the leaf space of a foliation over a pro-finite manifold, a *secondary map* is a map preserving leaves, etc.).



n-dimensional submanifolds of E $\iff$ leaves of $J^\infty(E,n)$ (1)

So, the "solution space" of an equation can be seen as a secondary submanifold of the *empty equation*[2] $J^\infty(E,n)$, since the graphs of the solutions of the former correspond to the leaves of the latter (1). Nonetheless, it is quite evident that the same equation dictates restrictions also on non-maximal integral submanifolds: indeed, by definition, a non-maximal integral submanifold is contained into a leaf, and among leaves there are the solutions. In this paper we propose a very natural geometric framework where $(n-1)$-dimensional integral submanifolds (henceforth called *small leaves*) coexist with the maximal ones, study its structure, and reveal some interesting properties of its characteristic cohomology. Small leaves are nothing but the geometric counterparts of infinite-order Cauchy data (Section 8), taking prominent roles in the theory of nonlinear PDEs, calculus of variations, field theory, etc., and, in our approach, they can be treated on the same footing as solutions. To this end, it is compulsory to "nest" one jet space into another (2), much as, in another context, flag manifolds are constructed out of nested Grassmmannians.



$(n-1)$-dimensional submanifolds of a leaf $L \subset J^\infty(E,n)$ $\iff$ leaves of $J^\infty(L,n-1)$ (2)

Once Cauchy data and solutions of a PDE are framed in the same secondary context, it becomes natural to perform algebro-geometric manipulations which mix secondary notions of horizontal degree n with ones of horizontal degree $n-1$. For example, a boundary variational integral (i.e., a secondary function of horizontal degree $n-1$) can be combined with a variational integral (i.e., a secondary function of horizontal degree n), and from their interaction it arises, in a surprisingly straightforward way, a general notion of transversality conditions (Section 11).

[1] *We keep the distinction between foliations of finite and pro-finite manifolds, due to the failure of the key Frobenius theorem on the latter.*

[2] *This is the reason why, sometimes, a leaf of $J^\infty(E,n)$ is also called a "solution."*

Structure of the paper

In Section 2 we define special subsets of the jet bundle, needed to associate with a map between manifolds a map between the corresponding jet bundles. This will allow to speak of "projectable" and, in particular, of "horizontal" jets later on, and hence to be able to deal with the jet bundles over pro-finite manifolds.

Section 3 contains the well-known material about Grassmannian and flag manifolds, with the focus on the universal sequence associated to a Grassmannian and the canonical bundles over flag manifolds. These notions are at the heart of the definition of 1^{st} order jet bundles and flag jet bundles, respectively.

In Section 4 we introduce a class of equations (given, in coordinates, by (14) and (15)) which, for n independent variables, constitute the key ingredient to define higher-order jet bundles out of lower-order ones, and, for $n-1$ independent variables, lead straightforwardly to the notion of a jet of a Cauchy datum. These are just examples of equations of involutive planes of a distribution.

Inheritance of involutivity allows to mimic the definition of a flag manifold and to introduce, in a similar fashion, higher-order flag jet bundles $J^k(E,n,n-1)$. In Section 5, besides the conceptual definition, two natural coordinate systems are proposed, stemming from the fundamental equation (20), which will be useful later on for the description of the canonical bundles associated with $J^k(E,n,n-1)$.

The notion of the 1^{st} jet of an $(n-1)$-dimensional involutive plane is "almost" the same as the notion of a flag jet, were, as usual in the theory of jet bundles, "almost" means that the desired property holds correctly only on the inverse limit, i.e., for infinite jets. Section 6 clarifies this relationship through the fundamental diagram (26). The advantage with this new point of view is that involutive planes, unlike flag jets, are naturally understood as solutions of an equation, which we denote by $\breve{\mathcal{J}}_{n-1}(\mathcal{C})$.

Having introduced flag jets was not a mere exercise, since they carry a natural normal bundle, which is essential to discover the structure of the space of Cauchy data. The idea, sketched in Section 7 by introducing the bundle of infinite-order normal directions, is that the space of Cauchy data can be seen as the space of sections of an (infinite-dimensional) bundle over a fixed Cauchy value, whose fiber coordinates capture the ideas of "purely normal derivatives".

In Section 8, after having given the formal definitions of finite and infinite-order Cauchy data, it is shown how a higher-order Cauchy datum can be constructed "over" a lower-order one, by using a section of a suitable normal bundle, where "over" means "projecting onto". This is the next step towards the clarification of the structure of the space of infinite-order Cauchy data.

The central result of the paper, Theorem 9.1, can be found in the last section of theoretical character, Section 9. It implies that the pro-finite manifold $\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})$ gives rise to three distinct secondary manifolds, one whose points are the Cauchy data, another whose points are the solutions, and the last whose points are $(n-1)$-dimensional submanifolds of solutions, thus providing a natural common framework for these three apparently heterogeneous entities. Most importantly, it shows that the (secondary) fibers of the naturally defined maps from one secondary manifold to the other, are, in turn, very simple secondary manifolds, namely *empty equations*. Handy coordinates, indispensable for applications, are also introduced here. The final comments on Theorem 9.1 are collected in Section 10, together with the envisaged consequences and applications.

In the last Section 11, we present a variational problem simultaneously involving Lagrangians with n and $n-1$ independent variables, and we test it on the toy model given by a simple 1-dimensional variational problem with constrained endpoints.

Notation and conventions

Even if we did our best to avoid proprietary notation, we must warn the reader about a somewhat extreme "slang" we are going to use throughout this paper, in line with the most recent works on the subject (see, e.g., [17]). "P is an M-module" means that P is the module of sections of a bundle π over M. Then the meaning of expressions like $P\otimes_M Q$ and $\mathrm{Hom}_M(P,Q)$ is clear. We use both π_x and E_x as synonymous of $\pi^{-1}(x)$, where π is a fibration of E. By a "plane in V" we mean just a vector subspace of V; similarly, a "plane in E" is a subspace of some tangent space to the manifold E. The term "space" without modifier (like the one appearing in the title of this paper) always means "secondary manifold".

We use the term "leaf" for an n-dimensional integral submanifold of the Cartan distribution on $J^\infty(E,n)$, and we denote it by L. A codimension-one submanifold of a leaf is called a "small leaf", and denoted by Σ. The projection of any object $\mathcal{O}$ associated with J^k (with $k=0,\dots,\infty$) on a lower order jet J^l is denoted by $\mathcal{O}_l$; for instance, L_0 is the submanifold of E which corresponds to the leaf L (but, in this case, we even skip the index "0"). We allow l to take the value -1, assuming that E_{-1} is an arbitrary choice of the manifold of independent variables (in which case we are considering a so-called *affine chart* in $J^\infty(E,n)$), and we write "$\overset{\text{loc}}{=}$" when an equality holds in coordinates or affine charts, like, e.g., $J^\infty(E,n)\overset{\text{loc}}{=}J^\infty(\pi)$. If $\theta\in J^\infty$ is the jet of a section in some point, then θ_{-1} is precisely that point. We use the word "over" to indicate that one thing projects over another.

TE denotes the tangent bundle of E, and f_* denotes the differential of $f\colon E\to E'$. If E is fibered, VE is the vertical tangent bundle (VJ^k means "vertical with respect to $\pi_{k,k-1}$"). The $\mathbb{R}$-dual of a vector space is denoted by $V^\vee$, and the annihilator of a subspace W by $W^\dagger$. The same symbol $P^\dagger$ is used, in different contexts, for the adjoint module to P. We prefer to say that "L is a leaf of $\mathcal{E}^{(\infty)}$", rather than "L_k is a solution of $\mathcal{E}\subseteq J^k(E,n)$". The R-distribution on J^k (see [1]) is called R^k, Cartan distribution on J^k is denoted by C^k, while that on J^∞ simply $\mathcal{C}$.

Modifier "local" in front of "section" or "coordinates" is skipped as a rule. All PDEs are assumed to be formally integrable.

Greek and Latin indexes range in disjoint sets: this means that the sets of coefficients of $\{\omega_a\}$ and $\{\omega_\alpha\}$ cannot have any element in common. Latin indexes correspond to independent variables, and Greek ones to dependent variables. Derivations are denoted by a semicolon: $\omega_{a;\alpha}$ means $\partial\omega_a/\partial u^\alpha$. For iterated jet spaces we encapsulate into parentheses the inner jet variables, before taking the outer derivatives, like in $(u^\alpha_a)^\beta_b$, or $(u)_a$. Concerning multi-indices for partial derivatives, uppercase Latin letters will always denote elements of the abelian group $\mathbb{N}_0^{n-1}$, even if we use multiplicative notation for its operation and the symbol O for its zero (as in the "monoidal notation", see [17]); the pair (A,l), where $l\in\mathbb{N}_0$, is an element of $\mathbb{N}_0^n$, namely the one having the first $n-1$ entries in common with A, and the last one equal to l (hence, $|(A,l)|=|A|+l$). For example,

$$\frac{\partial^{|A|+l}}{\partial x^{A,l}}=\frac{\partial^{i_1+\dots+i_{n-1}+l}}{\partial(x^1)^{i_1}\cdots\partial(x^{n-1})^{i_{n-1}}\partial(x^n)^l},\qquad A=(i_1,\dots,i_{n-1})\in\mathbb{N}_0^{n-1},\quad l\in\mathbb{N}_0. \tag{3}$$

The number n is fixed throughout this paper, index α is always assumed to be ranging in $1,\dots,m$, and index a in $1,\dots,n-1$. The symbol Aa represents the multi-index A whose a^{th} entry has been increased by one.

All constructions are coordinate-free. Nonetheless, many concepts look more familiar when written down in coordinates, so the reader will find several remarks labeled "coordinates" after any intrinsic definition.

We have chosen the notation $J^k(E,n)$ for the space of k-jets of n-dimensional submanifolds of E, just to stress the analogy with the linear case, when one works with Grassmann manifolds $\mathrm{Gr}(V,n)$ instead. Alternatively, one may regard $J^k(E,n)$ as a sub-quotient of $J^k(\mathbb{R}^n,E)$, the space of k-jets of smooth maps from $\mathbb{R}^n$ to E *à la* Michor [9], namely the space of k-jets of embeddings, factorized by the group of diffeomorphisms of $\mathbb{R}^n$. Or, in a more "mechanical" perspective, $J^k(E,n)$ may be seen as the space of k-jets of regular, parameter-free n-velocities in E *à la* Krupka [8]. No matter which point of view is adopted, the definition is the same, viz.,

$$J^k(E,n)=\bigsqcup_{y\in E}J^k_y(E,n),\qquad J^k_y(E,n)=\frac{\{L : L\subseteq E \text{ is } n\text{-dimensional submanifold and } L\ni y\}}{\sim^k_y},$$

where $\sim^k_y$ is the equivalence relation $L_1\sim^k_y L_2\Leftrightarrow L_1$ is tangent to L_2 at y with order k. The equivalence class of L w.r.t. $\sim^k_y$ is denoted by $[L]^k_y$. If $E=\{(x^i,u^\alpha)\}$, and $L=\mathrm{graph}(s)$, where $s=(s^1,\dots,s^m)$, with $s^\alpha=s^\alpha(x^1,\dots,x^n)$, then the jet coordinate $u^\alpha_{i_1\dots i_n}$ is defined as

$$u^\alpha_{i_1\dots i_n}\left([L]^k_{s(x)}\right)=\frac{\partial^{i_1+\dots+i_n}s}{\partial(x^1)^{i_1}\cdots\partial(x^n)^{i_n}}(\boldsymbol{x}),\qquad \boldsymbol{x}=(x^1,\dots,x^n).$$

It is worth recalling that the R-distribution is nothing but the jet-theoretic incarnation of the tautological (or universal) bundle associated to a Grassmann manifold; it associates with a point $\theta\in J^k$ the n-dimensional subspace of $T_{\theta_{k-1}}J^{k-1}$

spanned by

$$\partial_l|_{\theta_{k-1}} + \sum_{i_1+i_2+\cdots+i_n\leqslant k-1} u^\alpha_{i_1\cdots i_l+1\cdots i_n}(\theta)\,\partial_{u^\alpha_{i_1 i_2\cdots i_n}}\Big|_{\theta_{k-1}}, \qquad l=1,\dots,n, \tag{4}$$

where ∂_l is a short for ∂_{x^l}. The R-distribution "generates" the Cartan distribution, in the sense that $\mathcal{C}^k_\theta = \pi^{-1}_{k,k-1*}(R^k_\theta)$; as such, besides the n vectors (4), it takes also all the $\pi_{k,k-1}$-vertical vectors to span it.

2. Jet maps

Obvious Definition 2.2 below is given just to simplify subsequent constructions. Let $f\colon E\to E'$ be a smooth map.

Proposition 2.1.
The subset $\breve{J}^k(E,n) \stackrel{\text{def}}{=} \{\theta\in J^k(E,n) : R_{\theta_1}\cap\ker f_* = 0\}$ *is an open sub-bundle. Moreover, the natural map* $f_*\colon \breve{J}^k(E,n)\to J^k(E',n)$ *is smooth.*

Proof. Notice that $R_{\theta_1}\cap\ker f_*$ is the kernel of the restriction

$$f_*\restriction_{R_{\theta_1}}\colon R_{\theta_1}\to T_{f(\theta_0)}E' \tag{5}$$

of f_* to $R_{\theta_1}\subseteq T_{\theta_0}E$. In turn, $R_{\theta_1} = \langle\, \partial_i|_{\theta_0} + u^\alpha_i(\theta)\,\partial_{u^\alpha}|_{\theta_0} : i=1,\dots,n\,\rangle$, where $\{x^i,u^\alpha\}$ are local coordinates on E, and ∂_i is a short for ∂_{x^i}. Hence, $\theta\in J^k(E,n)$ belongs to $\breve{J}^k(E,n)$ if and only if (5) is injective, i.e., if and only if the n tangent vectors $f_*\left(\partial_i|_{\theta_0} + u^\alpha_i(\theta)\,\partial_{u^\alpha}|_{\theta_0}\right)$ are linearly independent in $T_{f(\theta_0)}E'$, which means that the n-multivector

$$Y(\theta) = f_*\left(\partial_1|_{\theta_0} + u^\alpha_1(\theta)\,\partial_{u^\alpha}|_{\theta_0}\right)\wedge\cdots\wedge f_*\left(\partial_n i|_{\theta_0} + u^\alpha_n(\theta)\,\partial_{u^\alpha}|_{\theta_0}\right)\in T_{f(\theta_0)}^{\wedge n}E'$$

must be nonzero. The result follows from the fact that Y depends smoothly on θ, and that $Y(\theta)\neq 0$ is an open condition. □

Definition 2.2.
We call $\breve{J}^k(E,n)$ the bundle of *f-mappable jets*, and f_* the induced *jet map*.

Example 2.3.
If π is a bundle of E over E_{-1}, then π-mappable jets are just jets of sections of π, i.e., $\breve{J}^k(E,n) = J^k(\pi)$. In this case, the map π_* is not very interesting, since $J^k(E_{-1},n)$ is a one-point manifold.

Example 2.4.
If f is an embedding, then all jets are f-mappable. In particular, if s is a section of π, then all jets in $J^l(E_{-1},r)$ are $j_k(s)$-mappable, so that there are well-defined smooth maps

$$j_k(s)_*\colon J^l(E_{-1},r)\to J^l\big(J^k(E,n),r\big).$$

Similarly, for any n-dimensional submanifold $L\subset E$, and $r\leqslant n$, there is a well-defined smooth map

$$j_k(L)_*\colon J^l(L,r)\to J^l\big(J^k(E,n),r\big). \tag{6}$$

Map (6) is the key to "lift" an r-dimensional submanifold of E to a special submanifold of $J^k(E,n)$, namely an *involutive* one (Section 4).

Remark 2.5.
It should be stressed that, as a rule, there is no natural embedding

$$J^k(E, n-1) \hookrightarrow J^k(E, n) \tag{7}$$

and (6) has to be regarded as the closest way one has to (7), when the necessity arises to force jets of $(n-1)$-dimensional submanifolds into the jet bundle of n-dimensional submanifolds. Nonetheless, (7) can be accomplished in a local, non-canonical way. Namely, equip E_{-1} with a metric g. Then each small submanifold $\Sigma \subseteq E$ can be seen as the graph of a section σ of π over Σ_{-1}. So, $[\Sigma]^k_y = [\sigma]^k_{y_{-1}}$, and σ can be extended to a constant section $\iota_{y_{-1}}(\sigma)$ along the orthogonal direction to $T_{y_{-1}}\Sigma_{-1}$. Then (7) is given by $[\Sigma]^k_y \mapsto [\iota_{y_{-1}}(\sigma)]^k_{y_{-1}}$.

Example 2.6.
All elements of $J^k(E, n)$, seen as a subset of $J^1\left(J^{k-1}(E, n), n\right)$, are $\pi_{k-1,k-2}$-mappable, and $(\pi_{k-1,k-2})_* = \pi_{k,k-1}$.

Example 2.7.
Let $E = \lim E_k$ be a pro-finite manifold. Then $J^r(E, n) = \lim \breve{J}^r(E_k, r)$.

3. Grassmannians and flag manifolds

The following basic facts about flags manifolds and Grassmannians belong to the common knowledge, so that it is hard to point an appropriate reference. Concerning the link between Grassmannians and jet spaces, a nice exposition can be found in the classical book [4].

Definition 3.1.
$\mathrm{Gr}(V, n) \stackrel{\text{def}}{=} \{L \subseteq V : L \text{ is an } n\text{-dimensional plane in } V\}$ is the Grassmannian of n-dimensional planes in V.

Recall that over $\mathrm{Gr}(V, n)$ it grows the so-called *universal sequence* of vector bundles

$$\begin{array}{ccccc} R(V,n) & \hookrightarrow & \mathrm{Gr}(V,n)\times V & \twoheadrightarrow & N(V,n) \\ & {\scriptstyle R}\searrow & \downarrow{\scriptstyle \tau} & \swarrow{\scriptstyle N} & \\ & & \mathrm{Gr}(V,n), & & \end{array} \tag{8}$$

where τ is the trivial bundle, R is the *tautological bundle*, and N is the *normal bundle*. By definition, $R_L = L$ (hence the name "tautological") and $N_L = V/L$, for all $L \in \mathrm{Gr}(V, n)$. In particular, $\operatorname{rank} R = n$ and $\operatorname{rank} N = \dim V - n$, and it holds the non-canonical bundle isomorphism

$$\tau_{\mathrm{Gr}(V,n)} \cong \mathrm{Hom}(R, N) = R^\vee \otimes_{\mathrm{Gr}(V,n)} N, \tag{9}$$

incidentally showing that $\dim \mathrm{Gr}(V, n) = (\dim V - n)\, n$.

Let now $\xi\colon E \to M$ be a vector bundle.

Lemma 3.2.
A smooth bundle $\mathrm{Gr}(E, n)$ *over* M *exists, and a short exact sequence of vector bundles over* $\mathrm{Gr}(E, n)$,

$$\begin{array}{ccccc} R(E,n) & \hookrightarrow & \mathrm{Gr}(E,n)\times_M E & \twoheadrightarrow & N(E,n) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Gr}(E,n), & & \end{array} \tag{10}$$

such that $\mathrm{Gr}(E,n)_x = \mathrm{Gr}(E_x,n)$ *and the restriction of* (10) *to a point* $x \in M$ *equals* (8)*, with* $V = E_x$*, and the following bundle isomorphism holds:*

$$V\,\mathrm{Gr}(E,n) \;\cong\; \mathrm{Hom}_{\mathrm{Gr}(E,n)}\big(R(E,n),N(E,n)\big) \;=\; R(E,n)^{\vee}\otimes_{\mathrm{Gr}(E,n)} N(E,n).$$

Proof. The first statement follows straightforwardly (by using transition functions) from the fact that the universal sequence (8) is well-behaved w.r.t. linear transformations of V, i.e., each $\phi \in \mathrm{GL}(V)$ induces a diffeomorphism $\underline{\phi}$ of $\mathrm{Gr}(V,n)$, and bundle automorphisms of $R(V,n)$, $\mathrm{Gr}(V,n)\times V$, and $N(V,n)$, which cover $\underline{\phi}$.

The second one is a consequence of (9), since $V_x\,\mathrm{Gr}(E,n)$ coincides with $T\,\mathrm{Gr}(E_x,n)$. □

Example 3.3.

$\mathrm{Gr}(TE,n)$ is one possible definition of $J^1(E,n)$ (see, e.g, [4]). An alternative one, given in term of tangency classes, can be found, e.g., in [1].

Example 3.4 (definition of flag manifolds).

$\mathrm{Gr}(R(V,n),n-1)$ is the flag manifold $\mathrm{Gr}(V,n,n-1)$. The corresponding canonical sequence

$$\begin{array}{ccccc} R(R(V,n),n-1) & \hookrightarrow & \mathrm{Gr}(R(V,n),n-1)\times_{\mathrm{Gr}(V,n)} R(V,n) & \twoheadrightarrow & N(R(V,n),n-1) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Gr}(R(V,n),n-1) & & \end{array}$$

is simply denoted by

$$\begin{array}{ccccc} r & \hookrightarrow & R & \twoheadrightarrow & n \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Gr}(V,n,n-1). & & \end{array} \tag{11}$$

By definition, if $\theta = (L,\Sigma) \in \mathrm{Gr}(V,n,n-1)$, $r_\theta = \Sigma$, $R_\theta = L$, and $n_\theta = L/\Sigma$.

Example 3.4 shows that $\mathrm{Gr}(V,n,n-1)$ is naturally fibered over $\mathrm{Gr}(V,n)$, and that $\mathrm{Gr}(V,n,n-1)_L = \mathrm{Gr}(L,n-1)$, for all $L \in \mathrm{Gr}(V,n)$.

Fact 3.5 (canonical fibrations of flag manifolds).

$\mathrm{Gr}(V,n,n-1)$ *is naturally fibered over both* $\mathrm{Gr}(V,n-1)$ *and* $\mathrm{Gr}(V,n)$*, i.e.,*

$$\begin{array}{ccc} & \mathrm{Gr}(V,n,n-1) & \\ {\scriptstyle p^1}\swarrow & & \searrow{\scriptstyle n^1} \\ \mathrm{Gr}(V,n) & & \mathrm{Gr}(V,n-1) \end{array} \tag{12}$$

where $n^1_\Sigma = \mathbb{P}(\Sigma^\dagger)$ *for all* $\Sigma \in \mathrm{Gr}(V,n-1)$ *and* $n^1_L = \mathbb{P}(L^\vee)$ *for all* $L \in \mathrm{Gr}(V,n)$.

The definition given by Example 3.4, the sequence (11) and the fibrations (12) are easily generalized to flags with more indices and complete flags, but they will not play a relevant role in our analysis. The aim of this section was to stress that, even if the family of all n-dimensional planes in V has a natural smooth manifold structure, the same is not true if in the same family enter $(n-1)$-dimensional planes, since, roughly speaking, the latter are more numerous than the former. Then one is forced to introduce a certain redundancy in the information about n-dimensional planes, to get something smooth: the result is $\mathrm{Gr}(V,n,n-1)$. A redundancy conceptually similar, but technically more involved, will have to be introduced in the context of nonlinear PDEs, in order to treat leaves and small leaves "as members of the same family".

4. The equation of involutive planes

Let P be an E-module, and suppose that $\Delta = \ker \Omega$ is a distribution given by means of the P-valued 1-form Ω. Let also $\Pi \in \Lambda^2(\Delta^\vee) \otimes_E \frac{\Delta^{(1)}}{\Delta}$ be the curvature form of Δ.

Definition 4.1.
A tangent plane R to E is called *involutive* if

$$\Omega\restriction_R = 0, \qquad \Pi\restriction_R = 0. \tag{13}$$

The totality of r-dimensional involutive planes of E is the *equation of involutive r-dimensional planes* of E, and denoted by $\mathcal{I}_r(\Delta)$.

Example 4.3 below should convince the reader about the smoothness of the submanifold $\mathcal{I}_r(\Delta)$ of $J^1(E,r)$.

Remark 4.2.
Definition 4.1 is hereditary for linear subspaces, since so are conditions (13).

Example 4.3 (coordinates).
Let $E = \{(x^i, u^\alpha)\}$, and Δ be given by means of 1-forms $\Delta = \bigcap_{A\in\mathbb{A}} \ker \omega^A$, with $\omega^A = \omega^A_i dx^i + \omega^A_\alpha du^\alpha$. Then

$$\mathcal{I}_r(\Delta) = \left\{\theta \in J^1(E,r) : \omega^A\restriction_{R_\theta} = 0,\ d\omega^A\restriction_{R_\theta} = 0\right\}$$

is locally given by the vanishing of the functions

$$f^A_i = \omega^A_i + \omega^A_\alpha u^\alpha_i, \tag{14}$$
$$f^A_{ij} = \omega^A_{[i;j]} + \omega^A_{[[i;\alpha]} u^\alpha_{j]} + \omega^A_{[\alpha;\beta]} u^\alpha_j u^\beta_i. \tag{15}$$

Remark 4.4.
Let $\mathcal{E} \subset J^1(E,r)$ be given just by the vanishing of (14) alone. Then $\mathcal{I}_r(\Delta) = \pi_{2,1}(\mathcal{E}^{(1)})$, i.e., (15) are differential consequences of (14).

Example 4.5.
If $\mathcal{C}^k$ is the contact distribution on $J^k(E,n)$, then $\mathcal{I}_n(\mathcal{C}^k)$ is the closure of $J^{k+1}(E,n)$ in $J^1\big(J^k(E,n),n\big)$. Adherence points correspond to the so-called singular R-planes (firstly studied by Vinogradov in the context of singular and multivalued solutions [11, 14]).

If E is fibered (see Example 2.3), then $\breve{\mathcal{I}}_r(\Delta) \stackrel{\text{def}}{=} \mathcal{I}_r(\Delta) \cap \breve{J}^1(E,r)$ is an open and dense subset of $\mathcal{I}_r(\Delta)$.

Definition 4.6.
$\breve{\mathcal{I}}_r(\Delta)$ is the equation of *horizontal* involutive r-dimensional planes.

Example 4.7.
Let $\mathcal{C}^k$ be as in Example 4.5. Then $\breve{\mathcal{I}}_n(\mathcal{C}^k) = J^{k+1}(E,n)$.

Remark 4.8.
Leaves of $\mathcal{I}_r^{(\infty)}(\Delta)$ are in one-to-one correspondence with r-dimensional involutive submanifolds of Δ (see Remark 4.4). In other words, $\mathcal{I}_r^{(\infty)}(\Delta)$ is the secondary manifold whose points are the r-dimensional involutive submanifolds of Δ.

5. Flags jet bundles

Remark 4.2 motivates the key Definition 5.2 below. Let $n = n_d > n_{d-1} > \ldots > n_2 > n_1 > 0$ be integers, and consider the fibered product

$$X \stackrel{\text{def}}{=} J^k(E,n) \times_{J^{k-1}(E,n)} \check{J}^1\big(J^{k-1}(E,n), n_{d-1}\big) \times \cdots \times_{J^{k-1}(E,n)} \check{J}^1\big(J^{k-1}(E,n), n_1\big).$$

A point $\Theta \in X$ can be seen as a d-tuple of planes in $J^{k-1}(E,n)$, whose dimension decreases from n_d to n_1, only whose first entry is required to be involutive.

Proposition 5.1.

Denote by R^i_Θ the i^{th} plane in Θ, i.e., the one of dimension n_i. Then the subset

$$J^k(E, n_d, n_{d-1}, \ldots, n_2, n_1) \stackrel{\text{def}}{=} \big\{\Theta \in X : R^i_\Theta \supseteq R^{i-1}_\Theta,\ i = 2, \ldots, d\big\} \tag{16}$$

is a smooth sub-bundle of X.

This can be easily checked in coordinates (see Remark 5.4 below).

Definition 5.2.

$J^k(E, n_d, n_{d-1}, \ldots, n_2, n_1)$ defined as (16) is the k-order *flag jet bundle* over $J^{k-1}(E,n)$. $J^k(E, n, n-1, \ldots, 2, 1)$ is the k-order *complete flag jet bundle*.

Fact 5.3.

$J^k(E, n, n-1, \ldots, 2, 1)$ projects naturally over any $J^k(E, i)$.

From now on, the focus will be on $J^k(E, n, n-1)$. An element $\Theta \in J^k(E, n, n-1)$ is written as a pair (R_Θ, r_Θ).

Remark 5.4 (coordinates I).

Let $\Theta \in J^k(E,n) \times_{J^{k-1}(E,n)} \check{J}^1(J^{k-1}(E,n), n-1)$, and consider its coordinate expression

$$\Theta = \big(x^a,\ t,\ \underset{|A|+l\leqslant k}{u^\alpha_{A,l}},\ t_a,\ \underset{|A'|+l'\leqslant k-1}{(u^\alpha_{A',l'})_a}\big)$$

into an adapted chart. Then

$$R_\Theta = \Big\langle \partial_a + u^\alpha_{Aa,l}\partial_{u^\alpha_{A,l}} : a = 1, \ldots, n-1 \Big\rangle + \big\langle \partial_t + u^\alpha_{A,l+1}\partial_{u_{A,l}} \big\rangle, \tag{17}$$

$$r_\Theta = \Big\langle \partial_a + t_a\partial_t + (u^\alpha_{A',l'})_a \partial_{u^\alpha_{A,l}} : a = 1, \ldots, n-1 \Big\rangle, \tag{18}$$

are the corresponding planes in $J^{k-1}(E,n)$. Observe that (17) contains (18) if and only if each generator of the latter is a linear combination of generators of the former, viz.,

$$\partial_a + t_a\partial_t + (u^\alpha_{A,l})_a \partial_{u^\alpha_{A,l}} = \partial_a + u^\alpha_{Aa,l}\partial_{u^\alpha_{A,l}} + t_a\big(\partial_t + u^\alpha_{A,l+1}\partial_{u_{A,l}}\big) \qquad \text{for all } a, \tag{19}$$

where $|A| + l \leqslant k-1$. In their turn, vector equalities (19) are equivalent to the system of equations

$$(u^\alpha_{A,l})_a = u^\alpha_{Aa,l} + t_a u^\alpha_{A,l+1}, \qquad |A| + l \leqslant k-1. \tag{20}$$

Hence,

$$x^a,\ t,\ \underset{|A|+l\leqslant k}{u^\alpha_{A,l}},\ t_a \tag{21}$$

can be assumed as coordinates on $J^k(E, n, n-1)$. A rough interpretation of (20) is the following: in $J^k(E, n, n-1)$ the independent variable t has become a dependent one, so that $u^\alpha_{A,l}$ depends on x^a not only directly (first summand in the right-hand side), but also through t (second summand).

Remark 5.5 (coordinates II).
Equations (20) furnish another coordinate system on $J^k(E,n,n-1)$, which will be handier than (21) in the study of normal bundles (see Remark 9.2 later on), namely

$$x^a,\ t,\quad \underset{|A|+l\leqslant k-1}{u^\alpha_{A,l}},\ u^\alpha_{\mathrm{O},k},\ t_a,\ \underset{|A'|+l'=k-1}{(u^\alpha_{A',l'})_a}\ . \tag{22}$$

Remark 5.6.
Observe that, equations (14) coincide with (20), for the forms $\omega^\alpha_{A,l} \overset{\text{def}}{=} du^\alpha_{A,l} - u^\alpha_{Aa,l}dx^a - u^\alpha_{A,l+1}dt$, $|A|+l\leqslant k-2$, defining $\mathcal{C}^{k-1}$. On the other hand, equations (15), which read $(u^\alpha_{A[a,l)_{b]}} = (u^\alpha_{A,l+1})_{[a}t_{b]}$, $|A|+l\leqslant k-2$, are algebraic consequences of (20).

Lemma 5.7.
Coordinates (21) *represent a local diffeomorphism*

$$J^k(E,n,n-1) \overset{\text{loc}}{=} J^k(E,n)\times_{E_{-1}} J^1(E_{-1},n-1). \tag{23}$$

Proof. Let $(\theta,\theta')\in J^k(E,n)\times_{E_{-1}}J^1(E_{-1},n-1)$, with $\theta=[s]^k_x$, where s is a section of $\pi\colon E\to E_{-1}$ and $\theta'\in J^1_x(E_{-1},n-1)$. Consider the jet map $j_k(s)_*\colon J^1(E_{-1},n-1)\to J^1\big(J^{k-1}(E,n),n-1\big)$ (Definition 2.2). It is easy to see that $j_k(s)_*(\theta')$ is a small plane contained in R_θ, whose definition is independent on the choice of s. Correspondence (23) is given precisely by

$$\big(R_\theta, j_k(s)_*(\theta')\big) \quad\Longleftrightarrow\quad (\theta,\theta'). \tag{24}$$

□

Paraphrasing (24), θ has been used to "lift" the small plane $R_{\theta'}$ in E_{-1}, i.e., the 1st jet of a Cauchy surface, to a small involutive (horizontal) plane in $J^{k-1}(E,n)$ (see also Example 2.4). However, since $R_{\theta'}$ is small, for the purpose of lifting, it is not necessarily the whole jet $\theta=[s]^k_x$, but rather the $k-1^{\text{st}}$ jet of s, plus the k^{th} derivatives of s along $R_{\theta'}$. So, elements of $J^k(E,n,n-1)$ cannot yet be called 1^{st} jets of $k-1^{\text{st}}$ order Cauchy surfaces (see Section 8 below) since they contain extra information. As we show in Section 6 below, this extra information is discarded by the natural projection of $J^k(E,n,n-1)$ over $\breve{\mathcal{I}}_{n-1}(\mathcal{C}^{k-1})$.

Corollary 5.8.
$\dim J^k(E,n,n-1) = \dim J^k(E,n,n-1) + \dim J^1(E_{-1},n-1) - n$.

This is a direct consequence of Remark 5.4.

6. Flag jets and involutive planes

The notions of a flag jet (introduced in Section 5) and that of an involutive plane (introduced in Section 4) are tightly interrelated in view of two simple facts. The first is that an element $r\in\breve{\mathcal{I}}_{n-1}(\mathcal{C}^k)$ can be seen as a "relative" flag of planes in J^k, in a sense elucidated by Lemma 6.1.

Lemma 6.1.
The map $\breve{\mathcal{I}}_{n-1}(\mathcal{C}^k)\xrightarrow{q^k} J^{k-1}(E,n,n-1)$, $\mathcal{C}^k_\theta\ni r\mapsto(R_\theta,r_{k-1})$, *is a bundle.*

Proof. By definition, $r\subseteq\mathcal{C}_\theta$ is horizontal, so r_{k-1} is an $(n-1)$-dimensional subspace of R_θ, i.e., r determines the flag (R_θ,r_r) on J^{k-1}. Smoothness follows from Remark 6.8. □

The second is that a flag projects over the space of involutive small planes, as shown by Lemma 6.2 which follows directly from Remark 4.2.

Lemma 6.2.
The canonical bundle $J^k(E,n) \times_{J^{k-1}(E,n)} \breve{J}^1\left(J^{k-1}(E,n), n-1\right) \to \breve{J}^1\left(J^{k-1}(E,n), n-1\right)$ *restricts to a bundle*

$$J^k(E,n,n-1) \xrightarrow{n^k} \breve{\mathfrak{I}}_{n-1}(\mathcal{C}^{k-1}). \tag{25}$$

As we shall see, taking the inverse limit, "relative" becomes "absolute", and the two sides of (25) will coincide (Theorem 6.9). The key tool is provided by diagram (26) below.

Lemma 6.3.
The bundle $J^k(E,n) \times_{J^{k-1}(E,n)} \breve{J}^1\left(J^{k-1}(E,n), n-1\right) \to J^{k-1}(E,n) \times_{J^{k-2}(E,n)} \breve{J}^1\left(J^{k-2}(E,n), n-1\right)$, $\Theta = (R_\Theta, r_\Theta) \mapsto \Theta_{k-1} \overset{\text{def}}{=} ((R_\Theta)_{k-1}, (r_\Theta)_{k-1})$, *restricts to a bundle*

$$J^k(E,n,n-1) \xrightarrow{\pi^{\text{flag}}_{k,k-1}} J^{k-1}(E,n,n-1).$$

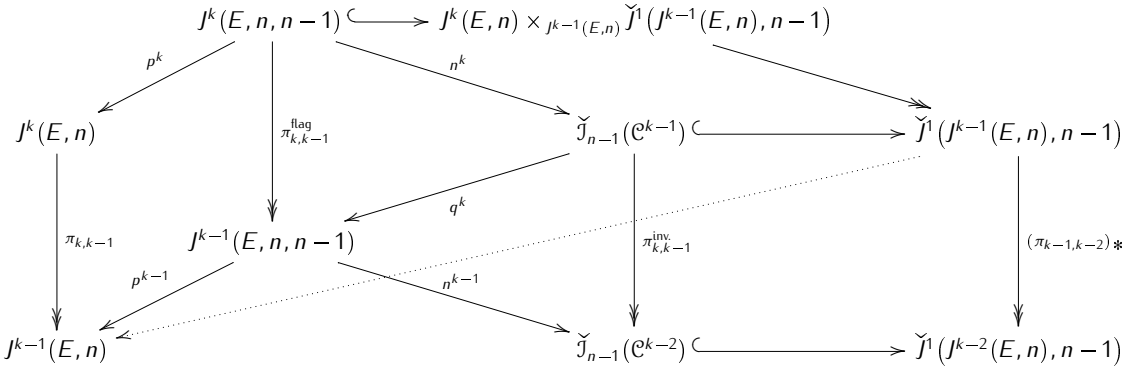
Let $(\pi_{k-1,k-2})_*$ be the jet map of $\pi_{k-1,k-2}$ (see Definition 2.2).

Lemma 6.4.
The bundle $\breve{J}^1\left(J^{k-1}(E,n), n-1\right) \xrightarrow{(\pi_{k-1,k-2})_*} \breve{J}^1\left(J^{k-2}(E,n), n-1\right)$ *restricts to a bundle*

$$\breve{\mathfrak{I}}_{n-1}(\mathcal{C}^{k-1}) \xrightarrow{\pi^{\text{inv.}}_{k,k-1}} \breve{\mathfrak{I}}_{n-1}(\mathcal{C}^{k-2}).$$

Remark 6.5.
The bundle of $J^k(E,n) \times_{J^{k-1}(E,n)} \breve{J}^1\left(J^{k-1}(E,n), n-1\right)$ over the first factor determines a bundle $p^k\colon J^k(E,n,n-1) \to J^k(E,n)$.

In view of the above lemmas, it makes sense to construct the below diagram (26), where unlabeled arrows are canonical embeddings/bundles.



(26)

Corollary 6.6.
Diagram (26) is commutative.

Proof. Commutativity of the rightmost square follows from Lemma 6.4, much as commutativity of the upper parallelepiped is a consequence of Lemma 6.2. Commutativity of the leftmost parallelepiped follows from Lemma 6.3 and Example 2.6. Commutativity of the central triangle follows from Lemmas 6.1 and 6.3, while Lemma 6.4, together with Lemma 6.1 and 6.2, guarantees commutativity of the lower triangle. Finally, by applying the composition $p^{k-1}\circ q^k$ to an element $r\in\breve{\mathcal{I}}_{n-1}(\mathcal{C}^k)$, in view of Lemma 6.1 and Remark 6.5, one gets the unique $\theta\in J^{k-1}(E,n)$ such that $r\subseteq\mathcal{C}^{k-1}_\theta$, i.e., $\pi_{1,0}(r)$. This proves that $p^{k-1}\circ q^k$ is the restriction of $\pi_{1,0}\colon \breve{J}^1\big(J^{k-1}(E,n),n-1\big)\to J^{k-1}(E,n)$. □

Remark 6.7 (coordinates on $\breve{\mathcal{I}}_{n-1}(\mathcal{C}^{k-1})$).

A point

$$\theta=\Big(x^a,\,t,\,\underset{|A|+l\leqslant k-1}{u^\alpha_{A,l}},\,t_a,\,\underset{|A'|+l'\leqslant k-1}{(u^\alpha_{A',l'})_a}\Big)\in J^1\big(J^{k-1}(E,n),n-1\big)$$

determines the small plane r_θ (see (18)), which, in view of Remark 5.6, is involutive iff (20) are satisfied. So,

$$x^a,\,t,\,\underset{|A|+l\leqslant k-1}{u^\alpha_{A,l}},\,t_a,\,\underset{|A'|+l'=k-1}{(u^\alpha_{A',l'})_a} \tag{27}$$

can be taken as coordinates on $\breve{\mathcal{I}}_{n-1}(\mathcal{C}^{k-1})$. Hence, by comparing (27) with (22), one sees that the $u^\alpha_{\mathrm{O},k}$ are fiber coordinates of n^k. Observe also that

$$n^k_\theta=\{\Theta\in J^k(E,n)\colon R_\Theta\supset r_\theta\} \tag{28}$$

is a subset of $\pi^{-1}_{k,k-1}(\theta_0)$, which is parametrized by top derivatives, i.e. coordinates

$$u^\alpha_{A,l},\qquad |A|+l=k, \tag{29}$$

and, thanks again to (20), any $u^\alpha_{A,l}$ in (29) with $A\neq\mathrm{O}$ can be expressed in terms of $u^\alpha_{A',l+1}$, with $|A'|=|A|-1$, so that the $u^\alpha_{\mathrm{O},k}$ must be coordinates along n^k.

Remark 6.8 (fiber coordinates of q^k).

Comparing (27) with (21), one sees that $(u_{A,l})^\alpha_a$, $|A|+l=k-1$, are coordinates along the fibers of q^k.

An easy consequence of Corollary 6.6 is Theorem 6.9 below, which establishes that the tower of flag jets carry the same information as the tower of equations of involutive small planes, i.e., that the tangent space to a small leaf of $J^\infty(E,n)$ is the same as a flag.

Theorem 6.9.

$\varprojlim \pi^{\mathrm{flag}}_{k,k-1}\cong\mathcal{I}_{n-1}(\mathcal{C})$.

7. Normal bundles and finite-order Cauchy data

Importance of Theorem 6.9 for the geometrical theory of Cauchy data is twofold. First, it shows that the flag jet construction is just an alternative description of the 1st order nonlinear PDE $\mathcal{I}_{n-1}(\mathcal{C})$, which can be discarded if one is merely interested in the secondary manifold $\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})$ (i.e., the space of Cauchy data according to Definition 8.2). One the other hand, $\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})$ would be rather difficult to work with, without some important insights on its structure (see Theorem 9.1), which follows from the factorization

$$\pi^{\mathrm{flag}}_{k,k-1}=q^k\circ n^k. \tag{30}$$

Remark 7.1.
Since, in view of Corollary 6.6, coordinates along $\pi^{\mathrm{flag}}_{k,k-1}$ are the same as those along $\pi_{k,k-1}$, i.e., $u^\alpha_{A,l}$, with $|A|+l=k$, a suggestive paraphrase of (30) is that it allows to add to the coordinates of $J^{k-1}(E,n)$, separately, first the k^{th} derivatives with at least one internal direction (i.e., the coordinates $(u_{A,l})^\alpha_a$, $|A|+l=k-1$, along q^k) and, then, the k^{th} purely normal derivative (i.e., the coordinates $u^\alpha_{O,k}$ along n^k), thus obtaining $J^k(E,n)$. However, "internal" and "normal", are to be understood in a *universal* sense, that it, valid for any given (infinitesimal) "space-time splitting" of the independent variables, i.e., a point of $J^1(E_{-1},n-1)$ (Lemma 5.7). This is why (locally), the above passage from the bundle $J^{k-1}(E,n)$ to the bundle $J^k(E,n)$ is valid only if we fiber-multiply them by $J^1(E_{-1},n-1)$.

Remark 7.2.
For $k=1$, diagram (26) yields

$$\begin{array}{ccc} & J^1(E,n,n-1) & \\ {\scriptstyle p^1}\swarrow & & \searrow{\scriptstyle n^1} \\ J^1(E,n) & & J^1(E,n-1), \end{array}$$

i.e.,

$$\begin{array}{ccc} & \mathrm{Gr}(TE,n,n-1) & \\ {\scriptstyle p^1}\swarrow & & \searrow{\scriptstyle n^1} \\ \mathrm{Gr}(TE,n) & & \mathrm{Gr}(TE,n-1), \end{array}$$

so that $p^1_R=\mathbb{P}(R^\vee)$ and $n^1_r=\mathbb{P}(r^\dagger)$ (see Fact 3.5) for any $R\in J^1(E,n)$ and $r\in J^1(E,n-1)$. In this sense, (26) is but a generalization of the canonical double fibered structure of flag manifolds.

Definition 7.3.
n^k is the k^{th} *normal bundle*.

Unlike n^1, which is a smooth bundle with abstract fiber $\mathbb{RP}^m$ (see Remark 7.2), n^k, with $k\geqslant 2$, are affine bundles, of dimension m. Indeed, (28) can be made more precise.

Corollary 7.4.
Let $r\subseteq\mathcal{C}^{k-1}_\theta$ be a point of $\breve{\mathcal{I}}_{n-1}(\mathcal{C}^{k-1})$. Then n^k_r is an affine subspace of $V_\theta J^{k-1}$ modeled by

$$S^{k-1}\left(\left(\frac{R_\theta}{(r)_{k-2}}\right)^\vee\right)\otimes_{\mathbb{R}} n^1_{r_0}.$$

Corollary 7.4 provides a link between the *true* normal bundle n^1, i.e., the one which formalizes the idea of the normal derivative to an embedded manifold, and the higher-order ones. This raises the possibility to join together all normal bundles. Indeed, in virtue of Theorem 6.9, $\mathcal{I}_{n-1}(\mathcal{C})$ inherits a canonical sequence, to be thought of as the limit of sequences (11) over finite-order flag jet bundles,

$$\begin{array}{ccccc} r & \hookrightarrow & \mathcal{C} & \twoheadrightarrow & \dfrac{\mathcal{C}}{r} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{I}_{n-1}(\mathcal{C}), & & \end{array}$$

and n^1 can be considered as a bundle over $\mathcal{I}_{n-1}(\mathcal{C})$ since the latter is, in turn, a bundle over $J^1(E,n-1)$.

Definition 7.5.
$\vec{n} \stackrel{\text{def}}{=} \prod_{k\in\mathbb{N}_0} S^k\left(\frac{\mathcal{C}}{r}\right)^{\vee} \otimes n_1$ is the *bundle of (infinite-order) normal directions*.

Remark 7.6.
Let $r \subseteq \mathcal{C}_\theta$ be a point of $\mathcal{I}_{n-1}(\mathcal{C})$. Then the k^{th} homogeneous component of $\vec{n}_r$, denoted by $\vec{n}^k_r$, is precisely the linear space over which $n^{k+1}_{r_k}$ is modeled.

8. Finite and infinite order Cauchy data

We show now that the familiar definition of a Cauchy datum of order k is naturally framed in diagram (26). To begin with, let us call a small submanifold $\Sigma \subseteq E$ a *Cauchy value*, or a 0^{th} *order Cauchy datum*. The reason is obvious: Σ_{-1} is a *Cauchy surface* in the manifold E_{-1} of independent variables, and Σ may be (locally) thougth of as the graph (i.e., the set of values) of an ($\mathbb{R}^m$-valued) function on Σ_{-1}. Then, it is natural to give the next Definition 8.1, for $k = 0, 1, \ldots, \infty$.

Definition 8.1.
A small involutive submanifold $\Sigma \subseteq J^k(E, n)$ is called a k^{th} *order Cauchy datum* (or, simply, a *Cauchy datum*, if $k = \infty$). Σ_0 is the Cauchy value corresponding to Σ and Σ_{-1}, if any, is the corresponding Cauchy surface.

We introduce now a secondary manifold whose secondary points (i.e., leaves) are in a natural one-to-one correspondence with small leaves of $J^k(E, n)$.

Definition 8.2.
$\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}^k)$ is the *space of k^{th} order Cauchy data*. When $k = \infty$, we obtain $\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})$, simply called the *space of Cauchy data*.

Observe that, thanks to jet projections, a Cauchy datum Σ determines a tower of k^{th} order Cauchy data Σ_k:

$$\Sigma_{-1} \leftarrow \Sigma_0 \leftarrow \ldots \leftarrow \Sigma_{k-1} \leftarrow \Sigma_k \leftarrow \ldots \leftarrow \Sigma. \tag{31}$$

Since terms of (31) project diffeomorphically one onto the other, so do the terms of the sequence (32) below:

$$(\Sigma_{-1})_{(1)} \leftarrow (\Sigma_0)_{(1)} \leftarrow \ldots \leftarrow (\Sigma_{k-1})_{(1)} \leftarrow (\Sigma_k)_{(1)} \leftarrow \ldots \leftarrow \Sigma_{(1)}. \tag{32}$$

On the other hand, the first prolongation $\Sigma_{(1)}$ is a small submanifold in $J^\infty(E, n, n-1)$, and, thanks to flag-jet projections (see Lemma 6.3), it determines a tower

$$(\Sigma_{(1)})_1 \leftarrow (\Sigma_{(1)})_2 \leftarrow \ldots \leftarrow (\Sigma_{(1)})_{k-1} \leftarrow (\Sigma_{(1)})_k \leftarrow \ldots \leftarrow \Sigma_{(1)}, \tag{33}$$

where $(\Sigma_{(1)})_k = \pi^{\text{flag}}_{\infty,k}(\Sigma_{(1)})$. Again, terms of (33) project diffeomorphically one onto the other. Lemma 8.3 below clarifies the relationship between the two towers, (32) and (33), having the common inverse limit $\Sigma_{(1)}$.

Lemma 8.3.
$(\Sigma_{(1)})_k$ is the graph of a section of n^k over $(\Sigma_{k-1})_{(1)}$.

Proof. If $\Sigma_{(1)} = \{(\mathcal{C}_\theta, T_\theta\Sigma) : \theta \in \Sigma)\}$, then $(\Sigma_{k-1})_{(1)} = \{T_{\theta_{k-1}}\Sigma_{k-1} : \theta_{k-1} \in \Sigma_{k-1})\}$, while

$$(\Sigma_{(1)})_k = \{(R_{\theta_k}, T_{\theta_{k-1}}\Sigma_{k-1}) : \theta_{k-1} \in \Sigma_{k-1}\}. \tag{34}$$

It remains to be noticed that, in (34), R_{θ_k} belongs to the fiber of n^k over $T_{\theta_{k-1}}\Sigma_{k-1}$. □

Lemma 8.3 shows that a section ν^k of n^k is the only "additional information" needed to produce a k^{th} order Cauchy datum out of a $(k-1)^{\text{st}}$ order one. Schematically,

$$\Sigma_{k-1} \mapsto (\Sigma_{k-1})_{(1)} \mapsto \nu^k((\Sigma_{k-1})_{(1)}) \mapsto p^k\big(\nu^k((\Sigma_{k-1})_{(1)})\big). \tag{35}$$

Fact 8.4.

Fix a section ν^k for any n^k. Then for any Cauchy value Σ_0 there is a unique Cauchy datum Σ over Σ_0 such that

$$(\Sigma_{(1)})_k = \operatorname{graph}\big(\nu^k|_{(\Sigma_{k-1})_{(1)}}\big), \qquad k = 1, 2, \dots \tag{36}$$

The proof goes inductively making use of (35).

Remark 8.5.

A valuable generalization of Fact 8.4 would be that the space of Cauchy data over a given Cauchy value is the same as the space of sections of $\vec{n}$. This cannot be achieved, since higher-order homogeneous components of $\vec{n}$ cannot be defined without the knowledge of lower-order Cauchy data. However, much as affine spaces are modeled by linear ones, the space of Cauchy data we are interested in can be "modeled" by the space of sections of $\vec{n}$, in a sense clarified by Proposition 8.6 below, which is fundamental to prove the structural Theorem 9.1.

Proposition 8.6.

Let Σ_0 be a Cauchy value, and fix a Cauchy datum Σ over it. Then sections of $\vec{n}|_{\Sigma_{(1)}}$ are in one-to-one correspondence with Cauchy data over Σ_0.

Proof. First of all, thanks to the chain of diffeomorphisms (32), the bundle $\vec{n}|_{\Sigma_{(1)}}$ can be identified with $\vec{n}|_{(\Sigma_k)_{(1)}}$, for any k. Hence, a section ν of $\vec{n}|_{\Sigma_{(1)}}$ may be thought of as a family $\{\nu^k\}_{k\in\mathbb{N}}$, where ν^k is a section of $\vec{n}^k|_{(\Sigma_{k-1})_{(1)}}$.

Because of Lemma 8.3, $(\Sigma_{(1)})_1$ is the graph of a section σ^1 of n^1 over $(\Sigma_0)_{(1)}$, so, in view of Remark 7.6, it makes sense to define

$$\Sigma_1' = \operatorname{graph}(\sigma^1 + \nu^0). \tag{37}$$

Now (37) can be used as the induction basis to subsequently "adjust" the given Cauchy datum by means of the sections of $\vec{n}$ (much as in the proof of Fact 8.4). Indeed (see also (36)), $(p^1(\Sigma_1'))_{(1)}$ is a small submanifold over $(\Sigma_0)_{(1)}$, and as such identifies diffeomorphically with $(\Sigma_1)_{(1)}$. Hence, ν^1 can be understood as a bundle over $(p^1(\Sigma_1'))_{(1)}$, and (37) can be used again to define Σ_2'. Continuing the iteration, one defines the Cauchy datum Σ'. □

9. The space of infinite-order Cauchy data

Let $\iota\colon \mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}) \to J^\infty\big(J^\infty(E,n), n-1\big)$ be the canonical inclusion, and

$$p \overset{\text{def}}{=} \pi_{\infty,0}\restriction_{\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})}, \qquad \text{where} \quad \pi_{\infty,0}\colon J^\infty\big(J^\infty(E,n), n-1\big) \to J^\infty(E,n),$$
$$n \overset{\text{def}}{=} (\pi_{\infty,0})_*\restriction_{\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})}, \qquad \text{where} \quad \pi_{\infty,0}\colon J^\infty(E,n) \to E$$

(see Definition 2.2 for the meaning of $(\pi_{\infty,0})_*$). Maps ι, p and n are conveniently depicted in the star-shaped diagram (38).

$$\begin{array}{ccccc} & & J^\infty\big(J^\infty(E,n),n-1\big) & & \\ & & \uparrow{\scriptstyle \iota} & & \\ & & \mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}) & & \\ & \swarrow{\scriptstyle p} & & \searrow{\scriptstyle n} & \\ J^\infty(E,n) & & & & J^\infty(E,n-1). \end{array} \tag{38}$$

Introduce the lifted distributions

$$\boldsymbol{C} \stackrel{\text{def}}{=} p^*(\mathcal{C}), \qquad \boldsymbol{D} \stackrel{\text{def}}{=} n^*(\mathcal{D}), \qquad \mathcal{D} \stackrel{\text{def}}{=} \iota^*(\widetilde{\mathcal{D}}),$$

where $\mathcal{D}$ (resp., $\widetilde{\mathcal{D}}$) is the ($(n-1)$-dimensional) structural distribution on $J^\infty(E,n-1)$ (resp., $J^\infty\big(J^\infty(E,n),n-1\big)$). Notice that, unlike $\boldsymbol{D}$, which has dimension $n-1$, both $\boldsymbol{C}$ and $\boldsymbol{D}$ are infinite-dimensional, though they are well-behaved, in the sense that, homotopically, they are finite-dimensional.

Observe that p maps a $\mathcal{D}$-leaf (i.e., a Cauchy datum) $\Sigma_{(\infty)}$ into the small leaf Σ of $\mathcal{C}$. The same Cauchy datum is mapped by n into the leaf $(\Sigma_0)_{(\infty)}$ of $\mathcal{D}$ (which is the corresponding Cauchy value). Hence, the secondary maps

$$\big(\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}),\mathcal{D}\big) \xrightarrow{p} \big(J^\infty(E,n),\mathcal{C}\big), \qquad \big(\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}),\mathcal{D}\big) \xrightarrow{n} \big(J^\infty(E,n-1),\mathcal{D}\big),$$

are well defined. Recall that the inclusion $L \subseteq J^\infty(E,n)$ determines an inclusion $J^\infty(L,n-1) \subseteq J^\infty(J^\infty(E,n),n-1)$ (see Example 2.4).

Theorem 9.1 (structural).

Let L (resp., Σ') be a leaf of $J^\infty(E,n)$ (resp., $J^\infty(E,n-1)$). Then the following identifications:

$$p^{-1}(L) = J^\infty(L,n-1), \tag{39}$$

$$n^{-1}(\Sigma') = J^\infty\big(\vec{n}|_{\Sigma_{(1)}}\big), \tag{40}$$

hold, where Σ is a Cauchy datum over $(\Sigma')_0$.

Proof. (40) is just a paraphrase of Proposition 8.6. Consider now the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}) & \hookrightarrow & J^\infty\big(J^\infty(E,n),n-1\big) \\ & \swarrow{\scriptstyle p} & \downarrow & & \downarrow{\scriptstyle \pi_{\infty,0}} \\ J^\infty(E,n) & \xleftarrow[p^\infty]{} & \mathcal{I}_{n-1}(\mathcal{C}) & \hookrightarrow & J^1\big(J^\infty(E,n),n-1\big), \end{array} \tag{41}$$

where p^∞ is the limit of the p^k and the vertical unlabeled arrow is $\pi_{\infty,0}\restriction_{\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})}$.

Observe that planes in $J^1(L,n-1)$ are involutive, being contained in the tangent planes to L, which is $\mathcal{C}$-integral. Hence, $J^1(L,n-1)$ is also a subset of $\mathcal{I}_{n-1}(\mathcal{C})$. Moreover,

$$(p^\infty)^{-1}(L) = \{(T_\theta L, r) : r \subseteq T_\theta L,\ \theta \in L\}$$

is a bundle over L whose fiber at $\theta \in L$ equals $\mathrm{Gr}(T_\theta L, n-1)$ (see Remark 7.2), hence it coincides with $J^1(L, n-1)$. It remains to be observed that

$$\pi_{\infty,0}^{-1}\big(J^1(L,n-1)\big) \cap \mathcal{I}_{n-1}^{(\infty)}(\mathcal{C}) = J^\infty(L,n-1), \tag{42}$$

where the latter is understood (see again Example 2.4) as a subset of $J^\infty\big(J^\infty(E,n),n-1\big)$. Inclusion "$\supseteq$" is obvious, since L is involutive and so are all its small submanifolds. Conversely, a point $[\Sigma']_\theta^\infty$ in the left-hand side of (42) must be such that $\theta = [L_0]_{\theta_0}^\infty$. On the other hand, being not maximal, Σ' must be contained into a leaf $L'_{(\infty)}$, where L' has the same infinite jet as L_0 at θ_0. So, there exist a small submanifold $\Sigma \subseteq L_0$, which is tangent to infinite order to Σ' at θ_0. Correspondingly, a small submanifold (denoted by the same symbol) $\Sigma \subseteq L$ exists, such that $[\Sigma]_\theta^\infty = [\Sigma']_\theta^\infty$. □

Observe that, unlike (40), (39) is canonical.

Remark 9.2 (coordinates).

By definition, the equations of $\mathcal{I}_{n-1}^{(\infty)}(\mathcal{C})$ are the infinite prolongations of the equations of $\mathcal{I}_{n-1}(\mathcal{C})$, which are just (20), rewritten with arbitrarily long multi-indexes, viz.,

$$(u^\alpha_{A,l})_a = u^\alpha_{Aa,l} + t_a u^\alpha_{A,l+1}, \qquad A \in \mathbb{N}_0^{n-1}, \quad l \in \mathbb{N}_0. \tag{43}$$

It is a simple computation to show that all differential consequences of (43) read as

$$(u^\alpha_{A,l})_B = \sum_{B'_1 B'_2 \dots B'_s B'' = B} t_{B'_1} t_{B'_2} \cdots t_{B'_s} u^\alpha_{AB'',l+s}, \qquad A, B \in \mathbb{N}_0^{n-1}, \quad l \in \mathbb{N}_0. \tag{44}$$

In view of (44), the (infinite) set of functions

$$x^a, t, u^\alpha_{A,l}, t_B, \qquad A, B \in \mathbb{N}_0^{n-1}, \quad B \neq \mathrm{O}, \quad l \in \mathbb{N}_0, \tag{45}$$

can be taken as coordinates on $\mathcal{I}_{n-1}^{(\infty)}(\mathcal{C})$, the infinite-order analog of (27). By using (45) and standard coordinates on $J^\infty(E,n)$, it looks obvious that t_B, with $B \in \mathbb{N}_0^{n-1}$, $B \neq \mathrm{O}$, are the fiber coordinates of p. Now, similarly as for (22), use (44) to produce a new coordinate system

$$x^a, t, u^\alpha_{A,0}, (u^\alpha_{\mathrm{O},l})_B, t_B, \qquad A, B \in \mathbb{N}_0^{n-1}, \quad A \neq \mathrm{O}, \quad l \in \mathbb{N}, \tag{46}$$

from which one sees that $(u^\alpha_{\mathrm{O},l})_B$, with $B \in \mathbb{N}_0^{n-1}$ and $l \in \mathbb{N}$ are the fiber coordinates of n.

Coordinates (45) can be recovered from (46) by the formulas

$$u^\alpha_{A,l} = \sum_{B_1 B_2 \dots B_s B = A} (-1)^s t_{B_1} t_{B_2} \cdots t_{B_s} (u^\alpha_{\mathrm{O},l+s})_B, \qquad A \in \mathbb{N}_0^{n-1}, \quad l \in \mathbb{N}. \tag{47}$$

Remark 9.3 (affine case).

The infinite-order generalization of Lemma 5.7 reads

$$\mathcal{I}_{n-1}^{(\infty)}(\mathcal{C}) \overset{\mathrm{loc}}{=} J^\infty(E,n) \times_{E_{-1}} J^\infty(E_{-1}, n-1). \tag{48}$$

Correspondence (48) takes a point $([\sigma]_x^\infty, [\Sigma]_x^\infty)$, where σ is a section of $E \to E_{-1}$, to the point $j_\infty(\sigma)_*([\Sigma]_x^\infty)$, where $j_\infty(\sigma)_* \colon J^\infty(E_{-1}, n-1) \to J^\infty\big(J^\infty(E,n), n-1\big)$ is the jet map associated to $j_\infty(\sigma)$ (Definition 2.2). If coordinates (45) are split into $x^a, t, u^\alpha_{A,l}$ and t_B, one gets precisely the coordinates of a point in the right-hand side of (48).

Remark 9.4.

A global analog of (48) can be constructed by replacing, in the structural bundle $\mathcal{C}$ over $J^\infty(E,n)$, each fiber $\mathcal{C}_\theta$ by $J_0^\infty(\mathcal{C}_\theta, n-1)$. This shows that $\mathcal{I}_{n-1}^{(\infty)}(\mathcal{C})$ is an infinite-dimensional bundle over $J^\infty(E,n)$, with generic fiber $J_0^\infty(\mathbb{R}^n, n-1)$, in strict analogy with flag bundles (Fact 3.5).

Corollary 9.5.
Let L (resp., Σ') be a leaf of $J^\infty(E,n)$ (resp., $J^\infty(E,n-1)$). Then the following identifications of secondary manifolds:

$$\boldsymbol{p}^{-1}(L) = J^\infty(L,n-1), \tag{49}$$
$$\boldsymbol{n}^{-1}(\Sigma') = J^\infty\big(\vec{n}|_{\Sigma_{(1)}}\big), \tag{50}$$
$$\big(\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}),\boldsymbol{C}\big) = J^\infty(E,n), \tag{51}$$
$$\big(\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C}),\boldsymbol{D}\big) = J^\infty(E,n-1), \tag{52}$$

hold, where Σ is a Cauchy datum over $(\Sigma')_0$.

Corollary 9.6 (transversality).
Projections p and n are leafwise transversal each other, i.e.,

- *p projects diffeomorphically $n^{-1}(\Sigma)$ onto $J^\infty(E,n)$, for any leaf Σ of $J^\infty(E,n-1)$;*
- *n projects diffeomorphically $p^{-1}(L)$ onto $J^\infty(E,n-1)$, for any leaf L of $J^\infty(E,n)$.*

Proof. The second assertion is an immediate consequence of the fact that $J^\infty(L,n-1)$ is embedded into $J^\infty(E,n-1)$ (see the proof of Theorem 9.1). For the first assertion, it is convenient to use the local coordinates from Remark 9.2. Namely, let Σ be given by functions f, g^α,

$$\Sigma: \qquad ct_B = \frac{\partial^B}{\partial x^B} f, \quad u^\alpha_A = \frac{\partial^A}{\partial x^A} g^\alpha. \tag{53}$$

Then $n^{-1}(\Sigma)$ is given, in the coordinates (46), by the same equations (53). Passing now to the coordinates (45),

$$n^{-1}(\Sigma): \begin{cases} u^\alpha_{A,l} = \dfrac{\partial^A}{\partial x^A} g^\alpha, & l = 0, \\ u^\alpha_{A,l} = \displaystyle\sum_{B_1B_2\dots B_sB=A} (-1)^s \frac{\partial^{B_1}}{\partial x^{B_1}} f \frac{\partial^{B_2}}{\partial x^{B_2}} f \cdots \frac{\partial^{B_s}}{\partial x^{B_s}} f\,(u^\alpha_{0,l+s})_B, & l \neq 0, \end{cases} \tag{54}$$

one sees that $n^{-1}(\Sigma)$ is parametrized by

$$(x^a, t, (u^\alpha_{0,l+s})_B), \tag{55}$$

while the other coordinates are obtained via (54). So, the projection $p(n^{-1}(\Sigma))$ is given by the same equations (54), in the standard coordinates $(x^a, t, u^\alpha_{A,l})$ of $J^\infty(E,n)$. Hence, $p(n^{-1}(\Sigma))$ is again parametrized by (55). □

Remark 9.7.
(50) and (49) might be seen as the secondary analog of the 1st order projections of flag manifolds (see Fact 3.5 and Remark 7.2).

10. Concluding remarks and perspectives

Secondary ODEs

Identifications (51) and (52) allow to regard $J^\infty(E,n)$ and $J^\infty(E,n-1)$ as secondary quotients of the same secondary manifold $\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})$. Indeed, $\boldsymbol{C}$ can be understood as the distribution generated by $\mathcal{D}$ and by a $\boldsymbol{p}$-vertical secondary distribution (and similarly for $\boldsymbol{D}$), as firstly pointed out by Vitagliano [18]. Since the leaves of $\boldsymbol{C}$ are canonically identified with the leaves of $\mathcal{C}$, any equation in n independent variables is the same as a (secondary) distribution on the space of admissible Cauchy data. Such a perspective seems to be evidence of a (formal) analogy with Hamiltonian formalism in mechanics.

Twisted characteristic cohomology

Theorem 9.1 is the natural departing point to define a twisted generalization of the characteristic cohomology of an equation (first of all, the empty one), where the coefficients belong to the $\boldsymbol{p}$- or $\boldsymbol{n}$-vertical characteristic cohomology of the corresponding space of Cauchy data, in analogy with the differential Leray–Serre spectral sequence associated with a fiber bundle. In particular, among terms of the twisted characteristic cohomology it can be found the one which corresponds to an "action-valued action", i.e., an action integral whose value on a leaf L (resp., Σ) is an action integral on $\boldsymbol{p}^{-1}(L)$ (resp., $\boldsymbol{n}^{-1}(\Sigma)$). In Section 11 below we propose a toy model for such an action, and derive the corresponding Euler–Lagrange equations.

The theory of twisted characteristic cohomology should be a source of simplification techniques in Calculus of Variations, and of methods to compute characteristic cohomology of nonlinear PDEs, much as the Künnet formula does in Algebraic Topology.

Fact 10.1, stemming from Theorem 9.1, provides a basic understanding of the characteristic cohomology of the space of Cauchy data.

Fact 10.1.
The $\mathcal{D}$-spectral sequence is 1-line.

To see it, embed $J^\infty(E,n-1)$ into $J^\infty(E,n)$ (see Remark 2.5), and then observe that $\mathcal{I}^{(\infty)}_{n-1}(\mathcal{C})$ is locally the space of horizontal infinite jets $\bar{J}^\infty\left(\vec{n}|_{J^\infty(E,n-1)_{(1)}}\right)$ (see Proposition 8.6).

Invariance of the framework

From a mere set-theoretical point of view, $n^{-1}(\Sigma')$ is but the inverse image of the submanifold $\Sigma'_0 \subseteq E$ via the projection $\pi_{\infty,0}\colon J^\infty(E,n) \to E$. The main virtue of Theorem 9.1 is to reveal that $n^{-1}(\Sigma')$ is an empty equation, a fact which is essential if one is interested in special subsets of $n^{-1}(\Sigma')$, which arise from the analysis of nonlinear PDEs, and compute their characteristic cohomology, by using the traditional geometrical and cohomological methods for PDEs. In a sense, the whole machinery developed in this paper was aimed at the proof of (40), but perhaps a key feature of our treatment was not given enough attention. Namely, the whole framework is *invariant*, i.e., well-behaved with respect to transformations, which gives a total freedom in the choice of coordinates for computational purposes (as in the toy model proposed in the last Section 11).

Higher codimension and complete flags

It is advisable to develop the theory for higher codimension flag jets, i.e., replace $n-1$ by any $n_0 < n$ in the constructions presented here. The so-obtained formalism may have interesting applications, e.g., in the context of quasi-local Hamiltonians (see, e.g., [6] concerning quasi-local mass in General Relativity). If complete flags are taken as the departing point, then the theory for the twisted characteristic cohomology of the so-obtained space of complete jet flag should be particularly rich, and play the same role, in the context of nonlinear PDEs, as the CW-complexes in Algebraic Topology.

11. An applicative example

In view of Theorem 9.1, every leaf of $J^\infty(E,n-1)$ produces an empty equation over the leaf itself,

$$\underset{\text{leaf of } J^\infty(E,n-1)}{\Sigma'} \longmapsto \underset{\text{space of infinite jets of the infinite normal bundle}}{n^{-1}(\Sigma')}. \tag{56}$$

Moreover, thanks to Corollary 9.6, the empty equation $n^{-1}(\Sigma')$ can be seen as a closed subset of $J^\infty(E,n)$. Hence, if some equation and/or variational principle is imposed on $J^\infty(E,n)$, it will reflect on $n^{-1}(\Sigma')$. This phenomenon has been originally noticed by Vinogradov in 1984 (see [13, Section 8.5]), but its cohomological analysis was carried out in

detail by Vinogradov and the author in the 2006 paper [16] (see also [10]), where the relationship between the $\mathcal{C}$-spectral sequence associated with $n^{-1}(\Sigma')$ and the relative $\mathcal{C}$-spectral sequence of the surrounding jet space $J^\infty(E,n)$ is clarified.

The example developed below, which shows how a variational principle determines a natural equation on $n^{-1}(\Sigma')$, is also a case where two action integrals of different horizontal degree are summed up.

Suppose that E is a closed domain in $\mathbb{R}^{n+m}$, such that an m-dimensional submanifold G of $\mathbb{R}^{n+m}$ exists, and E is a tubular neighborhood of it. Then E is (globally) a bundle over G with fiber D^n, and (locally) a bundle over D^n with fiber $\mathbb{R}^m$. Observe that the (graphs of the) sections of the latter belong to the larger class of submanifolds

$$\mathcal{A} \stackrel{\text{def}}{=} \big\{L \subseteq E : L \cap \partial E = \partial L,\ L \text{ is oriented and connected}\big\}.$$

Put also $\partial\mathcal{A} \stackrel{\text{def}}{=} \{\partial L : L \in \mathcal{A}\}$. Observe that $\mathcal{A}$ is nothing but a subset of the space $J^\infty(E,n)$, made of leaves which are well-behaved with respect to integration (in the terminology of Calculus of Variations, they would be referred to as "admissible", see also [13, Section 8.5] on this concern), and $\partial\mathcal{A}$ is a subset of the space $J^\infty(\partial E, n-1)$. Let

$$\boldsymbol{S} \in \overline{H}^n\big(J^\infty(E,n), \pi_{\infty,0}^{-1}(\partial E)\big), \qquad \boldsymbol{S}_\partial \in \overline{H}^{n-1}\big(J^\infty(E,n-1)\big), \tag{57}$$

be two action integrals, i.e., secondary real-valued functions on $\mathcal{A}$ and $\partial\mathcal{A}$, respectively,

$$\boldsymbol{S}\colon L \in \mathcal{A} \mapsto j_\infty(L)^*\boldsymbol{S} \in H^n(L,\partial L) \cong \mathbb{R}, \qquad \boldsymbol{S}_\partial\colon \Sigma \in \partial\mathcal{A} \mapsto j_\infty(\Sigma)^* \in H^{n-1}(\Sigma) \cong \mathbb{R},$$

where the last identifications are an elementary fact of differential topology (see [2]). (57) define a secondary function $\mathcal{A} \ni L \stackrel{\boldsymbol{S}_{\text{tot}}}{\longmapsto} \boldsymbol{S}(L) + \boldsymbol{S}_\partial(\partial L) \in \mathbb{R}$. Expectedly, the set of critical points of $\boldsymbol{S}_{\text{tot}}$ is smaller than a mere (suitably defined) intersection of the critical points of $\boldsymbol{S}$ and $\boldsymbol{S}_\partial$, because an "interaction term" arises. Namely, for any $L \in \mathcal{A}$, consider the module of cosymmetries (see [1]) $(\varkappa^\dagger)_L \stackrel{\text{def}}{=} \varkappa^\dagger\big(p\big(n^{-1}((\partial L)_{(\infty)})\big)\big)$ of $p\big(n^{-1}((\partial L)_{(\infty)})\big)$, the canonical splitting

$$\varkappa^\dagger\big(J^\infty(E,n), \pi_{\infty,0}^{-1}(\partial E)\big) \stackrel{\text{loc}}{=} \varkappa^\dagger(J^\infty(E,n)) \oplus (\varkappa^\dagger)_L,$$

and the corresponding decomposition[3] of the relative Euler–Lagrange differential[4] of $\boldsymbol{S}$,

$$\boldsymbol{d}_{\text{rel}}\boldsymbol{S} = (\boldsymbol{d}\boldsymbol{S}, (\boldsymbol{d}_{\text{rel}}\boldsymbol{S})_L). \tag{58}$$

It turns out that $(\boldsymbol{d}_{\text{rel}}\boldsymbol{S})_L = 0$ is a differential equation in $p\big(n^{-1}((\partial L)_{(\infty)})\big)$, i.e., imposed on the sections of $\vec{n}|_{((\partial L)_{(\infty)})_{(1)}}$, which formalizes precisely the above idea of interaction.

Theorem 11.1.

$$L \text{ is critical for } \boldsymbol{S}_{\text{tot}} \quad \Longleftrightarrow \quad \begin{cases} L_\infty \in \{\boldsymbol{d}\boldsymbol{S} = 0\}^{(\infty)} \subseteq J^\infty(E,n), \\ L_{(\infty)} \cap \pi_{\infty,0}^{-1}(\partial L) \in \{(\boldsymbol{d}_{\text{rel}}\boldsymbol{S})_L = 0\}^{(\infty)} \subseteq J^\infty\big(\vec{n}|_{((\partial L)_{(\infty)})_{(1)}}\big), \\ (\partial L)_{(\infty)} \in \{\boldsymbol{d}\boldsymbol{S}_\partial = 0\}^{(\infty)} \subseteq J^\infty(\partial E, n-1). \end{cases} \tag{59}$$

[3] *In [13, Section 8.5], equations $(\boldsymbol{d}_{\text{rel}}\boldsymbol{S})_L$ are denoted by $\Gamma(\overline{\omega})$, where $\overline{\omega}$ is a representative of $\boldsymbol{S}$, while in [16] they are denoted by $\theta'_{\overline{\omega}}$.*

[4] *Introduced in [16, Section 3.4], where it is denoted by $\boldsymbol{E}_{\text{rel}}$.*

Proof. Obviously, L is critical for $\boldsymbol{S}_{\text{tot}}$ if an only if L is a solution of the relative Euler–Lagrange equations,

$$\boldsymbol{d}_{\text{rel}}\boldsymbol{S} = 0, \tag{60}$$

and ∂L is a solution of $\boldsymbol{d}\boldsymbol{S}_\partial = 0$, i.e., the last equation of the list (59). It remains to observe that the first two equations are synthetically expressed by (60), thanks to (58). □

When (57) are volume integrals, critical points of $\boldsymbol{S}_{\text{tot}}$ are the *least-volume and least-boundary-area* submanifolds of E.

Apparently, (59) is just a clean way to write down the so-called natural boundary conditions in the Calculus of Variations, and all the machinery exploited to obtain (59) is but a paraphrase of the classical analytical manipulations on variational integral (see, e.g., [3, 5]) exploited to derive the natural boundary conditions. In fact, a very important feature of (59), their invariance, does not show at a superficial look. Such a property allows, for instance, to derive the correct expression of the transversality conditions, for any "tubular" manifold — which are not known to date — just by a wise choice of coordinates. We will not go into the details of the general construction, but present a simple toy model with $n = m = 1$.

Example 11.2 (the problem of Columbus).
Given the curves Γ_1 and Γ_2 in $\mathbb{R}^2$, consider the problem of finding, among the (non self-intersecting) (smooth) curves which start from a point of Γ_1 and end to a point of Γ_2 (without crossing $\Gamma_1 \cup \Gamma_2$ in any other point), those whose length is (locally) minimal. Obviously, a curve γ is a solution of the problem at hand if and only if

(EL) γ is a straight line;

(TC) γ hits at a right angle $\Gamma_1 \cup \Gamma_2$.

The problem can be formalized by means of a Lagrangian density $f\,dx$, where $f = f(x, y, y')$, on a tubular submanifold $E \subseteq \mathbb{R}^2$, with $\partial E = \Gamma_1 \cup \Gamma_2$. In this setting, condition (TC) for a curve $\gamma = (x, y(x))$ read

$$\left(f - \frac{\partial f}{\partial y'}\right) x^\Gamma + \frac{\partial f}{\partial y'}\, y^\Gamma = 0, \tag{61}$$

where (x^Γ, y^Γ) is a vector tangent to ∂E.

Proof. Equation (61) can be obtained in few lines (see [3]). We propose an alternative way, which stresses the role of invariance of (59). To this end, choose a diffeomorphism between E and the cylinder $[0,1] \times \mathbb{R}$, and denote by $\omega = g\,dx$ the pull-back of $f\,dx$ to such a cylinder. Then $\boldsymbol{S} = [\omega]$ is an element of $\overline{H}^1\big(J^\infty(\pi), \pi_\infty^{-1}(\{0,1\})\big)$, where $\pi\colon [0,1] \times \mathbb{R} \to [0,1]$, and

$$(\boldsymbol{d}_{\text{rel}}\boldsymbol{S})_{\{0,1\}} = \left.\frac{\partial g}{\partial y'}\right|_{\pi_\infty^{-1}(\{0,1\})}. \tag{62}$$

By pulling back (62) on E, one obtains (61). □

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