

Isometry groups of non standard metric products

Research Article

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Abstract: We consider isometry groups of a fairly general class of non standard products of metric spaces. We present sufficient conditions under which the isometry group of a non standard product of metric spaces splits as a permutation group into direct or wreath product of isometry groups of some metric spaces.

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1. Introduction

Let (X_i, d_i) , $i = 1, \dots, n$, be metric spaces. To define a metric on their cartesian product $X = \prod_{i=1}^n X_i$ one can use, for instance, one of the following equalities:

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = d_1(x_1, y_1) + \dots + d_n(x_n, y_n), \quad \tilde{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{d_1^2(x_1, y_1) + \dots + d_n^2(x_n, y_n)}.$$

There are different generalizations of these constructions. In the case $n = 2$ they include μ -products [1], f -products [5], warped products [3], etc. Following A. Bernig, T. Foertsch, V. Schroeder [2] we consider *non standard metric products* or Φ -products of metric spaces. Let us recall the precise definition. A function $\Phi: [0, \infty)^n \rightarrow [0, \infty)$ is called *admissible* if it satisfies the following conditions:

(A) $\Phi(p_1, p_2, \dots, p_n) = 0$ iff $p_1 = p_2 = \dots = p_n = 0$;

(B) $\Phi(q_1, \dots, q_n) \leq \Phi(r_1, \dots, r_n) + \Phi(p_1, \dots, p_n)$ for any $q_i, r_i, p_i \in [0, \infty)$ such that $q_i \leq r_i + p_i$, $1 \leq i \leq n$.

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Then the function

$$d_\Phi((x_1, \dots, x_n), (y_1, \dots, y_n)) = \Phi(d_1(x_1, y_1), \dots, d_n(x_n, y_n))$$

is a metric on X [2].

Definition 1.1.

The metric space (X, d_Φ) is called the Φ -product of metric spaces $X_1, \dots, X_n$.

Wreath products of metric spaces [6] arise as a special case of Φ -products of metric spaces. The aim of this article is to describe the isometry group of the Φ -product of $X_1, \dots, X_n$, under certain conditions for Φ . We will show the relation

$$(\text{Isom } X, X) \geq (\text{Isom } X_1, X_1) \times \dots \times (\text{Isom } X_n, X_n). \quad (1)$$

For $n = 2$ we describe a family of functions Φ for which the relation (1) is an equality. More generally, we show that for certain Φ -products of two metric spaces, its isometry group splits as a permutation group into the direct product of isometry groups of naturally defined subspaces. We also present sufficient conditions on Φ under which the isometry group of the Φ -product of $X_1, \dots, X_n$ is isomorphic as a permutation group to the wreath product of the isometry groups of spaces $X_1, \dots, X_n$.

2. Preliminaries

We will need the following.

Proposition 2.1.

Let $\Phi: [0, \infty)^n \rightarrow [0, \infty)$ be an admissible function. Then

$$\Phi(q_1, \dots, q_{i-1}, 0, q_{i+1}, \dots, q_n) \leq \Phi(q_1, \dots, q_{i-1}, q_i, q_{i+1}, \dots, q_n),$$

for all $q_i \in [0, \infty)$, $1 \leq i \leq n$.

Proof. If we replace q_j, r_j, p_j by $q_j, q_j, 0$, $1 \leq j \leq n$, $j \neq i$, and q_i, r_i, p_i by $0, q_i, 0$ respectively in condition (B), then we obtain

$$\Phi(q_1, \dots, q_{i-1}, 0, q_{i+1}, \dots, q_n) \leq \Phi(q_1, \dots, q_{i-1}, q_i, q_{i+1}, \dots, q_n) + \Phi(0, \dots, 0) = \Phi(q_1, \dots, q_{i-1}, q_i, q_{i+1}, \dots, q_n). \quad \square$$

Let q be a positive real number. It is easy to see that the function

$$\tilde{\Phi}(p_1, p_2, \dots, p_n) = \begin{cases} 0 & \text{if } p_1 = p_2 = \dots = p_n = 0, \\ q & \text{otherwise,} \end{cases}$$

is admissible. Therefore, for arbitrary metric spaces $X_1, \dots, X_n$ one can consider their $\tilde{\Phi}$ -product. The isometry group of $(X, d_{\tilde{\Phi}})$ is isomorphic as a permutation group to the symmetric group $S_{|X|}$. This is the largest possible isometry group of Φ -products of $X_1, \dots, X_n$.

In general, we obtain the following statement describing a candidate for the smallest possible isometry group of a Φ -product.

Proposition 2.2.

Let X be a Φ -product of metric spaces $X_1, \dots, X_n$, $n \geq 2$. Then the transformation group $(\text{Isom } X, X)$ contains a subgroup isomorphic to the direct product of transformation groups

$$(\text{Isom } X_1, X_1) \times \dots \times (\text{Isom } X_n, X_n).$$

Thus, the direct product $\prod_{i=1}^n (\text{Isom } X_i, X_i)$ is contained in the isometry group of the Φ -product of $X_1, \dots, X_n$ for any admissible Φ . In the next section, we consider conditions on Φ so that the isometry group of the space (X, d_Φ) is the smallest possible.

3. Direct product of transformation groups and Φ -products of metric spaces

In this section, we consider Φ -products $(X_1 \times X_2, d_\Phi)$ of two metric spaces $(X_1, d_1), (X_2, d_2)$. We will use the following notation. For each $a_1 \in X_1$, $a_2 \in X_2$ let

$$X_{a_1}^2 = \{(a_1, x_2) : x_2 \in X_2\}, \quad X_{a_2}^1 = \{(x_1, a_2) : x_1 \in X_1\}$$

be subspaces of $(X_1 \times X_2, d_\Phi)$. The points of $X_{a_1}^2$ are in natural one-to-one correspondence with the points of X_2 , while the points of spaces $X_{a_2}^1$ are in natural one-to-one correspondence with the points of X_1 . With these identifications, let the group $\text{Isom } X_{a_1}^2$ act on X_2 and $\text{Isom } X_{a_2}^1$ act on X_1 .

Lemma 3.1.

- (i) For arbitrary $a_1, b_1 \in X_1$ the spaces $X_{a_1}^2$ and $X_{b_1}^2$ are isometric.
- (ii) For arbitrary $a_2, b_2 \in X_2$ the spaces $X_{a_2}^1$ and $X_{b_2}^1$ are isometric.

Proof. (i) Let $g: X_{a_1}^2 \rightarrow X_{b_1}^2$ be a one-to-one correspondence between $X_{a_1}^2$ and $X_{b_1}^2$ given by the equality $g((a_1, x_2)) = (b_1, x_2)$ for all $x_2 \in X_2$. For different y_2 and z_2 from X_2 the equalities

$$d_\Phi(g((a_1, y_2)), g((a_1, z_2))) = d_\Phi((b_1, y_2), (b_1, z_2)) = \Phi(d_1(b_1, b_1), d_2(y_2, z_2)) = \Phi(0, d_2(y_2, z_2))$$

hold. As $d_\Phi((a_1, y_2), (a_1, z_2)) = \Phi(0, d_2(y_2, z_2))$,

$$d_\Phi(g((a_1, y_2)), g((a_1, z_2))) = d_\Phi((a_1, y_2), (a_1, z_2)).$$

Therefore, g is an isometry between $X_{a_1}^2$ and $X_{b_1}^2$.

(ii) The proof of this statement is similar to the proof of (i). □

Note that groups $(\text{Isom } X_{a_1}^2, X_2)$ and $(\text{Isom } X_2, X_2)$ (respectively $(\text{Isom } X_{a_2}^1, X_1)$ and $(\text{Isom } X_1, X_1)$) are not necessarily isomorphic. Moreover, the spaces $X_{a_1}^2$ and X_2 (resp. $X_{a_2}^1$ and X_1) are not necessarily isometric, see Example 3.5 below.

Denote by C_i the set of values of the metric d_i , $i = 1, 2$. Assume that

$$\inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0) > \sup_{q_2 \in C_2} \Phi(0, q_2), \quad \inf_{q_2 \in C_2, q_2 \neq 0} \Phi(0, q_2) > \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0). \quad (2)$$

Note that the estimates

$$\sup_{q_1 \in C_1} \Phi(q_1, 0) < \infty, \quad \sup_{q_2 \in C_2} \Phi(0, q_2) < \infty \quad \text{and} \quad \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0) > 0, \quad \inf_{q_2 \in C_2, q_2 \neq 0} \Phi(0, q_2) > 0$$

follow from the inequalities (2).

Theorem 3.2.

Let $\Phi: [0, \infty)^2 \rightarrow [0, \infty)$ be an admissible function, and let $(X_1, d_1), (X_2, d_2)$ be metric spaces. Assume that Φ satisfies (2) and the following condition holds:

$$\Phi(q_1, q_2) = \Phi(q_1, 0) + \Phi(0, q_2). \quad (3)$$

Then

$$(\text{Isom } X, X) \simeq (\text{Isom } X_{a_2}^1, X_1) \times (\text{Isom } X_{a_1}^2, X_2)$$

for any $(a_1, a_2) \in X_1 \times X_2$.

Proof. We shall show that each $(g_1, g_2) \in \text{Isom } X_{a_2}^1 \times \text{Isom } X_{a_1}^2$ is an isometry of (X, d_Φ) . The element (g_1, g_2) acts on $X_1 \times X_2$ coordinate-wise. Therefore, it suffices to show that this transformation preserves the metric d_Φ :

$$d_\Phi((x_1, x_2)^{(g_1, g_2)}, (y_1, y_2)^{(g_1, g_2)}) = d_\Phi((x_1^{g_1}, x_2^{g_2}), (y_1^{g_1}, y_2^{g_2})) = \Phi(d_1(x_1^{g_1}, y_1^{g_1}), d_2(x_2^{g_2}, y_2^{g_2})).$$

From (3) we have

$$\begin{aligned} \Phi(d_1(x_1^{g_1}, y_1^{g_1}), d_2(x_2^{g_2}, y_2^{g_2})) &= \Phi(d_1(x_1^{g_1}, y_1^{g_1}), 0) + \Phi(0, d_2(x_2^{g_2}, y_2^{g_2})) \\ &= \Phi(d_1(x_1^{g_1}, y_1^{g_1}), d_2(a_2, a_2)) + \Phi(d_1(a_1, a_1), d_2(x_2^{g_2}, y_2^{g_2})) \\ &= d_\Phi((x_1^{g_1}, a_2), (y_1^{g_1}, a_2)) + d_\Phi((a_1, x_2^{g_2}), (a_1, y_2^{g_2})). \end{aligned}$$

As $g_1 \in \text{Isom } X_{a_2}^1, g_2 \in \text{Isom } X_{a_1}^2$, the following equalities hold:

$$d_\Phi((x_1^{g_1}, a_2), (y_1^{g_1}, a_2)) = d_\Phi((x_1, a_2), (y_1, a_2)), \quad d_\Phi((a_1, x_2^{g_2}), (a_1, y_2^{g_2})) = d_\Phi((a_1, x_2), (a_1, y_2)).$$

Therefore,

$$\begin{aligned} d_\Phi((x_1, x_2)^{(g_1, g_2)}, (y_1, y_2)^{(g_1, g_2)}) &= d_\Phi((x_1, a_2), (y_1, a_2)) + d_\Phi((a_1, x_2), (a_1, y_2)) \\ &= \Phi(d_1(x_1, y_1), 0) + \Phi(0, d_2(x_2, y_2)) = d_\Phi((x_1, x_2), (y_1, y_2)). \end{aligned}$$

Hence,

$$(\text{Isom } X, X) \geq (\text{Isom } X_{a_2}^1, X_1) \times (\text{Isom } X_{a_1}^2, X_2).$$

Let $\varphi \in \text{Isom } X$. We show that there exist $g_1 \in \text{Isom } X_{a_2}^1$ and $g_2 \in \text{Isom } X_{a_1}^2$, such that φ acts on $X_1 \times X_2$ as $(g_1, g_2) \in \text{Isom } X_{a_2}^1 \times \text{Isom } X_{a_1}^2$ does. Let (y_1, y_2) be a point from $X_1 \times X_2$, u_1 a point from X_1 . Then

$$d_\Phi((y_1, y_2), (u_1, y_2)) = \Phi(d_1(y_1, u_1), d_2(y_2, y_2)) = \Phi(d_1(y_1, u_1), 0) \leq \sup_{q_1 \in C_1} \Phi(q_1, 0).$$

Denote by (z_1, z_2) the value $\varphi(y_1, y_2)$ and by (w_1, w_2) the value $\varphi(u_1, y_2)$. Using (3) we get

$$d_\Phi(\varphi(y_1, y_2), \varphi(u_1, y_2)) = d_\Phi((z_1, z_2), (w_1, w_2)) = \Phi(d_1(z_1, w_1), d_2(z_2, w_2)) = \Phi(d_1(z_1, w_1), 0) + \Phi(0, d_2(z_2, w_2)).$$

Assume that $z_2 \neq w_2$. Then using (2) we obtain

$$\begin{aligned} \Phi(d_1(z_1, w_1), 0) + \Phi(0, d_2(z_2, w_2)) &\geq \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0) + \inf_{q_2 \in C_2, q_2 \neq 0} \Phi(0, q_2) > \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0) + \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0) \\ &= \sup_{q_1 \in C_1} \Phi(q_1, 0). \end{aligned}$$

We have

$$d_\Phi(\varphi(y_1, y_2), \varphi(u_1, y_2)) > \sup_{q_1 \in C_1} \Phi(q_1, 0).$$

But φ is an isometry of the space X . Hence $z_2 = w_2$, i.e. $\varphi(u_1, y_2) = (w_1, z_2)$. Then the mapping φ acts as an isometry between subspaces of the form $X_{a_2}^1$, $a_2 \in X_2$.

Denote by g_1 the restriction of φ on $X_{y_2}^1$. Then $z_1 = g_1(y_1)$. We shall show that φ acts on each subspace of the form $X_{a_2}^1$, $a_2 \in X_2$, as g_1 . Fix arbitrary $(b_1, b_2) \in X_1 \times X_2$. Assume that $\varphi(b_1, b_2) = (h_1, h_2)$. We shall show that $h_1 = g_1(b_1)$. Indeed, in the opposite case from (2) it follows

$$d_\Phi((b_1, y_2), (b_1, b_2)) = \Phi(0, d_2(y_2, b_2)) < \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0)$$

and

$$\begin{aligned} d_\Phi(\varphi(b_1, y_2), \varphi(b_1, b_2)) &= d_\Phi((b_1^{g_1}, z_2), (h_1, h_2)) = \Phi(d_1(b_1^{g_1}, h_1), d_2(z_2, h_2)) = \Phi(d_1(b_1^{g_1}, h_1), 0) + \Phi(0, d_2(z_2, h_2)) \\ &\geq \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0). \end{aligned}$$

Therefore, the isometry φ acts on each subspace $X_{a_2}^1$, $a_2 \in X_2$, as g_1 .

Let now u_2 be a point from X_2 . Then

$$d_\Phi((y_1, y_2), (y_1, u_2)) = \Phi(0, d_2(y_2, u_2)) \leq \sup_{q_2 \in C_2} \Phi(0, q_2) < \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0). \quad (4)$$

Suppose that $\varphi(y_1, u_2) = (v_1, v_2)$. Using (3) we obtain

$$d_\Phi(\varphi(y_1, y_2), \varphi(y_1, u_2)) = d_\Phi((z_1, z_2), (v_1, v_2)) = \Phi(d_1(z_1, v_1), d_2(z_2, v_2)) = \Phi(d_1(z_1, v_1), 0) + \Phi(0, d_2(z_2, v_2)).$$

Assume that $z_1 \neq v_1$. We have

$$d_\Phi(\varphi(y_1, y_2), \varphi(y_1, u_2)) \geq \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0). \quad (5)$$

Combining (4) and (5), we obtain $z_1 = v_1$. Therefore, φ acts as an isometry between subspaces of the form $X_{a_1}^2$, $a_1 \in X_1$.

Denote by g_2 the restriction of φ on $X_{y_1}^2$. Then $z_2 = g_2(y_2)$. Assume now that $h_2 \neq g_2(b_2)$. Using (2) we get

$$d_\Phi((y_1, b_2), (b_1, b_2)) = \Phi(d_1(y_1, b_1), 0) \leq \sup_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0), \quad (6)$$

$$\begin{aligned} d_\Phi(\varphi(y_1, b_2), \varphi(b_1, b_2)) &= d_\Phi((z_1, b_2^{g_2}), (h_1, h_2)) = \Phi(d_1(z_1, h_1), d_2(b_2^{g_2}, h_2)) = \Phi(d_1(z_1, h_1), 0) + \Phi(0, d_2(b_2^{g_2}, h_2)) \\ &\geq \inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0) + \sup_{q_2 \in C_2} \Phi(0, q_2) > \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0) + \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0) = \sup_{q_1 \in C_1} \Phi(q_1, 0). \end{aligned} \quad (7)$$

Combining (6) and (7), we obtain $h_2 = g_2(b_2)$. Therefore, the isometry φ acts on each subspace of the form $X_{a_1}^2$, $a_1 \in X_1$, as g_2 . From Lemma 3.1 it follows, that $g_1 \in \text{Isom } X_{a_2}^1$, $g_2 \in \text{Isom } X_{a_1}^2$ for any $(a_1, a_2) \in X_1 \times X_2$. Finally, for arbitrary $(x_1, x_2) \in X_1 \times X_2$ we have

$$\varphi(x_1, x_2) = (x_1, x_2)^{(g_1, g_2)} = (x_1^{g_1}, x_2^{g_2}). \quad \square$$

Corollary 3.3.

Let $\Phi: [0, \infty)^2 \rightarrow [0, \infty)$ be an admissible function, and let (X_1, d_1) , (X_2, d_2) be metric spaces. Assume that Φ satisfies (2) and the following condition holds:

$$\Phi(q_1, q_2) = \Phi(q_1, 0) + \Phi(0, q_2).$$

If $\text{Isom } X_{a_2}^1 = \text{Isom } X_1$, $\text{Isom } X_{a_1}^2 = \text{Isom } X_2$ for some $(a_1, a_2) \in X_1 \times X_2$, then

$$(\text{Isom } X, X) \simeq (\text{Isom } X_1, X_1) \times (\text{Isom } X_2, X_2).$$

Example 3.4.

Let (X_1, d_1) and (X_2, d_2) be uniformly discrete metric spaces of finite diameters D_1, D_2 respectively. And let r_1, r_2 be positive numbers, such that for arbitrary points $x_i, y_i \in X_i$, $x_i \neq y_i$, the inequalities $d_i(x_i, y_i) \geq r_i$ hold, $i = 1, 2$. Denote $\Phi_1(q_1, q_2) = q_1 + q_2$. Then the function $\Phi_1(q_1, q_2)$ is admissible. If the inequalities

$$r_1 > D_2 \geq r_2 > \frac{1}{2} D_1 \quad \text{or} \quad r_2 > D_1 \geq r_1 > \frac{1}{2} D_2$$

hold, then the inequalities (2) hold as well. Therefore

$$\text{Isom}(X_1 \times X_2, d_{\Phi_1}) \simeq \text{Isom } X_1 \times \text{Isom } X_2.$$

Example 3.5.

Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let

$$\Phi_2(q_1, q_2) = \begin{cases} 0 & \text{if } q_1 = q_2 = 0, \\ 4 & \text{if } q_1 \neq 0, q_2 = 0, \\ 3 & \text{if } q_1 = 0, q_2 \neq 0, \\ 7 & \text{in other cases.} \end{cases}$$

Then

$$\text{Isom}(X_1 \times X_2, d_{\Phi_2}) \simeq S_{|X_1|} \times S_{|X_2|}.$$

4. Wreath products of groups and Φ -products

In this section we consider Φ -products $(X_1 \times \cdots \times X_n, d_\Phi)$ of n metric spaces $(X_1, d_1), \dots, (X_n, d_n)$, $n \geq 2$. Let us recall the definition of the wreath product of transformation groups. Let $(G_1, X_1), \dots, (G_n, X_n)$ be a sequence of transformation groups. Following [7], the transformation group $(G, \prod_{i=1}^n X_i)$ is called the *wreath products of groups* $(G_1, X_1), \dots, (G_n, X_n)$ if for all elements $u \in G$ the following conditions hold:

- 1) if $(x_1, \dots, x_n)^u = (y_1, \dots, y_n)$, then for all i , $1 \leq i \leq n$, the value of y_i depends only on $x_1, \dots, x_i$;
- 2) for fixed $x_1, \dots, x_{i-1}$ the mapping $g_i(x_1, \dots, x_{i-1})$ defined by the equality

$$g_i(x_1, \dots, x_{i-1})(x_i) = y_i, \quad x_i \in X_i,$$

is a permutation on the set X_i which belongs to G_i . Denote the wreath products of groups $(G_1, X_1), \dots, (G_n, X_n)$ by $\wr_{i=1}^n (G_i, X_i)$.

It follows from this definition that each element $u \in G$ can be represented as a so-called table $u = [g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1})]$, where $g_1 \in G_1$, $g_i(x_1, \dots, x_{i-1}) \in G_i^{X_1 \times \cdots \times X_{i-1}}$, $2 \leq i \leq n$. An element $u \in G$ acts on $(x_1, \dots, x_n) \in \prod_{i=1}^n X_i$ by the rule

$$(x_1, \dots, x_n)^u = (x_1^{g_1}, x_2^{g_2(m_1)}, \dots, x_n^{g_n(x_1, \dots, x_{n-1})}).$$

We can consider the space δT of paths in a rooted level homogeneous tree T as some Φ -product of discrete metric spaces. Indeed, let T be a finite n -levels rooted tree with root v_0 . Recall that a rooted tree T is called *level homogenous* [4] if it is homogenous on every level. Such a tree is uniquely determined by its *level index*, i.e. by a finite sequence of cardinal numbers $[k_0; k_1; k_2; \dots; k_n]$, where k_i is the number of edges joining a vertex of the i -th level with vertices of the $(i+1)$ -st level. A *rooted path* is a finite sequence of vertices $(v_0, v_1, \dots, v_n)$ such that $\{v_i, v_{i+1}\} \in E(T)$ for every i , $0 \leq i \leq n-1$. The metric space δT is defined to be the set of all rooted paths of T equipped with a natural ultrametric

$$\rho(v_1, v_2) = \frac{1}{m+1},$$

where m is the length of the maximal common part of rooted paths γ_1 and γ_2 .

Let now $(X_1, d_1), \dots, (X_n, d_n)$ be discrete spaces, i.e., for different points $u, v \in X_i$, $d_i(u, v) = 1$, $1 \leq i \leq n$. And let $|X_i| = k_i$, $1 \leq i \leq n$. We can introduce the function $\Phi_3: [0, \infty)^n \rightarrow [0, \infty)$ putting

$$\Phi_3(q_1, \dots, q_n) = \begin{cases} q_1 & \text{if } q_1 \neq 0, \\ \frac{1}{2} q_2 & \text{if } q_1 = 0 \text{ and } q_2 \neq 0, \\ \dots & \\ \frac{1}{n} q_n & \text{if } q_1 = \dots = q_{n-1} = 0 \text{ and } q_n \neq 0, \\ 0 & \text{if } q_1 = \dots = q_n = 0. \end{cases}$$

It is clear that Φ_3 is admissible. Therefore, one can consider the Φ_3 -product of the spaces $X_1, \dots, X_n$.

It is easy to see that the space δT of paths in the rooted level homogeneous tree T and the Φ_3 -product of discrete metric spaces $X_1, \dots, X_n$ are isometric. It is well known that the isometry group of the space δT is isomorphic as a permutation group to the wreath product of symmetric groups S_{k_i} , $i = 1, \dots, n$. Therefore, the isometry group of the space $(X_1 \times \dots \times X_n, d_{\Phi_3})$ is isomorphic as a permutation group to the wreath product of isometry groups of discrete spaces X_i , $i = 1, \dots, n$. In this section we generalize this result by extending the class of metric spaces and introducing restrictions on the function Φ .

Let now (X_i, d_i) , $i = 1, \dots, n$, be arbitrary metric spaces. And let as before C_i be the set of values of the metric d_i , $1 \leq i \leq n$. Assume that there exist functions $f_i: [0, \infty) \rightarrow [0, \infty)$, $1 \leq i \leq n$, such that

$$\Phi(q_1, \dots, q_n) = \begin{cases} f_1(q_1) & \text{if } q_1 \neq 0, \\ f_2(q_2) & \text{if } q_1 = 0 \text{ and } q_2 \neq 0, \\ \dots & \\ f_n(q_n) & \text{if } q_1 = \dots = q_{n-1} = 0 \text{ and } q_n \neq 0, \\ 0 & \text{if } q_1 = \dots = q_n = 0, \end{cases} \quad (8)$$

for arbitrary $q_i \geq 0$, $1 \leq i \leq n$. For each i , $1 \leq i \leq n$, denote by $\hat{X}_i$ the space $(X_i, \hat{d}_i)$, where for $u, v \in X_i$,

$$\hat{d}_i(u, v) = \begin{cases} f_i(d_i(u, v)) & \text{if } u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that for all i , $1 \leq i \leq n-1$,

$$\inf_{q_i \in C_i, q_i \neq 0} f_i(q_i) > \sup_{q_{i+1} \in C_{i+1}} f_{i+1}(q_{i+1}). \quad (9)$$

Theorem 4.1.

Let $\Phi: [0, \infty)^n \rightarrow [0, \infty)$ be an admissible function such that conditions (8) and (9) hold. Then the isometry group of the Φ -product X of metric spaces $X_1, X_2, \dots, X_n$ is isomorphic as a permutation group to the wreath product of isometry groups of spaces $\hat{X}_i$, $i = 1, \dots, n$,

$$(\text{Isom}(X, d_\Phi), X) \simeq \wr_{i=1}^n (\text{Isom } \hat{X}_i, X_i).$$

Proof. Consider arbitrary

$$\varphi = [g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1})] \in \wr_{i=1}^n \text{Isom } \hat{X}_i.$$

We shall show that φ is an isometry of (X, d_Φ) . By the definition of the wreath product of permutation groups [7] the element φ acts on $\prod_{i=1}^n X_i$. Therefore, it is sufficient to show that φ preserves the metric d_Φ . Indeed,

$$\begin{aligned} d_\Phi(\varphi(x_1, x_2, \dots, x_n), \varphi(y_1, y_2, \dots, y_n)) &= d_\Phi\left((x_1^{g_1}, x_2^{g_2(x_1)}, \dots, x_n^{g_n(x_1, \dots, x_{n-1})}), (y_1^{g_1}, y_2^{g_2(y_1)}, \dots, y_n^{g_n(y_1, \dots, y_{n-1})})\right) \\ &= \Phi\left(d_1(x_1^{g_1}, y_1^{g_1}), d_2(x_2^{g_2(x_1)}, y_2^{g_2(y_1)}), \dots, d_n(x_n^{g_n(x_1, \dots, x_{n-1})}, y_n^{g_n(y_1, \dots, y_{n-1})})\right). \end{aligned}$$

Using (8), we have

$$d_\Phi(\varphi(x_1, x_2, \dots, x_n), \varphi(y_1, y_2, \dots, y_n)) = \begin{cases} f_1(d_1(x_1^{g_1}, y_1^{g_1})) & \text{if } x_1^{g_1} \neq y_1^{g_1}, \\ f_2(d_2(x_2^{g_2(x_1)}, y_2^{g_2(y_1)})) & \text{if } x_1^{g_1} = y_1^{g_1}, x_2^{g_2(x_1)} \neq y_2^{g_2(y_1)}, \\ \dots & \\ 0 & \text{if } x_1^{g_1} = y_1^{g_1}, \dots, x_n^{g_n(x_1, \dots, x_{n-1})} = y_n^{g_n(y_1, \dots, y_{n-1})}. \end{cases} \quad (10)$$

But $g_1 \in \text{Isom } \widehat{X}_1$. Then $x_1^{g_1} = y_1^{g_1}$ iff $x_1 = y_1$. Hence, $x_1^{g_1} = y_1^{g_1}$ iff $g_2(x_1) = g_2(y_1)$, and so on. With similar reasoning, using (10), we get

$$d_\Phi(\varphi(x_1, \dots, x_n), \varphi(y_1, \dots, y_n)) = \begin{cases} f_1(d_1(x_1^{g_1}, y_1^{g_1})) & \text{if } x_1 \neq y_1, \\ f_2(d_2(x_2^{g_2(x_1)}, y_2^{g_2(y_1)})) & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2, \\ \dots & \\ 0 & \text{if } x_1 = y_1, \dots, x_n = y_n. \end{cases}$$

As g_1 is an isometry of $\widehat{X}_1$, $f_1(d_1(x_1^{g_1}, y_1^{g_1})) = f_1(d_1(x_1, y_1))$. Since $g_i(x_1, \dots, x_{i-1}) \in \text{Isom } \widehat{X}_i$ the following equalities hold:

$$f_i(d_i(x_i^{g_i(x_1, \dots, x_{i-1})}, y_i^{g_i(x_1, \dots, x_{i-1})})) = f_i(d_i(x_i, y_i)), \quad 2 \leq i \leq n.$$

Therefore,

$$\begin{aligned} d_\Phi(\varphi(x_1, \dots, x_n), \varphi(y_1, \dots, y_n)) &= \begin{cases} f_1(d_1(x_1, y_1)) & \text{if } x_1 \neq y_1, \\ f_2(d_2(x_2, y_2)) & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2, \\ \dots & \\ f_n(d_n(x_n, y_n)) & \text{if } x_1 = y_1, \dots, x_{n-1} = y_{n-1}, x_n \neq y_n, \\ 0 & \text{if } x_1 = y_1, \dots, x_n = y_n, \end{cases} \\ &= d_\Phi((x_1, \dots, x_n), (y_1, \dots, y_n)). \end{aligned}$$

Let now φ be an isometry of (X, d_Φ) . Fix a point $(x_1, \dots, x_n)$ from X and assume that $\varphi(x_1, \dots, x_n) = (z_1, \dots, z_n)$. Let $\varphi(y_1, y_2, \dots, y_n) = (w_1, w_2, \dots, w_n)$ for some $y_i \in X_i$, $1 \leq i \leq n$, $(y_1, \dots, y_n) \neq (x_1, \dots, x_n)$. We have

$$d_\Phi((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \Phi(d_1(x_1, y_1), d_2(x_2, y_2), \dots, d_n(x_n, y_n)) = f_j(d_j(x_j, y_j)), \quad (11)$$

where j is the smallest number such that $y_1 = x_1, \dots, y_{j-1} = x_{j-1}, y_j \neq x_j$. Using (8), we obtain

$$\begin{aligned} d_\Phi(\varphi(x_1, x_2, \dots, x_n), \varphi(y_1, y_2, \dots, y_n)) &= d_\Phi((z_1, z_2, \dots, z_n), (w_1, w_2, \dots, w_n)) \\ &= \Phi(d_1(z_1, w_1), d_2(z_2, w_2), \dots, d_n(z_n, w_n)) = f_k(d_k(x_k, y_k)), \end{aligned} \quad (12)$$

where k is the smallest number such that $z_1 = w_1, \dots, z_{k-1} = w_{k-1}, z_k \neq w_k$. Combining (11), (12) and (9), we get $j = k$. This means that for all i , $1 \leq i \leq n$, the value y_i depends only on $x_1, \dots, x_i$ and for fixed $x_1, \dots, x_{i-1}$ the mapping φ acts on X_i as some isometry $g_i(x_1, \dots, x_{i-1})$. Therefore, there exists a table

$$[g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1})]$$

such that $g_1 \in \text{Isom } \widehat{X}_1$, $g_i(x_1, \dots, x_{i-1}) \in (\text{Isom } \widehat{X}_i)^{X_1 \times \dots \times X_{i-1}}$. And the table $[g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1})]$ acts on X as φ does. This completes the proof. $\square$

Corollary 4.2.

Let $\Phi: [0, \infty)^n \rightarrow [0, \infty)$ be an admissible function such that conditions (8) and (9) hold. If

$$\text{Isom}(X_i, f_i(d_i)) = \text{Isom}(X_i, d_i)$$

for all $i, 1 \leq i \leq n$, then

$$(\text{Isom } X, X) \simeq \wr_{i=1}^n (\text{Isom } X_i, X_i).$$

Example 4.3.

Let (X_1, d_1) and (X_2, d_2) be metric spaces of finite diameters D_1, D_2 . Assume that there exists a positive number r such that for arbitrary points $x_1, x_2 \in X_1, x_1 \neq x_2$, the inequality $d_1(x_1, x_2) \geq r$ holds. Let $\Phi_3(q_1, q_2) = \max(q_1, q_2)$. If $r > D_2$, then $\text{Isom}(X_1 \times X_2, d_{\Phi_3}) \simeq (\text{Isom } X_1) \wr (\text{Isom } X_2)$.

Example 4.4.

Let $X_i = \mathbb{Z}$ and d_i be the Euclidean distance, $1 \leq i \leq n$. It is easy to see that the function

$$\Phi_5(q_1, \dots, q_n) = \begin{cases} n+1 - \frac{1}{q_1+1} & \text{if } q_1 \neq 0, \\ n - \frac{1}{q_2+1} & \text{if } q_1 = 0 \text{ and } q_2 \neq 0, \\ \dots & \\ 2 - \frac{1}{q_n+1} & \text{if } q_1 = \dots = q_{n-1} = 0 \text{ and } q_n \neq 0, \\ 0 & \text{if } q_1 = \dots = q_n = 0, \end{cases}$$

is admissible and satisfies (8) and (9). Therefore, one can consider the Φ_5 -product $(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5})$ of $X_i, 1 \leq i \leq n$. The set of values of the metric d_{Φ_5} is bounded, while each d_i takes arbitrary large values.

It follows from Theorem 4.1 that the isometry group of $(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5})$ is isomorphic as a permutation group to the wreath product of isometry groups of $(X_i, \hat{d}_i), i = 1, \dots, n$, where for arbitrary $u, v \in X_i$,

$$\hat{d}_i(u, v) = \begin{cases} n+2-i - \frac{1}{d_i(u, v)+1} & \text{if } u \neq v, \\ 0 & \text{in other cases.} \end{cases}$$

Recall, metric spaces (Y, d_1) and (Y, d_2) are called *isomorphic* [8] if there exists a *scale*, that is a strictly increasing continuous function $s: \mathbb{R}^+ \rightarrow \mathbb{R}^+, s(0) = 0$, such that $d_1 = s(d_2)$. It is easy to observe that if metric spaces (Y, d_1) and (Y, d_2) are isomorphic then their isometry groups $\text{Isom}(Y, d_1)$ and $\text{Isom}(Y, d_2)$ are equal.

For each $i, 1 \leq i \leq n$, the spaces $(\mathbb{Z}, \hat{d}_i)$ and $(\mathbb{Z}, d_i)$ are isomorphic. Indeed, if we consider a scale $s: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by the equality

$$s(t) = \begin{cases} n+2-i - \frac{1}{t+1} & \text{if } t \geq 1, \\ \left(n + \frac{3}{2} - i\right)t & \text{if } 0 \leq t \leq 1, \end{cases}$$

then $\hat{d}_i = s(d_i)$ on $\mathbb{Z}$. Therefore, the isometry group of the space $(\mathbb{Z}, \hat{d}_i)$ is isomorphic to the infinite dihedral group D_∞ . Hence, the isometry group of $(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5})$ is isomorphic as a permutation group to the wreath product of n infinite dihedral groups D_∞ :

$$(\text{Isom}(\mathbb{Z} \times \dots \times \mathbb{Z}, d_{\Phi_5}), \mathbb{Z} \times \dots \times \mathbb{Z}) \simeq \wr_{i=1}^n (D_\infty, \mathbb{Z}).$$

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