Cent. Eur. J. Math. • 10(6) • 2012 • 1981-1994
DOI: 10.2478/s11533-012-0112-9

VERSITA

Central European Journal of **Mathematics**

Existence and uniqueness of solution for a class of nonlinear sequential differential equations of fractional order

Research Article

Małgorzata Klimek[1]*, Marek Błasik[1]†

1 Institute of Mathematics, Częstochowa University of Technology, Dąbrowskiego 73, 42-200 Częstochowa, Poland

Received 28 February 2012; accepted 29 May 2012

Abstract: Two-term semi-linear and two-term nonlinear fractional differential equations (FDEs) with sequential Caputo derivatives are considered. A unique continuous solution is derived using the equivalent norms/metrics method and the Banach theorem on a fixed point. Both, the unique general solution connected to the stationary function of the highest order derivative and the unique particular solution generated by the initial value problem, are explicitly constructed and proven to exist in an arbitrary interval, provided the nonlinear terms fulfil the corresponding Lipschitz condition. The existence–uniqueness results are given for an arbitrary order of the FDE and an arbitrary partition of orders between the components of sequential derivatives.

MSC: 26A33, 34A08, 45Jxx

Keywords: Fractional derivative • Fractional differential equation • Fixed point condition • Equivalent norms/metrics • Initial value problem

1. Introduction

The paper is devoted to studying and solving a class of nonlinear sequential differential equations of an arbitrary order. The results will be presented for a two-term semi-linear fractional differential equation (FDE)

$$D^{\alpha_2}x(t) = D^{\alpha_1}a_1(t)x(t) + \Psi(t, x(t)), \tag{1}$$

* *E-mail: mklimek@im.pcz.pl*
† *E-mail: marek.blasik@gmail.com*

where orders $\alpha_2 > \alpha_1 > 0$ are arbitrary positive real numbers, fractional derivatives $D^{\alpha_2}, D^{\alpha_1}$ are Caputo derivatives in a sequential formulation (type I – Definition 2.4) and the above equation is fulfilled for any $t \in [0, b]$. This problem will be generalised to a two-term nonlinear FDE of the form

$$D^{\alpha_2} x(t) = \Psi\big(t, x(t), D^{\alpha_1} x(t)\big). \tag{2}$$

We shall explicitly construct unique general solutions of equation (1)–(2), formulate an initial value problem and derive the corresponding particular solutions.

Fractional differential equations are an interesting and fast developing area of mathematical investigations, both in the theory and applications. We refer the reader to a summary of fractional differential equations theory in monographs and review papers [1, 6, 8–11, 16–18]. Many solution methods have been transferred here from differential and integral equations theory, including the application of fixed point theorems, integral transforms and operator theory. Here, we consider a class of equations written in terms of sequential Caputo derivatives. Essentially, the two types of composed Caputo derivatives given in Definition 2.4 can be considered. Equations with type II derivatives can be easily converted into a system of one-term FDEs and were studied earlier, compare the monograph by Diethelm [6] and references given therein. In the present paper, we shall solve equations with type I derivatives and show how the partition of order α_2 between the component derivatives influences the transformation into an equivalent fractional integral equation, the subsequent solution and formulation of an initial value problem (IVP). The solution method developed in the paper consists of the construction of new metrics on $C[0, b]$ (equivalent to the standard supremum metric), reformulation of equations (1)–(2) as fixed point conditions for contractive mappings on $C[0, b]$ with a new metric, and finally application of the Banach contraction principle. This approach is very effective, producing a unique solution in an arbitrarily long interval. Earlier results for sequential FDEs of order in $(0, 1)$ can be found in papers [4, 5, 12–15, 19].

The paper is organised as follows. In the next section we quote all necessary definitions from fractional calculus as well as theorems on the properties of fractional derivatives and integrals. Here, we also introduce a class of norms and metrics on the space of continuous functions, equivalent to each other and to the standard supremum norm. We also prove a technical lemma on fractional integration which will be further applied in the derivation of the general solution of considered SFDEs. Section 3 contains main results which include two theorems on the existence and uniqueness of the solution of a two-term SFDE. We shall separately consider the case of semi-linear and nonlinear equations as the transformation into a fixed point condition and formulation of the initial value problem (IVP) are different in both cases. Similarly to differential equations theory, each admissible stationary function of the highest order derivative generates a unique continuous solution of the considered equations (1)–(2). Then, initial value problems are formulated and solved by showing the explicit form of the stationary function connected to the given IVP. The paper closes with a short conclusion section.

2. Preliminaries

In this section, we recall basic definitions and theorems from fractional calculus, which we shall apply to formulate and solve a two-term SFDE. The left-sided Riemann–Liouville integral and Caputo derivative are defined as follows [8, 20].

Definition 2.1.

The left-sided Riemann–Liouville integral of order α, denoted as I^{α}_{0+}, is given by the following formula for $\operatorname{Re}\alpha > 0$:

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u)\, du}{(t-u)^{1-\alpha}},$$

where Γ is the Euler gamma function.

Definition 2.2.

Let $\operatorname{Re}\alpha \in (n-1, n)$. The left-sided Caputo derivative of order α, denoted as ${}^cD^{\alpha}_{0+}$, is given by

$${}^cD^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(u)\, du}{(t-u)^{\alpha-n+1}} = I^{n-\alpha}_{0+} f^{(n)}(t).$$

Property 2.3.

The following differentiation rule is valid, provided $\mathrm{Re}\,\alpha \in (n-1,n)$ and $\mathrm{Re}\,\beta > n-1$ in the case $\mathrm{Re}\,\beta \notin \mathbb{N}_0$:

$$ {}^{c}D^{\alpha}_{0+} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \cdot t^{\beta-\alpha}. $$

Definition 2.4.

Let $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m$. Sequential Caputo derivatives of type I and II are given respectively as follows: $D^{\alpha_1} = {}^{c}D^{\alpha_1}_{0+}$ and, for $j = 2, \ldots, m$,

$$ D^{\alpha_j} = D^{\alpha_{j-1}}\, {}^{c}D^{\alpha_j-\alpha_{j-1}}_{0+}, \tag{I} $$

$$ \mathcal{D}^{\alpha_j} = {}^{c}D^{\alpha_j-\alpha_{j-1}}_{0+}\, \mathcal{D}^{\alpha_{j-1}}. \tag{II} $$

Next, we quote the property describing the composition rules for fractional integrals and derivatives [8, 20].

Property 2.5.

Let $f \in C([0,b],R)$ and $\alpha > 0$. The following equalities hold at any point $t \in [0,b]$:

$$ {}^{c}D^{\alpha}_{0+} I^{\alpha}_{0+} f(t) = f(t), \qquad I^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\beta+\alpha}_{0+} f(t). \tag{3} $$

If additionally $\beta > \alpha$, then we have at any point $t \in [0,b]$,

$$ {}^{c}D^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\beta-\alpha}_{0+} f(t). $$

In the proof of existence and uniqueness results for equations (1)–(2) we shall apply the new metrics introduced below.

Definition 2.6.

The following formulas define a new norm and metric on $C[0,b]$ for $\kappa, \gamma \in \mathbb{R}_+$:

$$ \|g\|_{\gamma,\kappa} = \sup_{t\in[0,b]} \frac{|g(t)|}{E_{\gamma,1}(\kappa t^{\gamma})}, \qquad d_{\gamma,\kappa}(g,h) = \|g-h\|_{\gamma,\kappa}, $$

where $E_{\gamma,1}$ denotes a one-parameter Mittag-Leffler function given in general as the series

$$ E_{\gamma,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\gamma j+1)}. $$

Lemma 2.7.

Let $\gamma, \gamma_j, \kappa, \kappa_j \in \mathbb{R}_+$ for $j = 1,2$.

(i) *Metrics d_{γ_1,κ_1} and d_{γ_2,κ_2} on $C[0,b]$ are equivalent.*

(ii) *Each metric $d_{\gamma,\kappa}$ on $C[0,b]$ is equivalent to metric $d_{0,0}$ generated by the supremum norm.*

Proof. Let us observe that for the norm $\|\cdot\|_{\gamma,\kappa}$, the following inequalities are valid for any function $g \in C[0,b]$:

$$ \frac{\|g\|}{\max\limits_{t\in[0,b]} E_{\gamma,1}(\kappa t^{\gamma})} \le \|g\|_{\gamma,\kappa} = \sup_{t\in[0,b]} \frac{|g(t)|}{E_{\gamma,1}(\kappa t^{\gamma})} \le \sup_{t\in[0,b]} |g(t)| = \|g\|, $$

where we denoted as $\|\cdot\|$ the supremum norm on $C[0,b]$. Thus, norms $\|\cdot\|$ and $\|\cdot\|_{\gamma,\kappa}$ are equivalent and so are metrics $d_{0,0}$ and $d_{\gamma,\kappa}$. The first part of the lemma is a straightforward corollary. □

Remark 2.8.

Each function space $\langle C[0,b], d_{\gamma,\kappa}\rangle$ is a metric complete space.

The constructed equivalent norms and the respective metrics will be the main tools in the proof of some existence and uniqueness results for the general and particular solution of two-term sequential FDEs given in (1)–(2). The method of equivalent norms/metrics was originally introduced by Bielecki in [3] in the theory of differential equations. It allows one to express the corresponding differential equation as the fixed point condition of a mapping on the function space. The equivalent norm and metric are chosen so that the mapping becomes a contraction on a new complete metric space. This approach was extended to fractional differential equations by El-Raheem in [7], where he considered a one-term FDE of order $\alpha \in (0,1)$. Then, Lakshmikantham et al. in [16] applied in the modification/scaling of the norm a Mittag-Leffler function considering the same class of FDEs. Baleanu and Mustafa [2] applied the equivalent norm/metric method in order to derive a global solution for a one-term FDE of an arbitrary fractional order. The Bielecki method can also be extended to FDEs in a sequential version. Some results on multi-term SFDEs with a basic fractional derivative — Riemann–Liouville, Caputo or Hadamard — can be found in [12, 13]. Preliminary results on semi-linear SFDEs of type (1) are enclosed in [4, 14], where the general solution is derived for arbitrary orders of sequential Caputo derivatives but the initial value problem is solved in the case of an equation of order in $(0,1)$.

The proof of existence-uniqueness result for the general solution of equations (1)–(2) will be based on the technical lemma given below.

Lemma 2.9.

The following integration formula is valid for any $\beta, \kappa \in \mathbb{R}_+$:

$$I^{\beta}_{0+} E_{\beta,1}(\kappa t^{\beta}) = \frac{1}{\kappa}\left(E_{\beta,1}(\kappa t^{\beta}) - 1\right). \tag{4}$$

Let $\beta > \alpha > 0$. Then, there exists a constant A (dependent on α and β) such that the following inequality is valid for any value of parameter $\kappa \in \mathbb{R}_+$:

$$\sup_{t\in[0,b]} \frac{I^{\beta}_{0+} E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})}{E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})} \le \frac{b^{\alpha} A}{\kappa}. \tag{5}$$

Proof. As the series defining the Mittag-Leffler function is uniformly convergent in any finite subinterval of $\mathbb{R}$, we can integrate it term by term and obtain formula (4):

$$I^{\beta}_{0+} E_{\beta,1}(\kappa t^{\beta}) = I^{\beta}_{0+} \sum_{l=0}^{\infty} \frac{\kappa^l t^{\beta l}}{\Gamma(l\beta+1)} = \sum_{l=0}^{\infty} \frac{\kappa^l t^{\beta(l+1)}}{\Gamma((l+1)\beta+1)} = I^{\beta}_{0+} \sum_{l'=1}^{\infty} \frac{\kappa^{l'-1} t^{\beta l'}}{\Gamma(l\beta+1)} = I^{\beta}_{0+} \frac{1}{\kappa} \sum_{l'=1}^{\infty} \frac{\kappa^{l'} t^{\beta l'}}{\Gamma(l\beta+1)} = \frac{1}{\kappa}\left(E_{\beta,1}(\kappa t^{\beta}) - 1\right).$$

Now, applying the semigroup property (3) and formula (4), we transform the integral on the left-hand side of (5) and express it as a quotient:

$$\begin{aligned}
\frac{I^{\beta}_{0+} E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})}{E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})} &= \frac{I^{\alpha}_{0+}\left(E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha}) - 1\right)}{\kappa E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})} = \frac{t^{\alpha} E_{\beta-\alpha,\alpha+1}(\kappa t^{\beta-\alpha}) - t^{\alpha}/\Gamma(\alpha+1)}{\kappa E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})} \\
&= \frac{\displaystyle\sum_{l=0}^{\infty} \frac{\kappa^l t^{l(\beta-\alpha)} t^{\alpha}}{\Gamma(l(\beta-\alpha)+\alpha+1)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}}{\displaystyle\kappa \sum_{l=0}^{\infty} \frac{\kappa^l t^{l(\beta-\alpha)}}{\Gamma(l(\beta-\alpha)+1)}} = \frac{\displaystyle t^{\alpha} \sum_{l=1}^{\infty} \frac{\kappa^l t^{l(\beta-\alpha)}}{\Gamma(l(\beta-\alpha)+\alpha+1)}}{\displaystyle\kappa \sum_{l=0}^{\infty} \frac{\kappa^l t^{l(\beta-\alpha)}}{\Gamma(l(\beta-\alpha)+1)}}.
\end{aligned} \tag{6}$$

It is clear that there exists a unique integer number $s \in \mathbb{N}$ such that

$$\frac{\gamma_{\min} - 1}{s} \le \beta - \alpha < \frac{\gamma_{\min} - 1}{s-1}, \tag{7}$$

where $\gamma_{\min}$ denotes the local and global minimum of the Euler gamma function on the positive half-axis. The above estimation leads to the following inequalities for the Euler gamma function valid for $l \geq s$:

$$\Gamma((\beta-\alpha)l+\alpha+1) > \Gamma((\beta-\alpha)l+1).$$

Using the value of parameter s, determined for any two fixed values of α and β in (7), and the above inequalities for the Euler gamma function, we rewrite equality (6) in the form of an inequality for the supremum

$$\sup_{t\in[0,b]} \frac{I^{\beta}_{0+}E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})}{E_{\beta-\alpha,1}(\kappa t^{\beta-\alpha})} \leq \frac{b^{\alpha}}{\kappa}\left(\frac{P_s(\kappa t_{\kappa}^{\beta-\alpha})}{1+\widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha})}+1\right), \tag{8}$$

where we denoted as P_s and $\widehat{P}_s$ the following polynomials:

$$P_s(\kappa t_{\kappa}^{\beta-\alpha}) = \sum_{l=1}^{s-1}\frac{\kappa^l (t_{\kappa})^{l(\beta-\alpha)}}{\Gamma(l(\beta-\alpha)+\alpha+1)}, \qquad \widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha}) = \sum_{l=1}^{s-1}\frac{\kappa^l (t_{\kappa})^{l(\beta-\alpha)}}{\Gamma(l(\beta-\alpha)+1)}.$$

The values of the polynomials above and in formula (8) are calculated at a point t_{κ} at which the rational function $P_s(\kappa t_{\kappa}^{\beta-\alpha})/(1+\widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha}))$ attains its maximum in the finite interval $[0,b]$ (for any given value of parameter $\kappa\in\mathbb{R}_+$). Two cases should be considered. First, let us assume that $\kappa t_{\kappa}^{\beta-\alpha}$ is bounded above by 1:

$$0 < \kappa t_{\kappa}^{\beta-\alpha} \leq 1.$$

Using this assumption, we note that P_s is also bounded,

$$P_s(\kappa t_{\kappa}^{\beta-\alpha}) \leq \sum_{l=1}^{s-1}\frac{1}{\Gamma(l(\beta-\alpha)+1)} = A_s,$$

which yields the inequality

$$\frac{b^{\alpha}}{\kappa}\left(\frac{P_s(\kappa t_{\kappa}^{\beta-\alpha})}{1+\widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha})}+1\right) \leq \frac{b^{\alpha}}{\kappa}\left(P_s(\kappa t_{\kappa}^{\beta-\alpha})+1\right) \leq \frac{b^{\alpha}}{\kappa}(A_s+1). \tag{9}$$

When value $\kappa t_{\kappa}^{\beta-\alpha}$ is bounded below by constant 1, we can estimate the arguments of the polynomials as follows: for $l = 1,2,\dots,s-1$,

$$0 < (\kappa t_{\kappa}^{\beta-\alpha})^{l-s+1} \leq 1,$$

and these inequalities lead to the estimation of rational function $P_s(\kappa t_{\kappa}^{\beta-\alpha})/(1+\widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha}))$,

$$\frac{b^{\alpha}}{\kappa}\left(\frac{P_s(\kappa t_{\kappa}^{\beta-\alpha})}{1+\widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha})}+1\right) \leq \frac{b^{\alpha}}{\kappa}\left(\frac{\sum_{l=1}^{s-1} 1/\Gamma(l(\beta-\alpha)+\alpha+1)}{1/\Gamma((s-1)(\beta-\alpha)+1)}+1\right) = \frac{b^{\alpha}}{\kappa}\left(A_s\Gamma((s-1)(\beta-\alpha)+1)+1\right). \tag{10}$$

Summarising, estimations (9)–(10) lead to a general inequality, valid for all values of arguments $\kappa t_{\kappa}^{\beta-\alpha}$ (thus, also when we consider a sequence $\lim_{m\to\infty}\kappa_m = \infty$)

$$\frac{b^{\alpha}}{\kappa}\left(\frac{P_s(\kappa t_{\kappa}^{\beta-\alpha})}{1+\widehat{P}_s(\kappa t_{\kappa}^{\beta-\alpha})}+1\right) \leq \frac{b^{\alpha}}{\kappa}A,$$

where the constant A is defined as

$$A = \max\left\{A_s+1;\, A_s\Gamma((s-1)(\beta-\alpha)+1)+1\right\}.$$

Let us point out that from the construction of constant A, it follows that it depends solely on the values of orders α,β and parameter s given in (7). This ends the proof of Lemma 2.9. □

Similarly to differential equations theory, where the stationary functions of integer order derivatives are polynomials, we use in FDE theory the stationary functions of fractional derivatives. For sequential Caputo derivatives from (1)–(2), these functions are described in the lemma below. We shall apply them further in the transformation of FDEs into equivalent fractional integral equations.

Lemma 2.10.

(i) *Let* $\alpha_1 \in (n_1-1, n_1)$. *Then the following equivalence is valid in* $C[0,b]$:

$$D^{\alpha_1}\phi_1(t) = 0 \quad \Longleftrightarrow \quad \phi_1(t) = \sum_{j=0}^{n_1-1} c_j t^j.$$

(ii) *Let* $\alpha_1 \in (n_1-1, n_1)$ *and* $\alpha_2 - \alpha_1 \in (n_2-1, n_2)$. *Then the following equivalence is valid in* $C[0,b]$:

$$D^{\alpha_2}\phi_2(t) = 0 \quad \Longleftrightarrow \quad \phi_2(t) = \sum_{j=0}^{n_2-1} c_{j,0} t^j + I_{0+}^{\alpha_2-\alpha_1} \sum_{j=0}^{n_1-1} c_{j,1} t^j.$$

(iii) *Let* $\alpha_j - \alpha_{j-1} \in (n_j-1, n_j)$ *for* $j = 1, \dots, m$. *Then the following equivalence is valid in* $C[0,b]$:

$$D^{\alpha_m}\phi_m(t) = 0 \quad \Longleftrightarrow \quad \phi_m(t) = \sum_{j=0}^{n_m-1} c_{j,0} t^j + \sum_{k=1}^{m-1} I_{0+}^{\alpha_m-\alpha_k} \sum_{j=0}^{n_k-1} c_{j,k} t^j.$$

3. Main results

We shall study and solve the FDEs given in (1) and (2) in the case when the nonlinear terms fulfil the Lipschitz condition. The form of this condition is given in the definition below.

Definition 3.1.

(L1) A function $\Psi\colon [0,b]\times\mathbb{R} \to \mathbb{R}$ fulfils the Lipschitz condition if

$$|\Psi(t,x_1) - \Psi(t,x_2)| \le M(t)\cdot|x_1 - x_2|$$

for any $t \in [0,b]$ and $x_1, x_2 \in \mathbb{R}$.

(L2) A function $\Psi\colon [0,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ fulfils the Lipschitz condition if

$$|\Psi(t,x_1,y_1) - \Psi(t,x_2,y_2)| \le M_1(t)\cdot|x_1 - x_2| + M_2(t)\cdot|y_1 - y_2|$$

for any $t \in [0,b]$ and $(x_j, y_j) \in \mathbb{R}^2$, $j = 1, 2$.

Example 3.2.

The following simple rational and trigonometric functions fulfil (L1) Lipschitz condition:

$$\Psi(t,x) = \frac{b_1(t)}{1+|x|}, \qquad b_1 \in C[0,b], \qquad\qquad \Psi(t,x) = b_1(t)\sin(\beta_1(t)x) + b_2(t)\cos(\beta_2(t)x), \qquad b_j, \beta_j \in C[0,b], \quad j = 1,2,$$

with $M(t) = |b_1(t)|$ in the first and $M(t) = |b_1(t)\beta_1(t)| + |b_2(t)\beta_2(t)|$ in the second case. Similarly it can be shown that functions

$$\Psi(t,x,y) = \frac{b_1(t)}{1+|x|+|y|}, \qquad b_1 \in C[0,b],$$
$$\Psi(t,x,y) = b_1(t)\sin[\beta_1(t)(x+y)] + b_2(t)\cos[\beta_2(t)(x+y)], \qquad b_j, \beta_j \in C[0,b], \quad j = 1,2,$$

obey (L2) Lipschitz condition with $M_1(t) = M_2(t) = |b_1(t)|$ in the first and $M_1(t) = M_2(t) = |b_1(t)\beta_1(t)| + |b_2(t)\beta_2(t)|$ in the second case.

3.1. Two-term semi-linear SFDE

First, we give results on existence and uniqueness of the solution of semi-linear fractional differential equation (1), where the sequential Caputo derivatives of arbitrary order are included. The reason we solve this less general case separately is due to its transformation into a fixed point condition which allows us to consider the general initial value problem. The two theorems enclosed in this section yield a unique general solution generated by the stationary function of the highest order derivative and a unique particular solution fulfilling the respective initial conditions. In the construction of the solution we shall apply the following mapping on $C[0,b]$:

$$T_{\phi_2} g(t) = I_{0+}^{\alpha_2-\alpha_1} a_1(t) g(t) + I_{0+}^{\alpha_2} \Psi(t, g(t)) + \phi_2(t). \tag{11}$$

Theorem 3.3.
Let $\alpha_2 > \alpha_1 > 0$ and $a_1 \in C^{n_1-1}([0,b],\mathbb{R})$, function $\Psi \in C([0,b]\times\mathbb{R},\mathbb{R})$ fulfils Lipschitz condition (L1) given in Definition 3.1 with $M \in C[0,b]$. Each stationary function ϕ_2 of sequential derivative D^{α_2} generates a unique solution of equation (1) continuous in $[0,b]$. This solution is a limit of iterations of mapping T_{ϕ_2} defined in (11),

$$x(t) = \lim_{j\to\infty} (T_{\phi_2})^j \chi(t),$$

where χ is an arbitrary function continuous on $[0,b]$.

Proof. Equation (1) can be rewritten using the composition rules from Property 2.5 in the form of an equation for a stationary function of the highest order derivative,

$$D^{\alpha_2}\left[x(t) - I_{0+}^{\alpha_2-\alpha_1} a_1(t)x(t) - I_{0+}^{\alpha_2}\Psi(t,x(t))\right] = 0$$

which leads to the following fractional integral equation:

$$x(t) - I_{0+}^{\alpha_2-\alpha_1} a_1(t)x(t) - I_{0+}^{\alpha_2}\Psi(t,x(t)) = \phi_2(t) \tag{12}$$

equivalent to FDE (1) on the space of functions continuous on $[0,b]$. Function ϕ_2 in the above equation is an arbitrary stationary function of sequential derivative D^{α_2} described in Lemma 2.10. Each such stationary function generates a mapping on $C[0,b]$ defined as

$$T_{\phi_2} g(t) = I_{0+}^{\alpha_2-\alpha_1} a_1(t) g(t) + I_{0+}^{\alpha_2} \Psi(t, g(t)) + \phi_2(t)$$

for arbitrary $g \in C[0,b]$. Thus, equation (1) can be transformed into a fixed point condition for mapping T_{ϕ_2},

$$x(t) = T_{\phi_2} x(t). \tag{13}$$

Our aim is to prove that this mapping is a contraction on a respective space $\langle C[0,b], d_\kappa\rangle$. In the proof we shall apply metric d_κ generated by the norm given in Definition 2.6 with scaling function $E_{\alpha_2-\alpha_1,1}$,

$$\|g\|_{\alpha_2-\alpha_1,\kappa} = \sup_{t\in[0,b]} \frac{|g(t)|}{E_{\alpha_2-\alpha_1,1}(\kappa t^{\alpha_2-\alpha_1})}, \qquad d_\kappa(g,h) = \|g-h\|_{\alpha_2-\alpha_1,\kappa}.$$

From Lemma 2.7, it follows that $\langle C[0,b], d_\kappa\rangle$ is a complete metric space for each value $\kappa \in \mathbb{R}_+$. Let us estimate the d_κ-distance between the images for an arbitrary pair of functions $g, h \in C[0,b]$:

$$\begin{aligned}
\|T_{\phi_2} g - T_{\phi_2} h\|_{\alpha_2-\alpha_1,\kappa} &= \left\| I_{0+}^{\alpha_2-\alpha_1} a_1(g-h) + I_{0+}^{\alpha_2}(\Psi(t,g(t)) - \Psi(t,h(t)))\right\|_{\alpha_2-\alpha_1,\kappa} \\
&\le \left\| I_{0+}^{\alpha_2-\alpha_1} |a_1(t)|\cdot|g(t)-h(t)| + \overline{M}\cdot I_{0+}^{\alpha_2}|g(t)-h(t)|\right\|_{\alpha_2-\alpha_1,\kappa} \\
&\le \|g-h\|_{\alpha_2-\alpha_1,\kappa}\cdot \sup_{t\in[0,b]} \frac{|a_1|\cdot I_{0+}^{\alpha_2-\alpha_1} E_{\alpha_2-\alpha_1,1}(\kappa t^{\alpha_2-\alpha_1}) + \overline{M}\cdot I_{0+}^{\alpha_2} E_{\alpha_2-\alpha_1,1}(\kappa t^{\alpha_2-\alpha_1})}{E_{\alpha_2-\alpha_1,1}(\kappa t^{\alpha_2-\alpha_1})},
\end{aligned}$$

where we denoted $|a_1| = \sup_{t\in[0,b]} |a_1(t)|$ and $\overline{M} = \sup_{t\in[0,b]} |M(t)|$. Applying the technical Lemma 2.9 introduced in the previous section, we obtain

$$\|T_{\phi_2} g - T_{\phi_2} h\|_{\alpha_2-\alpha_1,\kappa} \le \frac{|a_1| + b^{\alpha_1}\overline{M}\cdot A}{\kappa} \cdot \|g-h\|_{\alpha_2-\alpha_1,\kappa}.$$

Let us observe that κ is a free parameter and the following inequality is valid for an arbitrary pair $g, h \in C[0,b]$:

$$d_\kappa(T_{\phi_2} g, T_{\phi_2} h) \le L_\kappa d_\kappa(g,h), \qquad L_\kappa = \frac{|a_1| + b^{\alpha_1}\overline{M}\cdot A}{\kappa}.$$

Setting the value of parameter κ so that $L_\kappa \in (0,1)$, we conclude that mapping T_{ϕ_2} is a contraction on $\langle C[0,b], d_\kappa\rangle$. Then, from the Banach theorem on a fixed point, it follows that a unique solution $x \in C[0,b]$ exists so that the fixed point condition (13) is fulfilled. This function also solves equations (1) and (12), and is given as the limit of iterations of mapping T_{ϕ_2}. □

The above theorem describes the general solutions of (1). As we note, they are generated by the stationary function of the highest order derivative, similarly as in the theory of classical differential equations. To determine the constants, one can use different sets of initial/boundary conditions. In this paper, we shall discuss the initial value problem (IVP) for equation (1) with derivatives of arbitrary real order. First, we consider the case $n_1 \le n_2$ and set the initial conditions as

$$D^k x(0) = C_k, \qquad k = 0, \dots, n_2 - 1, \tag{14}$$
$$D^j\, {}^cD_{0+}^{\alpha_2-\alpha_1} x(0) = \overline{C}_j, \qquad j = 0, \dots, n_1 - 1. \tag{15}$$

The case $n_1 > n_2$ is more complicated as the corresponding initial conditions are

$$D^k x(0) = C_k, \qquad k = 0, \dots, n_2 - 1, \tag{16}$$
$$D^j\, {}^cD_{0+}^{\alpha_2-\alpha_1} x(0) = \overline{C}_j, \qquad j = 0, \dots, n_2 - 1, \tag{17}$$
$$D^j \left[{}^cD_{0+}^{\alpha_2-\alpha_1} - a_1\right] x(0) = \overline{C}_j, \qquad j = n_2, \dots, n_1 - 1. \tag{18}$$

In the theorem below, we prove that the continuous solution of equation (1) exists and fulfils the respective initial conditions. This solution is unique provided a certain assumption on the mapping generated by the stationary function connected to the above IVP is fulfilled.

Theorem 3.4.

Let the assumptions of Theorem 3.3 be fulfilled. Then, a unique solution of equation (1), fulfilling IVP (14)–(15) or (16)–(18) respectively, exists in $C[0,b]$. This solution is a limit of iterations of mapping T_{ϕ_2}, generated by the following stationary function ϕ_2:

$$\phi_2(t) = \sum_{k=0}^{n_2-1} C_k \frac{t^k}{\Gamma(k+1)} + I_{0+}^{\alpha_2-\alpha_1} \sum_{j=0}^{n_1-1} \left(\overline{C}_j - \sum_{l=0}^{j} \binom{j}{l} a_1^{(j-l)}(0)\, C_l \right) \frac{t^j}{\Gamma(j+1)}, \qquad n_1 \le n_2, \tag{19}$$

$$\phi_2(t) = \sum_{k=0}^{n_2-1} C_k \frac{t^k}{\Gamma(k+1)} + I_{0+}^{\alpha_2-\alpha_1} \left[\sum_{j=0}^{n_2-1} \left(\overline{C}_j - \sum_{l=0}^{j} \binom{j}{l} a_1^{(j-l)}(0)\, C_l \right) \frac{t^j}{\Gamma(j+1)} + \sum_{j=n_2}^{n_1-1} \overline{C}_j \frac{t^j}{\Gamma(j+1)} \right], \qquad n_1 > n_2. \tag{20}$$

Proof. Using Theorem 3.3, we note that equation (1) is solved by the solution of the fractional integral equation

$$x(t) = I_{0+}^{\alpha_2-\alpha_1} a_1(t) x(t) + I_{0+}^{\alpha_2} \Psi(t, x(t)) + \phi_2(t). \tag{21}$$

In the case $n_1 \leq n_2$, we can differentiate the above equation and obtain for $0 \leq j < n_2$

$$D^j x(t) = I_{0+}^{\alpha_2-\alpha_1-j} a_1(t)x(t) + I_{0+}^{\alpha_2-j}\Psi(t,x(t)) + \sum_{k=j}^{n_2-1} c_{k,0}\frac{\Gamma(k+1)t^{k-j}}{\Gamma(k-j+1)} + I_{0+}^{\alpha_2-\alpha_1-j}\sum_{m=0}^{n_1-1} c_{m,1}t^m. \tag{22}$$

Calculating the value of the above at $t=0$, we arrive at the system of equations connecting the initial values and coefficients $c_{j,0}$ of the stationary function for $j = 0,\ldots,n_2-1$,

$$C_j = D^j x(0) = \Gamma(j+1)c_{j,0}. \tag{23}$$

Thanks to the properties of fractional integration, the derivatives calculated in formula (22) are continuous on $[0,b]$. This means that $x \in C^{n_2-1}(0,b)$.

The second part of the calculations begins with the fractional differentiation of both parts of (21),

$${}^cD_{0+}^{\alpha_2-\alpha_1} x(t) = a_1(t)x(t) + I_{0+}^{\alpha_1}\Psi(t,x(t)) + \sum_{m=0}^{n_1-1} c_{m,1}t^m.$$

Next, we differentiate both sides, $j = 0,\ldots,n_1-1$, and obtain

$$D^j\, {}^cD_{0+}^{\alpha_2-\alpha_1} x(t) = D^j a_1(t)x(t) + I_{0+}^{\alpha_1-j}\Psi(t,x(t)) + \sum_{m=j}^{n_1-1} c_{m,1}\frac{\Gamma(m+1)t^{m-j}}{\Gamma(m-j+1)}. \tag{24}$$

Assuming $t=0$, we again derive the system of equations connecting the initial values and coefficients $c_{j,1}$ of the stationary function for $j = 0,\ldots,n_1-1$,

$$\overline{C}_j = D^j\, {}^cD_{0+}^{\alpha_2-\alpha_1} x(0) = \sum_{l=0}^{j}\binom{j}{l} a_1^{(j-l)}(0)C_l + \Gamma(j+1)c_{j,1}. \tag{25}$$

Using relations (23)–(25), we obtain the explicit formula for stationary function (19) generating the solution fulfilling initial conditions (14)–(15), valid in the case $n_1 \leq n_2$.

In the case $n_1 > n_2$, the calculations and the results determining coefficients $c_{j,0}$ for $j = 0,\ldots,n_2-1$ coincide with the ones presented above. Thus, we only need to derive coefficients $c_{j,1}$. Using formula (24), for $j = 0,\ldots,n_2-1$, we again obtain relations (25). Then, we take $j = n_2,\ldots,n_1-1$ and we observe that $D^j x(0)$ is not determined by conditions (16)–(17). Therefore we reformulate (15) as follows:

$$D^j\big[{}^cD_{0+}^{\alpha_2-\alpha_1} - a_1(t)\big]x(t) = I^{\alpha_1-j}\Psi(t,x(t)) + \sum_{m=j}^{n_1-1} c_{m,1}\frac{\Gamma(m+1)t^{m-j}}{\Gamma(m-j+1)},$$

and in consequence, we have at $t=0$ the following formulas determining $c_{j,1}$ for $j = n_2,\ldots,n_1-1$:

$$\overline{C}_j = D^j\big[{}^cD_{0+}^{\alpha_2-\alpha_1} - a_1(t)\big]x(t)\Big|_{t=0} = \Gamma(j+1)c_{j,1}.$$

The above result, together with (23)–(25), fully determines stationary function (20) generating the unique continuous solution in the case $n_1 > n_2$. □

3.2. Nonlinear two-term SFDE

In this section, we consider a nonlinear fractional differential equation of the form (2) with sequential Caputo derivatives of arbitrary order. The transformation of the FDE into a fixed point condition on the space of continuous functions will be here more complex than for equation (1). To this aim, we should introduce the notion of an admissible stationary function, given below.

Definition 3.5.

The stationary function ϕ_2 described in Lemma 2.10 will be called admissible for equation (2) when derivative $D^{\alpha_1}\phi_2$ exists and is continuous on $[0,b]$.

Let us observe that two cases are relevant here. First, from Property 2.3 it follows that when $n_1 < n_2$, then all stationary functions of derivative D^{α_2} are admissible. In the case $n_1 \geq n_2$, we observe that the derivative is

$$D^{\alpha_1}\phi_2(t) = D^{\alpha_1}\sum_{j=0}^{n_1-1} c_{j,1}\, t^{j+\alpha_2-\alpha_1}\frac{\Gamma(j+1)}{\Gamma(j+\alpha_2-\alpha_1+1)}.$$

According to Property 2.3, we should have $c_{j,1}=0$ for $j=0,\dots,n_1-n_2-1$ and the derivative becomes

$$D^{\alpha_1}\phi_2(t) = \sum_{j=n_1-n_2}^{n_1-1} c_{j,1}\, t^{j+\{\alpha_2-\alpha_1\}-\{\alpha_1\}+n_2-n_1}\frac{\Gamma(j+1)}{\Gamma(j+\alpha_2-2\alpha_1+1)},$$

where $\{\beta\}$ denotes the fractional part of real number $\beta\in\mathbb{R}_+$, and we conclude that ϕ_2 is admissible if we assume $\{\alpha_2-\alpha_1\}\geq\{\alpha_1\}$. In the case $\{\alpha_2-\alpha_1\}<\{\alpha_1\}$, we can also study the solutions of equation (2) but we should extend the function space of solutions as they will belong to the space of weighted continuous functions [8]. We leave this problem for a subsequent paper. Assuming that the stationary function ϕ_2 is admissible, we are able to prove the following result on existence and uniqueness of the solution of equation (2) by using in the construction the mapping $\widehat{T}_{\phi_2}$:

$$\widehat{T}_{\phi_2}g(t) = \Psi\big(t,\, I_{0+}^{\alpha_2}g(t)+\phi_2(t),\, I_{0+}^{\alpha_1}g(t)+D^{\alpha_1}\phi_2(t)\big), \tag{26}$$

defined on $C[0,b]$.

Theorem 3.6.

Let $\alpha_2>\alpha_1>0$, and $\{\alpha_2-\alpha_1\}\geq\{\alpha_1\}$ in the case $n_1\geq n_2$. Assume a function $\Psi\in C([0,b]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ fulfils the Lipschitz condition (L2) given in Definition 3.1 with $M_1,M_2\in C[0,b]$ and $\widehat{T}_{\phi_2}$ is defined as in (26). Then, each stationary function ϕ_2 of sequential derivative D^{α_2}, admissible in the sense of Definition 3.5, generates a unique solution of equation (2) continuous on $[0,b]$. This solution is determined by a limit of iterations of mapping $\widehat{T}_{\phi_2}$ in the following way:

$$\begin{aligned} x(t) &= I_{0+}^{\alpha_2}z(t)+\phi_2(t), \\ z(t) &= \lim_{j\to\infty}(\widehat{T}_{\phi_2})^j\chi(t), \end{aligned} \tag{27}$$

where χ is an arbitrary function continuous on $[0,b]$.

Proof. We can rewrite equation (2) in the vector form

$$D^{\alpha_2}x(t) = z(t), \tag{28}$$

$$z(t) = \Psi\big(t, x(t), D^{\alpha_1}x(t)\big). \tag{29}$$

The first of the above equations is, in $C[0,b]$, equivalent to the fractional integral equation

$$x(t) = I_{0+}^{\alpha_2} z(t) + \phi_2(t), \tag{30}$$

where by ϕ_2 we have denoted an arbitrary stationary function of derivative D^{α_2} described in Lemma 2.10 and admissible in the sense of Definition 3.5. Now, we differentiate the obtained integral equation by applying the composition rule from Property 2.5 and obtain

$$D^{\alpha_1} x(t) = I_{0+}^{\alpha_2-\alpha_1} z(t) + D^{\alpha_1}\phi_2(t). \tag{31}$$

We observe that in order to correctly calculate the required derivative, the derivative of stationary function $D^{\alpha_1}\phi_2$ should exist. It does, thanks to the assumption on the admissible stationary function, and it is continuous on $[0,b]$. Thus, the left-hand side of equation (29) also is continuous. For these stationary functions, system (28)–(29) can be formulated as an equivalent system of fractional integral equations

$$\begin{aligned} x(t) &= I_{0+}^{\alpha_2} z(t) + \phi_2(t), \\ z(t) &= \Psi\big(t,\, I_{0+}^{\alpha_2} z(t) + \phi_2(t),\, I_{0+}^{\alpha_2-\alpha_1} z(t) + D^{\alpha_1}\phi_2(t)\big), \qquad (32)\\ \widehat{T}_{\phi_2} g(t) &= \Psi\big(t,\, I_{0+}^{\alpha_2} g(t) + \phi_2(t),\, I_{0+}^{\alpha_2-\alpha_1} g(t) + D^{\alpha_1}\phi_2(t)\big). \qquad (33) \end{aligned}$$

Clearly, to derive the solution x we must solve the nonlinear fractional integral equation (32). Using the mapping (33), we observe that in fact it is a fixed point condition on $C[0,b]$,

$$z(t) = \widehat{T}_{\phi_2} z(t), \qquad t \in [0,b]. \tag{34}$$

In the proof, we shall again apply metric d_κ generated by the norm given in Definition 2.6 with scaling function $E_{\alpha_2-\alpha_1,1}$. Let us estimate the d_κ-distance between images $\widehat{T}_{\phi_2} g$ and $\widehat{T}_{\phi_2} h$ for an arbitrary pair of functions $g, h \in C[0,b]$:

$$\begin{aligned} \big\|\widehat{T}_{\phi_2} g - \widehat{T}_{\phi_2} h\big\|_{\alpha_2-\alpha_1,\kappa} &= \big\|\Psi\big(t,\, I_{0+}^{\alpha_2} g(t) + \phi_2(t),\, I_{0+}^{\alpha_2-\alpha_1} g(t) + D^{\alpha_1}\phi_2(t)\big) \\ &\quad - \Psi\big(t,\, I_{0+}^{\alpha_2} h(t) + \phi_2(t),\, I_{0+}^{\alpha_2-\alpha_1} h(t) + D^{\alpha_1}\phi_2(t)\big)\big\|_{\alpha_2-\alpha_1,\kappa} \\ &\le \big\|\overline{M}_1 I_{0+}^{\alpha_2}|g-h| + \overline{M}_2 I_{0+}^{\alpha_2-\alpha_1}|g-h|\big\|_{\alpha_2-\alpha_1,\kappa} \le \overline{M}_1 \big\|I_{0+}^{\alpha_2}|g-h|\big\|_{\alpha_2-\alpha_1,\kappa} + \overline{M}_2 \big\|I_{0+}^{\alpha_2-\alpha_1}|g-h|\big\|_{\alpha_2-\alpha_1,\kappa} \\ &\le \overline{M}_1 \|g-h\|_{\alpha_2-\alpha_1,\kappa} \cdot \sup_{t\in[0,b]} \frac{I_{0+}^{\alpha_2} E_{\alpha_2-\alpha_1}(\kappa t^{\alpha_2-\alpha_1})}{E_{\alpha_2-\alpha_1}(\kappa t^{\alpha_2-\alpha_1})} + \overline{M}_2 \|g-h\|_{\alpha_2-\alpha_1,\kappa} \cdot \sup_{t\in[0,b]} \frac{I_{0+}^{\alpha_2-\alpha_1} E_{\alpha_2-\alpha_1}(\kappa t^{\alpha_2-\alpha_1})}{E_{\alpha_2-\alpha_1}(\kappa t^{\alpha_2-\alpha_1})} \\ &\le \frac{\overline{M}_1 b^{\alpha_1} A + \overline{M}_2}{\kappa} \cdot \|g-h\|_{\alpha_2-\alpha_1,\kappa}, \end{aligned}$$

where $\overline{M}_i = \sup_{t\in[0,b]} |M_i(t)|$, $i = 1,2$. Concluding, we see that for any $\kappa \in \mathbb{R}_+$, the following inequality is fulfilled for an arbitrary pair $g, h \in C[0,b]$:

$$\big\|\widehat{T}_{\phi_2} g - \widehat{T}_{\phi_2} h\big\|_{\alpha_2-\alpha_1,\kappa} \le \widehat{L}_\kappa \cdot \|g-h\|_{\alpha_2-\alpha_1,\kappa}, \qquad \widehat{L}_\kappa = \frac{\overline{M}_1 b^{\alpha_1} A + \overline{M}_2}{\kappa}.$$

Setting the value of parameter κ so that $\widehat{L}_\kappa \in (0,1)$, we observe that mapping $\widehat{T}_{\phi_2}$ is the contraction on $\langle C[0,b], d_\kappa\rangle$. Then, from the Banach theorem on a fixed point, it follows that the unique solution $z \in C[0,b]$ of the fixed point condition (34) and equation (30) exists and is given as the limit of the iterations of $\widehat{T}_{\phi_2}$. Using (30), we obtain the unique solution of SFDE (2) generated by ϕ_2. □

We observe that the solution described in the above theorem depends on $n_1 + n_2$ constants in the case $n_1 \le n_2$ and on $2n_2$ constants when $n_1 > n_2$. To determine their value, we formulate an initial value problem (IVP) for nonlinear SFDE (2). First, we consider the case $n_1 \le n_2$ and find a unique continuous solution fulfilling the conditions

$$\begin{aligned} D^k x(0) &= C_k, \qquad k = 0,\dots,n_2-1, \\ D^j\, {}^c D_{0+}^{\alpha_2-\alpha_1} x(0) &= \overline{C}_j, \qquad j = 0,\dots,n_1-1. \end{aligned} \tag{35}$$

In the case $n_1 > n_2$, we know that in the explicit formula for admissible stationary function ϕ_2, some of the coefficients must vanish. Thus we formulate the respective IVP as follows:

$$\begin{aligned} D^k x(0) &= C_k, \qquad k = 0, \dots, n_2 - 1, \\ D^j\, {}^c D_{0+}^{\alpha_2-\alpha_1} x(0) &= 0, \qquad j = 0, \dots, n_1 - n_2 - 1, \\ D^j\, {}^c D_{0+}^{\alpha_2-\alpha_1} x(0) &= \overline{C}_j, \qquad j = n_1 - n_2, \dots, n_1 - 1. \end{aligned} \tag{36}$$

In the theorem below, we give a receipt for the unique continuous solution of (2) fulfilling the above initial conditions.

Theorem 3.7.

Let the assumptions of Theorem 3.6 be fulfilled. Then a unique solution of equation (2), obeying IVP (35) or respectively (36), exists in $C[0, b]$. This solution is given as

$$x(t) = I_{0+}^{\alpha_2} z(t) + \phi_2(t),$$

with the corresponding stationary function ϕ_2,

$$\phi_2(t) = \sum_{k=0}^{n_2-1} C_k \frac{t^k}{\Gamma(k+1)} + I_{0+}^{\alpha_2-\alpha_1} \sum_{j=0}^{n_1-1} \overline{C}_j \frac{t^j}{\Gamma(j+1)}, \qquad n_1 \le n_2, \tag{37}$$

$$\phi_2(t) = \sum_{k=0}^{n_2-1} C_k \frac{t^k}{\Gamma(k+1)} + I_{0+}^{\alpha_2-\alpha_1} \sum_{j=n_1-n_2}^{n_1-1} \overline{C}_j \frac{t^j}{\Gamma(j+1)}, \qquad n_1 > n_2, \tag{38}$$

where z is the limit of iterations of mapping $\widehat{T}_{\phi_2}$ given in (26) and (33).

Proof. From Theorem 3.6 and its proof, it follows that the unique solution $x \in C[0, b]$ of equation (2), generated by admissible stationary function ϕ_2, fulfils the fractional integral equation (27). First, we consider the case $n_1 \le n_2$ and differentiate both sides of this equality for $j = 1, \dots, n_2 - 1$:

$$D^j x(t) = I_{0+}^{\alpha_2-j} z(t) + D^j \phi_2(t). \tag{39}$$

Setting $t = 0$ in formula (39), we obtain equations connecting initial conditions C_j and coefficients $c_{j,0}$ of the stationary function,

$$C_j = D^j x(0) = \Gamma(j+1) c_{j,0}. \tag{40}$$

Thanks to properties of fractional integration and the assumption on admissibility of the stationary function, all derivatives calculated in formula (39) are continuous on $[0, b]$. Thus, $x \in C^{n_2-1}(0, b)$.

Similarly as the proof of Theorem 3.6, the second part of calculations begins with the fractional differentiation of both sides of (27),

$${}^c D_{0+}^{\alpha_2-\alpha_1} x(t) = I_{0+}^{\alpha_1} z(t) + \sum_{m=0}^{n_1-1} c_{m,1} t^m.$$

Next, we differentiate, $j = 0, \dots, n_1 - 1$, and obtain the relations

$$D^j\, {}^c D_{0+}^{\alpha_2-\alpha_1} x(t) = I_{0+}^{\alpha_1-j} z(t) + \sum_{m=j}^{n_1-1} c_{m,1} \frac{\Gamma(m+1)\, t^{m-j}}{\Gamma(m-j+1)} \tag{41}$$

which yield the system of equations connecting initial values $\overline{C}_j$ and coefficients $c_{j,1}$,

$$\overline{C}_j = D^j\, {}^c D_{0+}^{\alpha_2-\alpha_1} x(0) = \Gamma(j+1) c_{j,1}. \tag{42}$$

Let us observe that the system of equations (40)–(42) is fully determined — the obtained coefficients are unique, thus the stationary function given in (37) generates a unique continuous solution of (2) described fully in Theorems 3.6 and 3.7.

In the case $n_1 > n_2$, we again start with (39) and obtain the first set of relations (40). Then, we differentiate using the fractional derivative ${}^cD_{0+}^{\alpha_2-\alpha_1}$:

$$ {}^cD_{0+}^{\alpha_2-\alpha_1}x(t) = I_{0+}^{\alpha_1}z(t) + \sum_{m=n_1-n_2}^{n_1-1} c_{m,1}t^m. $$

We observe that only n_2 coefficients $c_{j,1}$ should be determined. We shall obtain the system of equations using the derivatives of integer order for $j = 0, \dots, n_1 - 1$, as in (41), and setting $t = 0$,

$$ \begin{aligned} 0 = \overline{C}_j = D^j\, {}^cD_{0+}^{\alpha_2-\alpha_1}x(0) = 0, \qquad & j = 0, \dots, n_1 - n_2 - 1, \\ \overline{C}_j = D^j\, {}^cD_{0+}^{\alpha_2-\alpha_1}x(0) = \Gamma(j+1)c_{j,1}, \qquad & j = n_1 - n_2, \dots, n_1 - 1. \end{aligned} \tag{43} $$

Formulas (40) and (43) uniquely determine the coefficients of the admissible stationary function (38) which generates the continuous solution of (2) solving IVP (36). □

4. Conclusions

In the paper, we constructed a unique continuous solution of the two-term semi-linear and two-term nonlinear FDE with sequential Caputo derivatives. The considered equations contain derivatives of arbitrary fractional order. Additionally, as follows from the equivalent fractional integral forms of equations (1)–(2), the derived solution belongs to the $C^{n_2-1}(0, b)$ class (remarks in proofs of Theorems 3.4 and 3.7).

We also discussed in detail how the partition of the fractional order between the orders of the component derivatives in (I) influences the formulation of the IVP (14)–(18), (35)–(36), and the form of the particular solution determined by the initial conditions. The introduced method of proof consists of the extension of the equivalent norms/metrics method known from differential equations theory and the subsequent use of the contraction principle. Here, we apply in the scaling norms and construction of a new metric, Mittag–Leffler functions dependent on the lowest order of fractional derivatives and on a free positive parameter. Using the new metric and technical Lemma 2.9, we were able to reformulate the problem as a fixed point condition of a contractive mapping on the complete metric function space. Analysing the presented results, we note that they can be generalised to derive the solutions of multi-term SFDEs and systems of SFDEs, provided the nonlinear terms obey the respective Lipschitz condition. Such a nonlinearity is not the general case but it yields an immediate global continuous solution of the considered equations as it exists in arbitrary long interval $[0, b]$. These problems will be studied in a subsequent paper.

Acknowledgements

This research was in part (M. Błasik) supported under CUT Grant Number BS/MN 1-105-301/11/P.

References

[1] Baleanu D., Diethelm K., Scalas E., Trujillo J.J., Fractional Calculus, Ser. Complex. Nonlinearity Chaos, 3, World Scientific, Singapore, 2012

[2] Băleanu D., Mustafa O.G., On the global existence of solutions to a class of fractional differential equations, Comput. Math. Appl., 2010, 59(5), 1835–1841

[3] Bielecki A., Une remarque sur la méthode de Banach–Cacciopoli–Tikhonov dans la théorie des équations différentielles ordinaires, Bull. Acad. Polon. Sci. Cl. III, 1956, 4, 261–264

[4] Błasik M., Klimek M., On application of contraction principle to solve two-term fractional differential equations, Acta Mechanica et Automatica, 2011, 5(2), 5–10

[5] Deng J., Ma L., Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Appl. Math. Lett., 2010, 23(6), 676–680

[6] Diethelm K., The Analysis of Fractional Differential Equations, Lecture Notes in Math., 2004, Springer, Berlin, 2010

[7] El-Raheem Z.F.A., Modification of the application of a contraction mapping method on a class of fractional differential equation, Appl. Math. Comput., 2003, 137(2-3), 371–374

[8] Kilbas A.A., Srivastawa H.M., Trujillo J.J., Theory and Applications of Fractional Differential Equations, North-Holland Math. Stud., 204, Elsevier, Amsterdam, 2006

[9] Kilbas A.A., Trujillo J.J., Differential equations of fractional order: methods, results and problems. I, Appl. Anal., 2001, 78(1-2), 153–192

[10] Kilbas A.A., Trujillo J.J., Differential equations of fractional order: methods, results and problems. II, Appl. Anal., 2002, 81(2), 435–493

[11] Klimek M., On Solutions of Linear Fractional Differential Equations of a Variational Type, Częstochowa University of Technology, Częstochowa, 2009

[12] Klimek M., On contraction principle applied to nonlinear fractional differential equations with derivatives of order $\alpha \in (0, 1)$, Banach Center Publ., 2011, 95, 325–338

[13] Klimek M., Sequential fractional differential equations with Hadamard derivative, Commun. Nonlinear Sci. Numer. Simul., 2011, 16(12), 4689–4697

[14] Klimek M., Błasik M., Existence–uniqueness result for nonlinear two-term sequential FDE, In: 7^{th} European Nonlinear Dynamics Conference (ENOC 2011), Rome, July 24–29, 2011, available at http://w3.uniroma1.it/dsg/enoc2011/proceedings/pdf/Klimek_Blasik.pdf

[15] Kosmatov N., Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Anal., 2009, 70(7), 2521–2529

[16] Lakshmikantham V., Leela S., Vasundhara Devi J., Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cambridge, 2009

[17] Miller K.S., Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, 1993

[18] Podlubny I., Fractional Differential Equations, Math. Sci. Engrg., 198, Academic Press, San Diego, 1999

[19] ur Rehman M., Khan R.A., Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations, Appl. Math. Lett., 2010, 23(9), 1038–1044

[20] Samko S.G., Kilbas A.A., Marichev O.I., Fractional Integrals and Derivatives, Gordon and Breach, Amsterdam, 1993