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Singular cardinals and strong extenders

Research Article

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Abstract: We investigate the circumstances under which there exist a singular cardinal μ and a short (κ, μ)-extender E witnessing "κ is μ-strong", such that μ is singular in $\mathrm{Ult}(V, E)$.

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Keywords: Strong cardinal • Extender • Inner model • Singular cardinal

1. Introduction

In the course of some work on his doctoral dissertation [1], Brent Cody encountered some issues which caused him to raise the following question:

Question 1.1.
If κ is μ-strong for some singular cardinal $\mu > \kappa$, is there a (κ, μ)-extender E which witnesses that κ is μ-strong and is such that

$$\mathrm{Ult}(V, E) \models \text{“}\mu \text{ is singular”}\,?$$

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For background on extenders and strong cardinals, we refer the reader to Mitchell's introductory article on inner model theory [7] and Kanamori's book on large cardinals [4]. When E is an extender we write $\mathrm{Ult}(V, E)$ for the transitive class model obtained by forming the ultrapower of V by E, and j_E for the associated elementary embedding from V to $\mathrm{Ult}(V, E)$. The term "inaccessible" is used throughout to mean strongly inaccessible.

Before describing our results we make a few preliminary remarks.

(1) Let κ be μ-strong for some cardinal $\mu > \kappa$ with $|V_\mu| = \mu$, and fix $j: V \to M$ witnessing this. Since $|V_\alpha| < \mu$ and $V_{\alpha+1} \in M$ for all $\alpha < \mu$, it follows easily that $M \models |V_\mu| = \mu$, and hence that if E is the (κ, μ)-extender approximating j then $V_\mu \subseteq \mathrm{Ult}(V, E)$. So there is a (κ, μ)-extender witnessing that κ is μ-strong.

(2) Suppose that there is a (κ, μ)-extender E witnessing that κ is μ-strong. Then we claim that $|V_\mu| = \mu$, so that in particular μ is strong limit. Otherwise there is $\alpha < \mu$ such that $|V_\alpha| \geq \mu$, and since $V_{\alpha+1} \in \mathrm{Ult}(V, E)$ it follows easily that $P(\mu) \in \mathrm{Ult}(V, E)$ and hence $E \in \mathrm{Ult}(V, E)$, which is impossible.

(3) For any $\mu > \kappa$ with $|V_\mu| = \mu$ and any embedding $j: V \to M$ with $\mathrm{crit}(j) = \kappa$, we claim that $V_\mu \subseteq M$ if and only if M contains all the bounded subsets of μ. One direction of this implication is immediate, and the other follows from the observation that elements of V_μ can be coded by bounded subsets of μ.

Suppose now that $\kappa < \mu$ for some singular μ with $|V_\mu| = \mu$, and E is a (κ, μ)-extender witnessing that κ is μ-strong. We will say that E is a *good witness* if μ is singular in $\mathrm{Ult}(V, E)$, and a *bad witness* if μ is regular (and hence inaccessible) in $\mathrm{Ult}(V, E)$.

The authors observed that if κ is ν-strong for some inaccessible $\nu > \kappa$, then there is a club set $C \subseteq \nu$ such that for every singular $\mu \in C$ there is a bad witness (see Fact 2.2 below). Brent Cody recently informed us that this result was already known, at least under the additional assumption of GCH, and had appeared in print in a paper of Friedman and Honzik [2, Observation 2.8]. However this result does not completely settle Question 1.1.

Our main results are:

(1) If there is a bad witness, then there is a normal measure on κ concentrating on α which are strong up to α^*, where α^* is the least inaccessible cardinal greater than α.

(2) If κ is ν-strong for some cardinal ν, then for every singular μ such that $\kappa < \mu < \nu$ and $|V_\mu| = \mu$ there is a good witness.

(3) (From suitable large cardinal assumptions)

   (3a) It is consistent that μ is singular, there is exactly one (κ, μ)-extender E witnessing that κ is μ-strong, and E is a good witness.

   (3b) It is consistent that μ is singular, there is exactly one (κ, μ)-extender E witnessing that κ is μ-strong, and E is a bad witness.

The last result uses models of the form $L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of non-overlapping extenders.

2. Proofs of the main results

2.1. Bad witnesses

We begin with an easy reflection argument, giving a lower bound in consistency strength for the existence of a bad witness.

Theorem 2.1.
If there is a bad witness for "κ is μ-strong", then there is a normal measure on κ concentrating on α which are strong up to α^*, where α^* is the least inaccessible cardinal greater than α.

Proof. Let E be a bad witness, so that μ is inaccessible in $\mathrm{Ult}(V,E)$. Since $V_\mu \subseteq \mathrm{Ult}(V,E)$, $\mathrm{Ult}(V,E)$ contains extenders witnessing that κ is ν-strong for every $\nu < \mu$, so

$$\mathrm{Ult}(V,E) \models \text{“}\mu \text{ is inaccessible and } \kappa \text{ is strong up to } \mu\text{”}.$$

If U is the normal measure derived from j_E then U concentrates on the set of α which are strong up to α^*. □

The following fact, giving an upper bound in consistency strength for the existence of a bad witness, was observed by Friedman and Honzik [2, Observation 2.8]. We give the proof because we need the idea later in Theorem 2.10.

Fact 2.2.

If κ is ν-strong for some inaccessible $\nu > \kappa$, then there is a club set $C \subseteq \nu$ such that for every singular $\mu \in C$ there is a bad witness.

Proof. Let E be a (κ,ν)-extender witnessing that κ is ν-strong, and let $j_E \colon V \to M_E = \mathrm{Ult}(V,E)$ be the corresponding ultrapower map. We note that since ν is inaccessible in V, ν is inaccessible in $\mathrm{Ult}(V,E)$. Recall that

$$M_E = \left\{ j_E(f)(a) : a \in [\nu]^{<\omega},\ \mathrm{dom}(f) = [\kappa]^{<\omega} \right\}.$$

For any $\mu < \nu$ we may form the (κ,μ)-extender $E \restriction \mu$ and the corresponding ultrapower map $j_{E\restriction\mu} \colon V \to M_{E\restriction\mu}$. It is easy to see that there is an elementary embedding k_μ from $M_{E\restriction\mu}$ to M_E such that $k_\mu \circ j_{E\restriction\mu} = j_E$. This embedding is given by the formula $k_\mu \colon j_{E\restriction\mu}(f)(a) \mapsto j_E(f)(a)$ for $a \in [\mu]^{<\omega}$. Note that $M_E = \bigcup_{\mu<\nu} \mathrm{rge}(k_\mu)$, so that in particular $\nu \in \mathrm{rge}(k_\mu)$ for all large enough $\mu < \nu$.

We now define a function F with domain ν, where

$$F(\mu) = \max \left\{ |V_\mu|, \sup(\mathrm{rge}(k_\mu) \cap \nu) \right\}.$$

Since ν is inaccessible and

$$\mathrm{rge}(k_\mu) \cap \nu \subseteq \left\{ j_E(f)(a) \,\middle|\, a \in [\mu]^{<\omega},\ f \colon [\kappa]^{<\omega} \to \kappa \right\},$$

$\mathrm{rge}(F) \subseteq \nu$. If μ is a closure point of F with $\nu \in \mathrm{rge}(k_\mu)$, then $|V_\mu| = \mu$ and $\nu \cap \mathrm{rge}(k_\mu) = \mu$. Let C be the club set of closure points μ of F such that $\nu \in \mathrm{rge}(k_\mu)$. For each $\mu \in C$, we have that $k_\mu(\mu) = \nu$, so that by elementarity μ is inaccessible in $M_{E\restriction\mu} = \mathrm{Ult}(V, E\restriction\mu)$. It follows easily that for every singular $\mu \in C$ the extender $E \restriction \mu$ is a bad witness. □

2.2. Good witnesses

Theorem 2.3.

If κ is ν-strong for some cardinal ν, then for every singular μ such that $\kappa < \mu < \nu$ and $|V_\mu| = \mu$ there is a good witness.

Proof. Fix an extender E witnessing that κ is ν-strong. We prove the claim by induction on μ. Suppose that μ is a minimal counterexample. As above we may form $j_E \colon V \to M_E$ and factor it through the ultrapower by $E \restriction \mu$, obtaining an embedding k_μ from $M_{E\restriction\mu}$ to M_E such that $k_\mu \circ j_{E\restriction\mu} = j_E$.

As usual $\mu \subseteq \mathrm{rge}(k_\mu)$, and we claim that in this case also $\mu \in \mathrm{rge}(k_\mu)$. To see this we observe that $V_\nu \subseteq M_E$, so that by a routine calculation

$$M_E \models \text{“}\mu \text{ is the least cardinal with no good witness”}.$$

Hence μ is definable from κ in M_E, which implies that $\mu \in \mathrm{rge}(k_\mu)$. It follows that $k_\mu(\mu) = \mu$. Since $V_\nu \subseteq M_E$ we see that μ is singular in M_E, and hence by elementarity μ is singular in $M_{E\restriction\mu} = \mathrm{Ult}(V, E\restriction\mu)$. So $E \restriction \mu$ is a good witness for μ, contradicting the choice of μ as the least counterexample. □

2.3. Unique witnesses

To prove the remaining results we need some analysis of extenders in models of the form $L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of non-overlapping extenders. We refer the reader to Mitchell's excellent survey paper [7] for a detailed account of these models; we adopt the terminology and conventions of that paper, in particular we note that $E(\kappa, \beta)$ is a total $(\kappa, \kappa+1+\beta)$-extender.

The key fact is the *Comparison Lemma* [7, Lemma 3.15], which states that (under the right hypotheses) two models with extender sequences can be iterated so that the images of the original extender sequences are "lined up". In our proofs we will freely use the Comparison Lemma and some immediate consequences. In all comparison iterations which appear below, the extender sequences are coherent, so that the critical points are strictly increasing.

We will also use the following fairly standard facts about models of the form $L[\vec{E}]$, all of which are due to Mitchell [5, 6]. For the convenience of the reader, we have included references or have sketched the proofs. Some of the arguments would be slicker with an appeal to the theory of core models, but we have chosen to avoid this in order to minimise the prerequisites. Let $V = L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of non-overlapping extenders. Then a straightforward consequence of [7, Theorem 3.24] is

Fact 2.4.
GCH holds.

[7, Lemma 3.19] implies

Fact 2.5.
If there is a $(\kappa, \kappa+1+\beta)$-extender E such that $o^{j_E(\vec{E})}(\kappa) = \beta$, then $\beta < o^{\vec{E}}(\kappa)$ and $E = E(\kappa, \beta)$. In particular this will hold whenever $j_E(\vec{E}) \restriction (\kappa, \beta) = \vec{E} \restriction (\kappa, \beta)$.

Fact 2.6.
Let E be an extender and $j_E \colon V \to M_E = \mathrm{Ult}(V, E)$ the corresponding ultrapower map. Then comparing V and M_E leads to iterations $i_0 \colon V \to N$ and $i_1 \colon M_E \to N$ with a common target model N, and $i_1 \circ j_E = i_0$.

Proof. A model $L[\vec{E}]$ is said to be ϕ-*minimal* if $L[\vec{E}]$ is a model of ϕ, but for no initial segment $\vec{E}'$ of $\vec{E}$ is $L[\vec{E}']$ a model of ϕ. Suppose the claim fails, and let $L[\vec{E}]$ be ϕ-minimal for the formula "there is an extender E such that the claim fails". Appealing to [7, Proposition 3.18] we obtain an immediate contradiction. □

Fact 2.7.
Let μ be a cardinal with $\kappa^{++} \leq \mu < o^{\vec{E}}(\kappa)$. Then for every ν with $\mu \leq \nu < o^{\vec{E}}(\kappa)$, every bounded subset of μ appears in $\mathrm{Ult}(V, E(\kappa, \nu))$.

Proof. We start by proving that if $\lambda > \kappa$ and $o^{\vec{E}}(\kappa) \geq \lambda^+$, then every subset of λ appears in $\mathrm{Ult}(V, E(\kappa, \beta))$ for some β with $\lambda < \beta < \lambda^+$. To see this, let $A \subseteq \lambda$ and find some large regular θ such that $A \in L_\theta[\vec{E}]$. Let $X \prec L_\theta[\vec{E}]$ be such that $|X| = \lambda$, $P(\kappa) \subseteq X$, $A \in X$ and $X \cap \lambda^+ \in \lambda^+$. Let M be the transitive collapse of X, so that $A \in M$ and $M = L_{\bar{\theta}}[\vec{F}]$ for some extender sequence $\vec{F}$.

It is clear that $\vec{E} \restriction \kappa = \vec{F} \restriction \kappa$, and since $P(\kappa) \subseteq M$ a straightforward induction using Fact 2.5 shows that for every $\zeta < o^{\vec{F}}(\kappa)$ we have $\zeta < o^{\vec{E}}(\kappa)$ and $E(\kappa, \zeta) = F(\kappa, \zeta)$. Let $\beta = o^{\vec{F}}(\kappa)$, and note that $\lambda < \beta < \lambda^+ \leq o^{\vec{E}}(\kappa)$. Compare the models M and $\mathrm{Ult}(V, E(\kappa, \beta))$, to obtain iterations $i_0 \colon M \to N_0$ and $i_1 \colon \mathrm{Ult}(V, E(\kappa, \beta)) \to N_1$. By coherence and the agreement between $\vec{E}$ and $\vec{F}$, together with the non-overlapping condition, we see that both i_0 and i_1 have critical points greater than λ.

In the comparison it is not possible that M out-iterates $\mathrm{Ult}(V, E(\kappa, \beta))$, for then M would out-iterate V and we could obtain a set of indiscernibles for V. It follows that $N_0 \subseteq N_1$. Since $A \in M$ and the critical points of i_0, i_1 are greater than λ we see successively that $A \in N_0$, $A \in N_1$ and finally $A \in \mathrm{Ult}(V, E(\kappa, \beta))$.

To finish the proof of Fact 2.7 let κ, μ, and ν be as above, and let B be a bounded subset of μ, so that without loss of generality $B \subseteq \lambda$ for some cardinal $\lambda < \mu$. It follows from what was proved above that $B \in \mathrm{Ult}(V, E(\kappa, \beta))$ where $\lambda < \beta < \lambda^+ \leq \mu$. Now $E(\kappa, \beta) \in \mathrm{Ult}(V, E(\kappa, \nu))$ and the models V and $\mathrm{Ult}(V, E(\kappa, \nu))$ agree past κ, so that easily their ultrapowers by $E(\kappa, \beta)$ agree past the image of κ, and so $B \in \mathrm{Ult}(V, E(\kappa, \nu))$ as claimed. □

Using these results, we can characterise the (κ, μ)-extenders which witness that κ is μ-strong.

Lemma 2.8.

Let $V = L[\vec{E}]$ where $\vec{E}$ is a non-overlapping coherent sequence of extenders. Let $\kappa < \mu = |V_\mu|$. Then:

(1) *For every $\overline{\mu}$ such that $\mu \leq \overline{\mu} < o^{\vec{E}}(\kappa)$, the extender $E(\kappa, \overline{\mu}) \restriction \mu$ witnesses that κ is μ-strong.*

(2) *If E is a (κ, μ)-extender witnessing that κ is μ-strong, then $E = E(\kappa, \overline{\mu}) \restriction \mu$ for some $\overline{\mu}$ such that $\mu \leq \overline{\mu} < o^{\vec{E}}(\kappa)$.*

Proof. The first claim is straightforward. Since μ is a limit cardinal greater than κ, appealing to Fact 2.7 we see that all bounded subsets of μ are in $\mathrm{Ult}(V, E(\kappa, \overline{\mu}))$, so that $V_\mu \subseteq \mathrm{Ult}(V, E(\kappa, \overline{\mu}))$. The extender $E(\kappa, \overline{\mu}) \restriction \mu$ is the (κ, μ)-extender approximating the embedding $j_{E(\kappa, \overline{\mu})}$, so it witnesses that κ is μ-strong.

We prove the second claim by contradiction. If it fails, we may assume that $L[\vec{E}]$ is ϕ-minimal where ϕ asserts "the second claim fails". Fix an extender E witnessing that κ is μ-strong, and form $j_E \colon V \to M_E = \mathrm{Ult}(V, E)$. Let $\vec{F} = j_E(\vec{E})$, so that $\vec{F}$ is a coherent non-overlapping sequence of extenders in $M_E = L[\vec{F}]$. Now compare V and M_E: by Fact 2.6 above we get iterations $i_0 \colon V \to N$ and $i_1 \colon M_E \to N$ such that $i_1 \circ j_E = i_0$.

Since $\mathrm{crit}(j_E) = \kappa$, we also have $\mathrm{crit}(i_0) = \kappa$, so that the first extender which is used in i_0 must be of the form $E(\kappa, \overline{\mu})$ for some $\overline{\mu} < o^{\vec{E}}(\kappa)$. Since $\mathrm{crit}(j_E) = \kappa$, the extender sequences $\vec{E}$ and $\vec{F}$ agree up to κ. An easy induction using Fact 2.5 shows that for every $\eta < o^{\vec{F}}(\kappa)$ we have $\eta < o^{\vec{E}}(\kappa)$ and $F(\kappa, \eta) = E(\kappa, \eta)$.

Since $V_\mu \subseteq M_E$, for every $\eta < \min\{\mu, o^{\vec{E}}(\kappa)\}$ we have $E(\kappa, \eta) \in M_E$, so by another appeal to Fact 2.5 we see that $\eta < o^{\vec{F}}(\kappa)$. Summarising, at the critical point κ we have that

- The sequence $\vec{F}(\kappa, -)$ is an initial segment of $\vec{E}(\kappa, -)$.
- The sequences $\vec{F}(\kappa, -)$ and $\vec{E}(\kappa, -)$ agree up to μ.

In the comparison of $L[\vec{E}]$ and $L[\vec{F}]$ an extender with critical point κ is applied at the first step in the iteration i_0 of $L[\vec{E}]$, so the only possibility is that $\mu \leq \overline{\mu} = o^{\vec{F}}(\kappa) < o^{\vec{E}}(\kappa)$.

In the first step of the comparison we applied $E(\kappa, \overline{\mu})$ on the V-side and did nothing on the M_E-side, obtaining models which have identical $\overline{\mu}$-sequences of extenders at critical point κ. Since we are using non-overlapping extender sequences and $\overline{\mu} \geq \mu$, it follows that in the remainder of the comparison all critical points are greater than μ; that is to say $\mathrm{crit}(i_1) > \mu$, and all critical points in i_0 past the first step are greater than μ. So now for any $a \in [\mu]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$, we see that

$$a \in j_E(X) \iff a \in i_0(X) \iff a \in j_{E(\kappa, \overline{\mu})}(X),$$

where the first equivalence holds because $\mathrm{crit}(i_1) > \mu$ and $i_1 \circ j_E = i_0$, and the second equivalence holds because the first step in i_0 is to apply $E(\kappa, \overline{\mu})$ and all subsequent critical points in i_0 are greater than μ.

It follows that $E = E(\kappa, \overline{\mu}) \restriction \mu$. We have shown that the second claim holds in $L[\vec{E}]$, an immediate contradiction. □

The following corollary is immediate:

Corollary 2.9.

Let $V = L[\vec{E}]$ where $\vec{E}$ is a non-overlapping coherent sequence of extenders. Let $o^{\vec{E}}(\kappa) = \mu + 1$ for some cardinal $\mu > \kappa$ with $|V_\mu| = \mu$. Then $E(\kappa, \mu)$ is the only (κ, μ)-extender witnessing that κ is μ-strong.

With Corollary 2.9 in hand, we can now prove the remaining results about Question 1.1.

Theorem 2.10.
It is consistent that μ is singular, there is exactly one (κ,μ)-extender E witnessing that κ is μ-strong, and E is a bad witness.

Proof. The proof is very similar to that of Fact 2.2. Suppose that $V = L[\vec{E}]$ for a non-overlapping coherent extender sequence $\vec{E}$, and $o^{\vec{E}}(\kappa) = \nu + 1$ for some inaccessible cardinal $\nu > \kappa$. Let $E = E(\kappa,\nu)$. By the arguments in the proof of Fact 2.2 we can find a singular cardinal μ such that $\kappa < \mu < \nu$, and j_E factors as $k \circ j_{E\restriction\mu}$ where $\mathrm{crit}(k) = \mu$ and $k(\mu) = \nu$. As we already argued, if we let $F = E\restriction\mu$ then F is a bad witness.

The novel point is that since k is elementary and $o^{j_E(\vec{E})}(\kappa) = \nu$, we have $o^{j_F(\vec{E})}(\kappa) = \mu$, so that $E(\kappa,\mu) = F$ by an appeal to Fact 2.5 above. Now we take the ultrapower of V by $E(\kappa,\mu+1)$, and obtain (by coherence and Corollary 2.9) a model N in which F is still a bad witness and F is the unique (κ,μ)-extender witnessing that κ is μ-strong. □

Theorem 2.11.
It is consistent that μ is singular, there is exactly one (κ,μ)-extender E witnessing that κ is μ-strong, and E is a good witness.

Proof. Suppose that $V = L[\vec{E}]$ for a non-overlapping coherent extender sequence $\vec{E}$, and that κ is minimal such that there exists $\mu > \kappa$ with $o^{\vec{E}}(\kappa) = \mu + 1$ and $\mu = |V_\mu|$. We claim that μ is singular. Otherwise μ is inaccessible, so there is $\overline{\mu} < \mu$ such that $\overline{\mu} = |V_{\overline{\mu}}|$. Now if $E = E(\kappa,\overline{\mu})$ then E witnesses "κ is $\overline{\mu}$-strong" and

$$\mathrm{Ult}(V,E) \models \text{“}\overline{\mu} = |V_{\overline{\mu}}| \text{ and } o^{j_E(\vec{E})}(\kappa) = \overline{\mu}\text{”},$$

so that by elementarity we obtain a contradiction to the minimal choice of κ.

By Corollary 2.9, if we let $F = E(\kappa,\mu)$ then F is the unique (κ,μ)-extender witnessing that κ is μ-strong. We claim that F is a good witness. If not then μ is inaccessible in $\mathrm{Ult}(V,F)$, but since $\mu = o^{j_F(\vec{E})}(\kappa)$ this would imply by the elementarity of j_F that there are many ordinals $\delta < \kappa$ such that $o^{\vec{E}}(\delta)$ is inaccessible, contradicting the minimal choice of κ. □

3. Conclusion and open questions

We have determined fairly tight upper and lower bounds in consistency strength for the existence of a bad witness, and have produced models in which the unique witness is good or bad as we please. The following questions are left open by the results in this paper:

(1) Determine the exact consistency strength of the existence of a bad witness. We note that (by a straightforward argument) if there is a bad witness there is a model of the form $L[\vec{E}]$ with a bad witness, so the question amounts to asking how long the sequence $\vec{E}$ must be before a bad witness appears.

(2) Is it consistent that μ is singular and there are exactly two (κ,μ)-extenders witnessing "κ is μ-strong", of which one is good and the other is bad?

Note added in proof: After reading a circulated draft of this paper, Moti Gitik [3] resolved both of the open questions. With his kind permission we give a brief outline of his answers.

It can be shown that if $\eta > \kappa$ is minimal such that for some (κ,η)-extender E we have that η is regular in $\mathrm{Ult}(V,E)$, then $\mathrm{cf}(\eta) = \kappa^+$. Moreover, if we fix such an E, and factor j_E through the ultrapower map j_U associated with the measure $U = \{X : \kappa \in j_E(X)\}$ so that $j_E = k \circ j_U$, we obtain the following situation: $\eta = \sup k``\eta^*$ where η^* is the least inaccessible cardinal greater than κ in $\mathrm{Ult}(V,U)$.

This gives a characterisation of the consistency strength of the existence of a bad witness, in terms of a hypothesis on the core model K and its extender sequence $\vec{E}$. The hypothesis is that $\vec{E}$ has a last extender E at κ, η is the strength of E, and $\eta = \sup k``\eta^*$ where k and η^* are defined as in the last paragraph. This resolves question (1).

Turning to question (2), let $\eta > \kappa$ be the least singular cardinal such that there is a bad witness. Let E be such a witness, and suppose that F is another extender at κ with $E \in \mathrm{Ult}(V, F)$. It can be shown using the minimal choice of η that η is singular in $\mathrm{Ult}(V, F)$. If we now consider a model of the form $L[\vec{E}]$ where κ and η are as above, and the penultimate extender at κ is a bad witness for η, then the same kind of analysis used to prove Theorems 2.10 and 2.11 shows that in V there are exactly two witnesses for η, one good and the other bad.

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