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Lorentzian similarity manifolds

Research Article

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Abstract: An $(m+2)$-dimensional *Lorentzian similarity* manifold M is an affine flat manifold locally modeled on $(G,\mathbb{R}^{m+2})$, where $G=\mathbb{R}^{m+2}\rtimes(\mathrm{O}(m+1,1)\times\mathbb{R}^{+})$. M is also a conformally flat Lorentzian manifold because G is isomorphic to the stabilizer of the Lorentzian group $\mathrm{PO}(m+2,2)$ of the Lorentz model $S^{m+1,1}$. We discuss the properties of compact Lorentzian similarity manifolds using developing maps and holonomy representations.

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1. Introduction

Let $\mathrm{A}(m+2)=\mathbb{R}^{m+2}\rtimes\mathrm{GL}(m+2,\mathbb{R})$ be the affine group of the $(m+2)$-dimensional Euclidean space $\mathbb{R}^{m+2}$. An $(m+2)$-manifold M is an *affinely flat* manifold if M is locally modeled on $\mathbb{R}^{m+2}$ with coordinate changes lying in $\mathrm{A}(m+2)$. When $\mathbb{R}^{m+2}$ is endowed with a Lorentz inner product, we obtain *Lorentz similarity geometry*

$$\mathrm{Sim}_L(\mathbb{R}^{m+2})=\mathbb{R}^{m+2}\rtimes(\mathrm{O}(m+1,1)\times\mathbb{R}^{+})$$

as a subgeometry of $\mathrm{A}(m+2)$. If an affinely flat manifold M is locally modeled on $\mathrm{Sim}_L(\mathbb{R}^{m+2})$, then M is said to be a *Lorentzian similarity manifold.* Lorentzian similarity geometry contains *Lorentzian flat geometry* $\big(\mathrm{E}(m+1,1),\mathbb{R}^{m+2}\big)$, where $\mathrm{E}(m+1,1)=\mathbb{R}^{m+2}\rtimes\mathrm{O}(m+1,1)$.

We start with the following result.

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Theorem A.

If M is a compact complete Lorentzian similarity manifold, then M is a Lorentzian flat space form. Furthermore, M is diffeomorphic to an infrasolvmanifold.

The first part of Theorem A has been proved by T. Aristide [1]. Once M is a compact Lorentzian flat space form, it is shown by Y. Carrière and F. Dal'bo [5] that M is diffeomorphic to an infrasolvmanifold. In particular, the Auslander–Milnor conjecture is true for compact complete Lorentzian similarity manifolds, cf. [24]. In this direction, we have obtained a new characterization of compact Lorentzian flat space forms.

Theorem B.

If M is a compact Lorentzian flat space form, then the fundamental group admits a nontrivial translation subgroup.

We prove Theorem B in Section 2. As an application, we study compact Lorentzian flat Seifert manifolds in Section 3; see [10, 23].

Let $\big(\mathrm{PO}(m+2,2), S^{m+1,1}\big)$ be a *conformally flat Lorentzian geometry*. If a point $\hat{\infty} \in S^{m+1,1}$ is defined as the projectivization of a null vector in $\mathbb{R}^{m+4}$, the stabilizer $\mathrm{PO}(m+2,2)_{\hat{\infty}}$ is isomorphic to $\mathrm{Sim}_L(\mathbb{R}^{m+2})$ for which there is a suitable conformal Lorentzian embedding of $\mathbb{R}^{m+2}$ into $S^{m+1,1}-\{\hat{\infty}\}$ that is equivariant with respect to $\mathrm{Sim}_L(\mathbb{R}^{m+2}) = \mathrm{PO}(m+2,2)_{\hat{\infty}}$, cf. [17]. In contrast to *conformally flat Riemannian geometry*, $\mathbb{R}^{m+2}$ is properly contained in the complement $S^{m+1,1}-\{\hat{\infty}\}$, cf. [2]. A Lorentzian similarity geometry $\big(\mathrm{Sim}_L(\mathbb{R}^{m+2}), \mathbb{R}^{m+2}\big)$ is a sort of subgeometry of conformally flat Lorentzian geometry $\big(\mathrm{PO}(m+2,2), S^{m+1,1}\big)$.

For $m = 2n$, there is the natural embedding $\mathrm{U}(n+1,1) \to \mathrm{O}(2n+2,2)$ such that $\big(\mathrm{U}(n+1,1), S^1\times S^{2n+1}\big)$ is a subgeometry of $\big(\mathrm{O}(2n+2,2), S^1\times S^{2n+1}\big)$. Here $S^1\times S^{2n+1}$ is a two-fold covering of $S^{2n+1,1}$. A $(2n+2)$-dimensional manifold M is said to be a *conformally flat Fefferman–Lorentz parabolic* manifold if M is uniformized with respect to $\big(\mathrm{U}(n+1,1), S^1\times S^{2n+1}\big)$, cf. [18]. In Section 5, we consider when the developing map of a compact conformally flat Fefferman–Lorentz parabolic manifold becomes a covering map onto its image; see [15]. Let

$$\mathbb{Z} \to \big(\mathrm{U}(n+1,1)^{\sim}, \widetilde{S}^{2n+1,1}\big) \xrightarrow{(Q,q)} \big(\mathrm{U}(n+1,1), S^1\times S^{2n+1}\big)$$

be the equivariant covering map. In Section 5 we prove

Theorem C.

Let M be a $(2n+2)$-dimensional compact conformally flat Fefferman–Lorentz parabolic manifold and

$$(\rho, \mathrm{dev})\colon\ (\pi_1(M), \widetilde{M}) \to \big(\mathrm{U}(n+1,1)^{\sim}, \mathbb{R}\times S^{2n+1}\big)$$

the developing pair. Suppose that the holonomy image $Q(\rho(\pi_1(M)))$ is discrete in $\mathrm{U}(n+1,1)$. If the developing map $q\circ\mathrm{dev}\colon \widetilde{M} \to S^1\times S^{2n+1}$ is not surjective and such that the complement $\Lambda = S^1\times S^{2n+1} - q\circ\mathrm{dev}(\widetilde{M})$ is S^1-invariant, then $q\circ\mathrm{dev}$ is a covering map onto the image.

For noncompact complete Lorentzian case, i.e., properly discontinuous actions of free groups on complete simply connected Lorentzian flat manifolds, the behavior changes drastically. See [2, 6, 13] for details.

2. Lorentzian similarity manifold

Consider the following exact sequence:

$$1 \to \mathbb{R}^{m+2}\rtimes\mathbb{R}^{+} \to \mathrm{Sim}_L(\mathbb{R}^{m+2}) \xrightarrow{P} \mathrm{O}(m+1,1) \to 1. \tag{1}$$

Lemma 2.1.

Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact complete Lorentzian similarity manifold where $\Gamma \leq \mathrm{Sim}_L(\mathbb{R}^{m+2})$. Suppose that $P(\Gamma)$ is discrete in $\mathrm{O}(m+1,1)$. If $\Delta = (\mathbb{R}^{m+2} \rtimes \mathbb{R}^+) \cap \Gamma$, then $\Delta \leq \mathbb{R}^{m+2}$ which is nontrivial.

Proof. Since $P(\Gamma)$ is discrete, it acts properly discontinuously on the $(m+1)$-dimensional hyperbolic space $\mathbb{H}^{m+1}_{\mathbb{R}} = (\mathrm{O}(m+1)\times \mathrm{O}(1)) \setminus \mathrm{O}(m+1,1)$. The (virtually) cohomological dimension vcd of $P(\Gamma)$ satisfies $\mathrm{vcd}\, P(\Gamma) \leq m+1$. On the other hand, the cohomological dimension $\mathrm{cd}\,\Gamma = m+2$, the intersection Δ of (1) is nontrivial. Let

$$1 \to \mathbb{R}^{m+2} \to \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \xrightarrow{p} \mathbb{R}^+ \to 1$$

be the exact sequence. If $p(\Delta)$ is nontrivial, then we may assume that there exists an element $\gamma = (a, \lambda) \in \Delta$ such that $p(\gamma) = \lambda < 1$. A calculation shows

$$\gamma^n = \left(\frac{1-\lambda^n}{1-\lambda} a, \lambda^n \right), \qquad n \in \mathbb{Z}.$$

The sequence of the orbits $\{\gamma^n \cdot \mathbf{0} : n \in \mathbb{Z}\}$ at the origin $\mathbf{0} \in \mathbb{R}^{m+2}$ converges when $n \to \infty$,

$$\gamma^n \cdot \mathbf{0} = \frac{1-\lambda^n}{1-\lambda} a + \lambda^n \cdot \mathbf{0} = \frac{1-\lambda^n}{1-\lambda} a \ \to \ \frac{1}{1-\lambda} a.$$

As Δ acts properly discontinuously on $\mathbb{R}^{m+2}$, $\{\gamma^n : n = 1, 2, \dots\}$ is a finite set. Since Δ is torsion-free, $\gamma = 1$ which is a contradiction. So $p(\Gamma)$ must be trivial. □

Proposition 2.2.

Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact complete Lorentzian similarity manifold. Then Γ is virtually solvable in $\mathrm{Sim}_L(\mathbb{R}^{m+2})$.

Proof. **(1)** When $P(\Gamma)$ is discrete, we obtain the following exact sequences from (1).

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^{m+2} & \longrightarrow & \mathrm{Sim}_L(\mathbb{R}^{m+2}) & \xrightarrow{L} & \mathrm{O}(m+1,1)\times\mathbb{R}^+ & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \xrightarrow{L} & L(\Gamma) & \longrightarrow & 1
\end{array} \tag{2}$$

If $\Delta \cong \mathbb{Z}^k$, then the span $\mathbb{R}^k$ of Δ in $\mathbb{R}^{m+2}$ is normalized by Γ. Let $\langle\cdot,\cdot\rangle$ be the Lorentz inner product on $\mathbb{R}^{m+2}$. The rest of the argument is similar to that of [12]. In fact, $L(\Gamma)$ of (2) induces a properly discontinuous affine action ρ on $\mathbb{R}^{m+2-k}$ with finite kernel $\mathrm{Ker}\,\rho$:

$$\rho \colon L(\Gamma) \to \mathrm{Aff}(\mathbb{R}^{m+2-k}),$$

cf. Lemma 3.1. If necessary, we can find a torsion-free normal subgroup of finite index in $\rho(L(\Gamma))$ by Selberg's lemma. Passing to a finite index subgroup if necessary, the quotient $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a compact complete affinely flat manifold.

Suppose that $\langle\cdot,\cdot\rangle\restriction_{\mathbb{R}^k}$ is nondegenerate. According to whether $\langle\cdot,\cdot\rangle\restriction_{\mathbb{R}^k}$ is positive definite or indefinite, $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a compact complete Lorentzian similarity manifold or Riemannian similarity manifold respectively.

If $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a Lorentzian similarity manifold, by induction hypothesis, $L(\Gamma)$ is virtually solvable. When $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a Riemannian similarity manifold, i.e., $\rho(L(\Gamma)) \leq \mathrm{Sim}(\mathbb{R}^{m+2-k})$ which is an amenable Lie group, a discrete subgroup $\rho(L(\Gamma))$ is virtually solvable by Tits' theorem (cf. [24]; furthermore, $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a Riemannian flat manifold by Fried's theorem [9]). In each case, Γ is virtually solvable.

If $\langle\cdot,\cdot\rangle\restriction_{\mathbb{R}^k}$ is degenerate, then $\mathbb{R}^k = \mathrm{R}$ consisting of a lightlike vector as a basis. The holonomy group $L(\Gamma)$ leaves invariant R. The subgroup of $\mathrm{O}(m+1,1)\times\mathbb{R}^+$ preserving R is isomorphic to $\mathrm{Sim}^*(\mathbb{R}^m)\times\mathbb{R}^+ = (\mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*))\times\mathbb{R}^+$ which is an amenable Lie group. As $L(\Gamma) \leq \mathrm{Sim}^*(\mathbb{R}^m)\times\mathbb{R}^+$, $L(\Gamma)$ is virtually solvable, and so is Γ.

(2) When $P(\Gamma)$ is indiscrete, it follows from [25, Theorem 8.24] that the identity component of the closure $\overline{P(\Gamma)}^0$ is solvable in $\mathrm{O}(m+1,1)$. It belongs to the maximal amenable subgroup up to conjugate:

$$\overline{P(\Gamma)}^0 \leq \mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*).$$

It is easy to check that the normalizer of $\overline{P(\Gamma)}^0$ is still contained in $\mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*)$ because the normalizer leaves invariant at most two points $\{0,\infty\}$ on the boundary $S^m = \partial\mathbb{H}^{m+1}_{\mathbb{R}}$ for which $\mathrm{O}(m+1,1)_\infty = \mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*)$. Hence $P(\Gamma) \leq \mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*)$. There is an exact sequence induced from (1):

$$1 \to \mathbb{R}^{m+2}\rtimes\mathbb{R}^+ \to P^{-1}(\mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*)) \xrightarrow{P} \mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*) \to 1$$

in which $P^{-1}(\mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*))$ is an amenable Lie subgroup. Hence, Γ is virtually solvable. □

Proposition 2.3.

Let M be a compact complete Lorentzian similarity manifold $\mathbb{R}^{m+2}/\Gamma$. Then M is diffeomorphic to an infrasolvmanifold U/Γ.

Proof. As $\Gamma \leq \mathbb{R}^{m+2}\rtimes(\mathrm{O}(m+1,1)\times\mathbb{R}^+)$ is a virtually solvable group, take the real algebraic hull $A(\Gamma) = U\cdot T$, where U is a unipotent radical and T is a reductive d-subgroup such that T/T^0 is finite. Then each element $r = u\cdot t \in U\cdot T$ acts on U by $\gamma x = utxt^{-1}$, $x\in U$. It follows from the result of [3] that Γ acts properly discontinuously on U so that U/Γ is compact. Furthermore, U/Γ is diffeomorphic to an infrasolvmanifold by [3, Theorem 1.2].

Since U/Γ is compact, we choose a compact subset $D\subset U$ such that $U = \Gamma\cdot D$. As Γ acts properly discontinuously on $\mathbb{R}^{m+2}$ and $U\cdot T \leq \mathbb{R}^{m+2}\rtimes(\mathrm{O}(m+1,1)\times\mathbb{R}^+)$, it is easily checked that U acts properly on $\mathbb{R}^{m+2}$. Since T is reductive, we may assume that $T\cdot\mathbf{0} = \mathbf{0}\in\mathbb{R}^{m+2}$. Define a map

$$\rho\colon U\to\mathbb{R}^{m+2}, \qquad \rho(x) = x\cdot\mathbf{0}.$$

Noting that U acts freely on $\mathbb{R}^{m+2}$, ρ is a simply transitive action. For $\gamma = u\cdot t\in\Gamma$, $\gamma x = utxt^{-1}$ as above. Then $\rho(\gamma x) = utxt^{-1}\cdot\mathbf{0} = utx\cdot\mathbf{0} = \gamma\rho(x)$. So ρ is Γ-equivariant, ρ induces a diffeomorphism on the quotients $U/\Gamma \cong \mathbb{R}^{m+2}/\Gamma$. □

In particular, a compact (and hence complete) Lorentzian flat space form is diffeomorphic to an infrasolvmanifold. Moreover, we have a new characterization on compact complete Lorentzian flat space forms. First, let $\{\ell_1, e_2, \dots, e_{m+1}, \ell_{m+2}\}$ be the basis on $\mathbb{R}^{m+2}$ such that

$$\langle \ell_1,\ell_1\rangle = \langle \ell_{m+2},\ell_{m+2}\rangle = 0, \qquad \langle e_i, e_j\rangle = \delta_{ij}, \qquad \langle \ell_1,\ell_{m+2}\rangle = 1.$$

The subgroup $\mathrm{Sim}(\mathbb{R}^m)$ of $\mathrm{O}(m+1,1)$ with respect to the above basis has the following form:

$$\mathrm{Sim}(\mathbb{R}^m) = \left\{ A = \begin{pmatrix} \lambda & x & -\lambda^{-1}|x|^2/2 \\ \mathbf{0} & B & -\lambda^{-1}B\,{}^t x \\ \mathbf{0} & \mathbf{0} & \lambda^{-1} \end{pmatrix} : \lambda\in\mathbb{R}^+,\, B\in\mathrm{O}(m),\, x\in\mathbb{R}^m \right\}. \tag{3}$$

Here $|x|$ is the orthogonal norm for $x\in\mathbb{R}^m$. See [18] for details.

Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact Lorentzian flat space form. As Γ is a virtually polycyclic group, cf. [12], we assume that Γ is a discrete polycyclic group in $\mathrm{E}(m+1,1) = \mathbb{R}^{m+2}\rtimes\mathrm{O}(m+1,1)$. Let $A(\Gamma) = U\cdot T$ be the real algebraic hull for Γ for which there is the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{R}^{m+2} & \longrightarrow & \mathrm{E}(m+1,1) & \xrightarrow{L} & \mathrm{O}(m+1,1) \\ & & \uparrow & & \\ & & A(\Gamma) & \xrightarrow{L} & L(A(\Gamma)) \\ & & \uparrow & & \\ & & \Gamma & \xrightarrow{L} & L(\Gamma). \end{array} \tag{4}$$

As $A(\Gamma)$ is solvable, it is contained in the maximal amenable group

$$\mathbb{R}^{m+2} \rtimes (\mathbb{R}^m \rtimes (T^k \times \mathbb{R}^+)) \leq \mathbb{R}^{m+2} \rtimes \mathrm{Sim}(\mathbb{R}^m). \tag{5}$$

Here T^k is a k-torus in $\mathrm{O}(m)$. Since $\mathbb{R}^{m+2} \rtimes \mathbb{R}^m$ is a maximal normal unipotent subgroup in the group (5), it follows

$$U \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m \qquad \Longrightarrow \qquad L(U) \leq \mathbb{R}^m. \tag{6}$$

Let $\mathrm{Fitt}(\Gamma)$ denote the Fitting subgroup which is the maximal nilpotent normal subgroup of Γ. Then $\mathrm{Fitt}(\Gamma) = U \cap \Gamma$. See, e.g., [3, 14]. It follows $\mathrm{Fitt}(\Gamma) \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m$. The Fitting hull $\mathrm{F}(\Gamma)$ is the Zariski-closure $\mathrm{A}(\mathrm{Fitt}(\Gamma))$ of $\mathrm{Fitt}(\Gamma)$ in U. Then $\mathrm{Fitt}(\Gamma)$ is a uniform subgroup of $\mathrm{F}(\Gamma)$ such that $V = U/\mathrm{F}(\Gamma)$ is a vector group.

Lemma 2.4.

Suppose that there exists an element $\gamma = (a, A) \in \Gamma$, where the form A in (3) has nontrivial $\lambda \neq 1$. Then at least one of the following holds.

(i) $\mathrm{Fitt}(\Gamma) \cap \mathbb{R}^{m+2}$ *is nontrivial.*

(ii) *There is an element $\gamma_1 \in \mathrm{Fitt}(\Gamma)$ such that*

$$\gamma_1 = \left(\begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & y & -|y|^2/2 \\ \mathbf{0} & I & -{}^t y \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \right) \in \mathbb{R}^{m+2} \rtimes \mathbb{R}^m.$$

Proof. Suppose that the holonomy homomorphism

$$L\colon \mathrm{Fitt}(\Gamma) \to L(\mathrm{Fitt}(\Gamma))\, (\leq \mathbb{R}^m)$$

is isomorphic (if not, then (i) holds). Then $\mathrm{Fitt}(\Gamma)$ is a free abelian group so that the Fitting hull $\mathrm{F}(\Gamma) \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m$ becomes a simply connected abelian Lie subgroup. Note that $\mathrm{F}(\Gamma)$ has a nontrivial summand in $\mathbb{R}^m$ of $\mathbb{R}^{m+2} \rtimes \mathbb{R}^m$. Every 1-parameter subgroup of $\mathrm{F}(\Gamma)$ has the following form in $\mathbb{R}^{m+2} \rtimes \mathbb{R}^m$:

$$\{\varphi_t\}_{t\in\mathbb{R}} = \left(\begin{bmatrix} f_1(t) \\ f_2(t) \\ t \end{bmatrix}, \begin{pmatrix} 1 & g(t) & -|g(t)|^2/2 \\ \mathbf{0} & I & -{}^t g(t) \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \right) \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m. \tag{7}$$

Case 1. If $\dim \mathrm{F}(\Gamma) \geq 2$, then we can choose a 1-parameter subgroup $\{\psi_t\}_{t\in\mathbb{R}}$ such that

$$\psi_t = \left(\begin{bmatrix} q_1(t) \\ q_2(t) \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & p(t) & -|p(t)|^2/2 \\ \mathbf{0} & I & -{}^t p(t) \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \right).$$

Since $\mathrm{F}(\Gamma)$ is abelian, the commutativity $g_s \circ \psi_t = \psi_t \circ g_s$ for any $\{g_s\} \leq \mathrm{F}(\Gamma)$ shows

$$g_s = \left(\begin{bmatrix} h_1(s) \\ h_2(s) \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & r(t) & -|r(t)|^2/2 \\ \mathbf{0} & I & -{}^t r(t) \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \right).$$

Hence

$$\mathrm{F}(\Gamma) \leq \begin{bmatrix} \mathbb{R}^{m+1} \\ 0 \end{bmatrix} \rtimes \mathbb{R}^m.$$

Case 2. Suppose that $\dim \mathrm{F}(\Gamma) = 1$. Then $\mathrm{Fitt}(\Gamma)$ is an infinite cyclic group $\{\gamma_1\}$. Passing to a subgroup of index 2 if necessary, Γ centralizes $\mathrm{Fitt}(\Gamma)$:

$$\gamma\gamma_1\gamma^{-1} = \gamma_1 \qquad \gamma \in \Gamma. \tag{8}$$

Let

$$\gamma_1 = \left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \begin{pmatrix} 1 & y & -|y|^2/2 \\ \mathbf{0} & I & -{}^t y \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \right), \qquad \gamma = \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{pmatrix} \lambda & x & -\lambda^{-1}|x|^2/2 \\ \mathbf{0} & B & -\lambda^{-1}B\,{}^t x \\ \mathbf{0} & \mathbf{0} & \lambda^{-1} \end{pmatrix} \right)$$

in which $\mathbb{R}^+ \ni \lambda \neq 1$ by the hypothesis. Then the equality (8) shows $\lambda^{-1}c_3 = c_3$. Hence $c_3 = 0$ for γ_1. □

Lemma 2.5.

A maximal connected abelian subgroup of $\mathbb{R}^{m+2} \rtimes \mathbb{R}^m$ which has a nontrivial summand in $\mathbb{R}^m$ is isomorphic to $\mathbb{R}^k \times \mathbb{R}^{m-k+1}$, $1 \leq k \leq m$.

Proof. By calculation, a maximal connected abelian subgroup with nontrivial summand in $\mathbb{R}^m$ is as follows:

$$\mathbb{R}^k \times \mathbb{R}^{m-k+1} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & \mathbf{0} & y_{k+1} \ \dots \ y_{m+1} & -|y|^2/2 \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ & & & y_{k+1} \\ \mathbf{0} & \mathbf{0} & I & \vdots \\ & & & y_{m+1} \\ 0 & 0 & 0 \ \dots \ 0 & 1 \end{pmatrix} \right\}.$$

□

Theorem 2.6.

The fundamental group Γ of a compact Lorentzian flat space form $\mathbb{R}^{m+2}/\Gamma$ admits a nontrivial translation subgroup.

Proof. Let $\gamma = (a, A) \in \Gamma$ be such that $A \in \mathrm{Sim}(\mathbb{R}^m)$. Take $\gamma_1 \in \mathrm{Fitt}(\Gamma)$ so that $\gamma_1 = (c, C) \in \mathbb{R}^{m+2} \rtimes \mathbb{R}^m$. As

$$\gamma\gamma_1\gamma^{-1} = \left(a + Ac - ACA^{-1}a,\ ACA^{-1}\right),$$

a calculation shows

$$\gamma^\ell\gamma_1\gamma^{-\ell} = \left(\left(I - A^\ell C A^{-\ell}\right) \sum_{i=0}^{\ell-1} A^i a + A^\ell c,\ A^\ell C A^{-\ell} \right). \tag{9}$$

We put

$$P(\ell) = \left(I - A^\ell C A^{-\ell}\right) \sum_{i=0}^{\ell-1} A^i a. \tag{10}$$

Case I. Suppose that A of (3) satisfies $\lambda \neq 1$ (say $\lambda < 1$). Conjugating γ by a translation, we can assume for $\gamma = (a, A)$ that $a = {}^t[a_1\, a_2\, 0]$. As we described, $A = (x, \lambda B) \in \mathrm{Sim}(\mathbb{R}^m)$, cf. (3), and $C = (y, I) \in \mathbb{R}^m$, a calculation shows that

$$A^\ell C A^{-\ell} = (\lambda^\ell B^\ell y, I) = \begin{pmatrix} 1 & \lambda^\ell B^\ell y & z \\ \mathbf{0} & I & -{}^t(\lambda^\ell B^\ell y) \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}, \tag{11}$$

where $z = -\lambda^{2\ell}|y|^2/2$. It is easy to see that if $\ell \to \infty$,

$$A^\ell C A^{-\ell} \to I. \tag{12}$$

Similarly for $A = (x, \lambda B) \in \mathrm{Sim}(\mathbb{R}^m)$,

$$A^\ell = \big((I - (\lambda B)^\ell)(I - \lambda B)^{-1} x,\, \lambda^\ell B^\ell\big) = \begin{pmatrix} \lambda^\ell & w & u \\ \mathbf{0} & B^\ell & -\lambda^{-\ell} B^{\ell\,t} w \\ \mathbf{0} & \mathbf{0} & \lambda^{-\ell} \end{pmatrix}, \tag{13}$$

where

$$w = (I - (\lambda B)^\ell)(I - \lambda B)^{-1} x, \qquad u = -\frac{\lambda^{-\ell}|w|^2}{2}.$$

Furthermore, a calculation shows

$$\begin{gathered} b_1 = (1-\lambda^\ell)(1-\lambda)^{-1} a_1 + \big((\ell-1)I - (I - (\lambda B)^\ell (I - \lambda B)^{-1})\big)(I - \lambda B)^{-1} x \cdot a_2, \\ \sum_{i=0}^{\ell-1} A^i a = {}^t[b_1\, b_2\, 0], \qquad b_2 = \sum_{i=0}^{\ell-1} B^i a_2. \end{gathered} \tag{14}$$

In our case, $B \in T^k \leq \mathrm{O}(m)$ for some $k \geq 0$, we may put

$$B = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & S_k \end{pmatrix}, \qquad S_k = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & \\ \sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_k & -\sin\theta_k \\ & & & \sin\theta_k & \cos\theta_k \end{pmatrix}.$$

Noting that $x \cdot y = \langle x, y \rangle$ is $\mathrm{O}(m)$-invariant, substitute (11), (14) into (10):

$$P(\ell) = \begin{bmatrix} -\lambda^\ell B^\ell y \cdot b_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda^\ell \langle (I + B + \cdots + B^\ell) y,\, a_2 \rangle \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda^\ell \left\langle \begin{pmatrix} (\ell+1) I_k & \mathbf{0} \\ \mathbf{0} & (I - S_k^{\ell+1})(I - S_k)^{-1} \end{pmatrix} y,\, a_2 \right\rangle \\ 0 \\ 0 \end{bmatrix}.$$

As $\lambda < 1$ can be sufficiently small (if necessary), it follows

$$\lambda^\ell(\ell+1) \to 0, \qquad \ell \to \infty.$$

Similarly $S_k^{\ell+1} \to I$. Hence if $\ell \to \infty$, we have

$$P(\ell) \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

We choose an element $\gamma_1 = (c, C) \in \mathrm{Fitt}(\Gamma)$ from Lemma 2.4 such that $c = {}^t[c_1\, c_2\, 0]$. By (13), when $\ell \to \infty$, we have

$$A^\ell c = \begin{bmatrix} \lambda^\ell c_1 + w \cdot c_2 \\ B^\ell c_2 \\ 0 \end{bmatrix} \to \begin{bmatrix} (I - \lambda B)^{-1} x \cdot c_2 \\ c_2 \\ 0 \end{bmatrix}. \tag{16}$$

Using (12), (15) and (16),

$$\gamma^\ell \gamma_1 \gamma^{-\ell} = \big(P(\ell) + A^\ell c,\, A^\ell C A^{-\ell}\big) \to \left(\begin{bmatrix} (I - \lambda B)^{-1} x \cdot c_2 \\ c_2 \\ 0 \end{bmatrix}, I \right) = \gamma_2.$$

Since $\mathrm{Fitt}(\Gamma)$ is closed (discrete) and normal in Γ, the limit γ_2 exists in $\mathrm{Fitt}(\Gamma)$. If $c_2 \neq 0$, then γ_2 is a nontrivial translation in $\mathbb{R}^{m+2}$. (Otherwise, $\gamma_2 = 1$. By discreteness of $\mathrm{Fitt}(\Gamma)$, $\gamma^\ell \gamma_1 \gamma^{-\ell} = 1$ for sufficiently large ℓ, or $\gamma_1 = 1$ which is impossible.) This proves **Case I**.

Case II. Suppose that the $\mathbb{R}^+$-summand λ is trivial for every element of Γ. Then it follows

$$\Gamma \le U \rtimes T \le \mathbb{R}^{m+2} \rtimes (\mathbb{R}^m \rtimes T^k).$$

In particular, we have $T \le T^k$ so that U/Γ is an infranilmanifold by Proposition 2.2. By the Auslander–Bieberbach theorem, Γ has a finite index maximal normal nilpotent subgroup Γ_0. By maximality, $\Gamma_0 = \mathrm{Fitt}(\Gamma)$. Note that $L(\mathrm{Fitt}(\Gamma))$ is abelian because $L(\mathrm{Fitt}(\Gamma)) \le L(U) \le \mathbb{R}^m$ by (6).

Suppose that $L\colon \mathrm{Fitt}(\Gamma) \to L(\mathrm{Fitt}(\Gamma))$ is isomorphic. (If not, $\mathbb{R}^{m+2} \cap \Gamma$ is nontrivial.) Then $\mathrm{Fitt}(\Gamma)$ is a free abelian subgroup of finite index in Γ. In particular, $\mathbb{R}^{m+2}/\mathrm{Fitt}(\Gamma)$ is a compact manifold with $\mathrm{Rank}\,\mathrm{Fitt}(\Gamma) = m + 2$. On the other hand, the Fitting hull $\mathrm{F}(\Gamma)\,(= U)$ becomes a unipotent abelian Lie subgroup in $\mathbb{R}^{m+2} \rtimes \mathbb{R}^m$. By Lemma 2.5, $\mathrm{F}(\Gamma) \le \mathbb{R}^k \times \mathbb{R}^{m-k+1}$. This implies $\mathrm{Rank}\,\mathrm{Fitt}(\Gamma) \le m + 1$ which is a contradiction. Therefore Γ admits a translation subgroup. □

T. Aristide has shown the following in [1]. The proof here is much the same as that of [1] except for the final part. We retain the notations of Theorem 2.6.

Theorem 2.7.

Every compact complete Lorentzian similarity manifold is a Lorentzian flat space form.

Proof. Let $\Gamma \le \mathrm{Sim}_L(\mathbb{R}^{m+2})$ be the fundamental group of a compact complete Lorentzian similarity manifold. Suppose that there is an element $\gamma = (a, \mu A) \in \Gamma$ such that $\mu \neq 1$ and

$$A = \begin{pmatrix} \lambda & x & -\lambda^{-1}|x|^2/2 \\ \mathbf{0} & B & -\lambda^{-1} B\,{}^t x \\ \mathbf{0} & \mathbf{0} & \lambda^{-1} \end{pmatrix} \in \mathrm{Sim}(\mathbb{R}^m)$$

from (3). As Γ acts freely on $\mathbb{R}^{m+2}$, we can assume $\mu = \lambda^{-1}$. Since

$$\begin{pmatrix} \lambda^{-1} B & -\lambda^{-2} B\,{}^t x \\ \mathbf{0} & \lambda^{-2} \end{pmatrix}$$

has no eigenvalue 1, conjugating by a translation we may assume that

$$\gamma = \left(\begin{bmatrix} a_1 \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & \lambda^{-1} x & -\lambda^{-2}|x|^2/2 \\ \mathbf{0} & \lambda^{-1} B & -\lambda^{-2} B\,{}^t x \\ \mathbf{0} & \mathbf{0} & \lambda^{-2} \end{pmatrix} \right). \tag{17}$$

Let $\gamma_1 = (c, C) \in \mathrm{Fitt}(\Gamma) \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m$ be an element such that

$$C = (y, I) = \begin{pmatrix} 1 & y & -|y|^2/2 \\ \mathbf{0} & I & -{}^t y \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}.$$

Similarly as in (9), it follows

$$\gamma^\ell \gamma_1 \gamma^{-\ell} = \left(\left(I - D^\ell C D^{-\ell}\right) \sum_{i=0}^{\ell-1} D^i d + D^\ell c,\ D^\ell C D^{-\ell} \right), \tag{18}$$

where

$$I - D^\ell C D^{-\ell} = \begin{pmatrix} 0 & -\lambda^\ell B^\ell y & \lambda^{2\ell}|y|^2/2 \\ \mathbf{0} & \mathbf{0} & {}^t(\lambda^\ell B^\ell y) \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix} \tag{19}$$

and

$$D^\ell = \lambda^{-\ell}\left((I - (\lambda B)^\ell)(I - \lambda B)^{-1} x,\ \lambda^\ell B^\ell\right) = \begin{pmatrix} 1 & (\lambda^{-\ell} I - B^\ell)(I - \lambda B)^{-1} x & u \\ \mathbf{0} & \lambda^{-\ell} B^\ell & w \\ \mathbf{0} & \mathbf{0} & \lambda^{-\ell} \end{pmatrix}, \tag{20}$$

where

$$2u = -\lambda^{-\ell}\left|(\lambda^{-\ell} I - B^\ell)(I - \lambda B)^{-1} x\right|^2, \qquad w = -\lambda^{-\ell} B^\ell \cdot {}^t\left((\lambda^{-\ell} I - B^\ell)(I - \lambda B)^{-1} x\right).$$

Since $d = {}^t[a_1\, 0]$ from (17), $\sum_{i=0}^{\ell-1} D^i d = {}^t[\ell a_1\, 0\, 0]$. In particular, we obtain $(I - D^\ell C D^{-\ell}) \sum_{i=0}^{\ell-1} D^{\mathrm{id}} = 0$. It follows from (18) that $\gamma^\ell \gamma_1 \gamma^{-\ell} = (D^\ell c, D^\ell C D^{-\ell})$.

We suppose $\mu = \lambda^{-1} < 1$ as usual. Put $c = {}^t[c_1\, c_2\, c_3]$. We evaluate at the origin $\mathbf{0} \in \mathbb{R}^{m+2}$:

$$\gamma^\ell \gamma_1 \gamma^{-\ell} \cdot \mathbf{0} = D^\ell c \to \begin{bmatrix} c_1 - (I - \lambda B)^{-1} x \cdot c_2 \\ 0 \\ 0 \end{bmatrix}, \qquad \ell \to \infty.$$

As Γ acts properly discontinuously and freely, there exists $k \in \mathbb{Z}$ such that $\gamma^k \gamma_1 \gamma^{-k} = \gamma_1$. Since

$$\gamma_1 = (c, C) = \left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \begin{pmatrix} 1 & y & -|y|^2/2 \\ \mathbf{0} & I & -{}^t y \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \right) \in \mathbb{R}^{m+2} \rtimes \mathbb{R}^m,$$

it follows $D^k C D^{-k} = C$, which shows $\lambda^{2k}|y| = |y|$ from (19). As $\lambda \neq 1$, $y = 0$. This implies $\gamma_1 = (c, I) \in \mathbb{R}^{m+2}$. Thus $\gamma^k \gamma_1 \gamma^{-k} = (D^k c, I)$ and so $D^k c = c$. Noting $|B^\ell c_2| = |c_2|$, it is easy to see that $c_3 = c_2 = 0$ which shows

$$\gamma_1 = \left(\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}, I \right).$$

Thus by (20),

$$\gamma \gamma_1 \gamma^{-1} = (Dc, I) = \left(\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}, I \right) = \gamma_1.$$

As long as $\lambda \neq 1$, this is true for any element γ_1 of $\mathrm{Fitt}(\Gamma)$, i.e., γ commutes with every element of $\mathrm{Fitt}(\Gamma)$. Consider the nilpotent subgroup $\Gamma_0 = \{\gamma, \mathrm{Fitt}(\Gamma)\}$ of Γ. Since $L(\Gamma) \leq \mathbb{R}^m \rtimes (T^k \times \mathbb{R}^+)$, it follows $[L(\Gamma), L(\Gamma)] \leq \mathbb{R}^m$, i.e., $[\Gamma, \Gamma] \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m$ which is a nilpotent normal subgroup. By maximality, $[\Gamma, \Gamma] \leq \mathrm{Fitt}(\Gamma)$. Moreover, this implies that Γ_0 is a nilpotent normal subgroup containing $\mathrm{Fitt}(\Gamma)$. Hence $\Gamma_0 = \mathrm{Fitt}(\Gamma)$ and so $\gamma \in \mathrm{Fitt}(\Gamma) \leq \mathbb{R}^{m+2} \rtimes \mathbb{R}^m$. This contradicts the hypothesis that $\mu = \lambda^{-1} \neq 1$. Therefore, every element of Γ has *trivial summand* in $\mathbb{R}^+$, i.e., $\Gamma \leq \mathrm{E}(m+1, 1)$. □

Remark 2.8.
There is a compact incomplete Lorentzian similarity $(m+2)$-manifold whose fundamental group is isomorphic to $\Gamma \times \mathbb{Z}$, where Γ is a torsion-free discrete cocompact isometry subgroup of the hyperboloid $\mathbb{H}^{m+1}_{\mathbb{R}}$. This is easily obtained by taking the interior of the cone in $\mathbb{R}^{m+2}$ which is identified with the product $\mathbb{H}^{m+1}_{\mathbb{R}} \times \mathbb{R}^+$ on which the holonomy group $\mathrm{O}(m+1,1) \times \mathbb{R}^+$ acts transitively. In particular, the virtual solvability of $\pi_1(M)$ does not follow from compactness for a Lorentzian similarity manifold M, cf. [4, 24].

3. Lorentzian flat Seifert manifolds

Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact complete Lorentzian similarity manifold. It follows from Proposition 2.6 that $\Gamma \cap \mathbb{R}^{m+2}$ is nontrivial, say $\mathbb{Z}^k$. Then Γ normalizes its span $\mathbb{R}^k$ of $\mathbb{Z}^k$ in $\mathbb{R}^{m+2}$. As $\mathbb{R}^k$ acts properly on $\mathbb{R}^{m+2}$ as translations, we have an equivariant principal bundle

$$(\mathbb{Z}^k, \mathbb{R}^k) \to (\Gamma, \mathbb{R}^{m+2}) \xrightarrow{\nu} (Q, \mathbb{R}^\ell),$$

where $\ell = m+2-k$ and $Q = \Gamma/\mathbb{Z}^k$. In this case each element γ of Γ has the form

$$\gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right), \tag{21}$$

where

$$\nu(\gamma) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \qquad A \in \mathrm{GL}(k,\mathbb{Z}), \qquad B \in \mathrm{GL}(\ell,\mathbb{R}).$$

If we put

$$\rho(\nu(\gamma)) = (b, B) \in \mathrm{A}(\ell), \tag{22}$$

then it is easy to see that $\rho\colon Q \to \mathrm{A}(\ell)$ is a well-defined homomorphism. The quotient group Q acts on $\mathbb{R}^\ell$ through ρ:

$$\alpha \cdot w = \rho(\nu(\gamma))w, \qquad \nu(\gamma) = \alpha \in Q, \quad w \in \mathbb{R}^\ell.$$

Recall the following lemma [12]:

Lemma 3.1.
The group $\rho(Q)$ is a properly discontinuous affine action on $\mathbb{R}^\ell$ such that

- *$\operatorname{Ker}\rho$ is a finite subgroup,*
- *$\mathbb{R}^\ell/\rho(Q)$ is a compact affine orbifold.*

Proof. We show that Q acts properly discontinuously. Consider the pushout

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \Gamma & \xrightarrow{\nu} & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^k & \longrightarrow & \mathbb{R}\cdot\Gamma & \xrightarrow{\nu} & Q & \longrightarrow & 1. \end{array}$$

As both $\mathbb{R}^k$ and Γ act freely and properly on $\mathbb{R}^{m+2}$ with $\mathbb{R}^k/\mathbb{Z}^k = T^k$, it follows that $\mathbb{R}^k \cdot \Gamma$ acts properly on $\mathbb{R}^{m+2}$. Since $\mathbb{R}^k \to \mathbb{R}^{m+2} \xrightarrow{\nu} \mathbb{R}^\ell$ is a principal bundle, choose a *continuous* section $s\colon \mathbb{R}^\ell \to \mathbb{R}^{m+2}$ of ν. Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be a sequence of Q such that

$$\alpha_i \cdot w_i \to z, \qquad w_i \to w, \qquad i \to \infty.$$

Choose a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ from Γ such that $\nu(\gamma_i) = \alpha_i$. As

$$\nu(\gamma_i s(w_i)) = \alpha_i \cdot w_i = \nu(s(\alpha_i w_i)),$$

there is a sequence $\{t_i\}_{i\in\mathbb{N}} \le \mathbb{R}^k$ such that

$$t_i\gamma_i s(w_i) = s(\alpha_i w_i), \qquad s(\alpha_i \cdot w_i) \to s(z), \qquad s(w_i) \to s(w).$$

Since $\mathbb{R}^k \cdot \Gamma$ acts properly on $\mathbb{R}^{m+2}$, there is an element $g \in \mathbb{R}^k \cdot \Gamma$ such that $t_i\gamma_i \to g$ and so $\alpha_i = \nu(t_i\gamma_i) \to \nu(g) \in \Gamma$. Thus Q acts properly discontinuously on $\mathbb{R}^\ell$.

We check that $\operatorname{Ker}\rho$ is finite. Let $1 \to \mathbb{Z}^k \to \Gamma_1 \to \operatorname{Ker}\rho \to 1$ be the induced extension by the inclusion $\operatorname{Ker}\rho \le Q$. Then Γ_1 acts invariantly in the inverse image $\mathbb{R}^k = \nu^{-1}(\mathrm{pt})$. As Γ acts freely and properly, the quotient $\mathbb{R}^k/\Gamma_1$ is a closed submanifold in M. Since $\mathbb{R}^k/\mathbb{Z}^k = T^k$ covers $\mathbb{R}^k/\Gamma_1$, $\operatorname{Ker}\rho$ is finite. □

By the definition [22], we obtain

Proposition 3.2.

$T^k \to M \to \mathbb{R}^\ell/\rho(Q)$ is an injective Seifert fiber space with typical fiber a torus T^k and exceptional fiber a Euclidean space form T^k/F.

In [10] Fried has found all simply transitive Lie group actions on 4-dimensional Lorentzian flat space $\mathbb{R}^4$ and applied them to classify 4-dimensional compact (complete) Lorentzian flat manifolds M up to a finite cover. As a consequence, each such M is finitely covered by a solvmanifold.

We take a different approach to determine 4-dimensional compact complete Lorentzian flat manifolds M from the existence of *causal actions*.

Definition 3.3.

A circle S^1 (respectively $\mathbb{R}$) is a *causal action on M* if the vector field induced by S^1 (respectively $\mathbb{R}$) is a timelike, spacelike or lightlike vector field on M; cf. [2, 16].

We have the following result which occurs in dimension 4 but not in general.

Proposition 3.4.

The fundamental group Γ of a compact complete Lorentzian flat manifold M has a finite index subgroup which contains a central translation subgroup. In particular, some finite cover of M admits a causal circle action.

Proof. Let $\mathbb{Z}^k = \Gamma \cap \mathbb{R}^4$ which is a nontrivial translation subgroup by Theorem 2.6. If $k = 1$, then $\mathbb{Z}$ is central in a subgroup of finite index in Γ.

Case 1. Suppose that $\mathbb{Z}^2 = \Gamma \cap \mathbb{R}^4$ (which is maximal). Let

$$G = \mathbb{R}^4 \rtimes \left(\mathbb{R}^2 \rtimes (\mathrm{SO}(2) \times \mathbb{R}^+)\right)$$

be the maximal connected solvable Lie subgroup of $\mathrm{E}(3,1)$. (See part **(2)** of the proof of Proposition 2.2.) Then Γ lies in the following exact sequences up to finite index:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & \mathrm{E}(3,1) & \xrightarrow{L} & \mathrm{O}(3,1) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \xrightarrow{L} & L(\Gamma) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mu_P} & & \downarrow{\scriptstyle \mu_P} & & \downarrow{\scriptstyle \mu_P} & & \\
1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & G & \xrightarrow{L} & \mathbb{R}^2 \rtimes (\mathrm{SO}(2)\times\mathbb{R}^+) & \longrightarrow & 1
\end{array} \tag{23}$$

Here μ_P is the conjugate homomorphism by some matrix $P \in \mathrm{GL}(4,\mathbb{R})$. For $\gamma \in \Gamma$, we write

$$\gamma = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right) \qquad \text{so that} \qquad L(\gamma) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

The conjugation homomorphism $\phi\colon L(\Gamma) \to \operatorname{Aut}\mathbb{Z}^2$ is given by

$$\phi(L(\gamma)) = A \in \mathrm{GL}(2,\mathbb{Z}).$$

As $L(\Gamma)$ is a free abelian group of rank 2, $\phi(L(\Gamma))$ belongs to A or N up to conjugacy, where $\mathrm{SL}(2,\mathbb{R}) = \mathrm{KAN}$. Since $\mathrm{GL}(2,\mathbb{Z})$ is discrete, $\phi(L(\Gamma))$ is isomorphic to $\mathbb{Z}$, and so $\operatorname{Ker}\phi = \mathbb{Z}$. Choose a generator γ_0 from $\operatorname{Ker}\phi$ and $\gamma \in \Gamma$ for which $\phi(L(\gamma))$ generates $\phi(L(\Gamma))$. Note that γ_0, γ and $\mathbb{Z}^2$ generate Γ.

Recall the homomorphism $\rho\colon L(\Gamma) \to \mathrm{A}(2)$ from (22) defined by $\rho(L(\gamma)) = (a_2, B)$. Since $\rho(L(\Gamma))$ is a properly discontinuous action of $\mathrm{A}(2)$ with compact quotient, the holonomy group of $\rho(L(\Gamma))$ is a *unipotent subgroup* of $\mathrm{GL}(2,\mathbb{R})$. In particular, each B has two eigenvalues 1 and so $L(\gamma)$ has at least two eigenvalues 1. From (23), $\mu_P(L(\Gamma)) \leq \mathbb{R}^2 \rtimes (\mathrm{SO}(2)\times\mathbb{R}^+)$ and

$$\mu_P(L(\gamma)) = PL(\gamma)P^{-1} = \begin{pmatrix} \lambda^{-1} & x & -\lambda|x|^2/2 \\ 0 & T & -\lambda T\,{}^t x \\ 0 & 0 & \lambda \end{pmatrix}, \tag{24}$$

where $T \in \mathrm{SO}(2)$. Since $L(\Gamma)$ is a free abelian group of rank 2, it follows either $\mu_P(L(\Gamma)) \leq \mathbb{R}^2$ or $\mu_P(L(\Gamma)) \leq \mathrm{SO}(2)\times\mathbb{R}^+$. If $\mu_P(L(\Gamma)) \leq \mathrm{SO}(2)\times\mathbb{R}^+$, applying $\gamma_0 \in \operatorname{Ker}\phi$,

$$PL(\gamma_0)P^{-1} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{25}$$

As $\phi(L(\gamma_0)) = A = I$ in this case, $L(\gamma_0)$ has all eigenvalues 1. (25) shows $\lambda = 1$, $T = I$. Hence $PL(\gamma_0)P^{-1} = I$ or $L(\gamma_0) = I$. So $\gamma_0 \in \Gamma \cap \mathbb{R}^4$ which contradicts a maximality of the translation subgroup $\mathbb{Z}^2$. It then follows $\mu_P(L(\Gamma)) \leq \mathbb{R}^2$. In this case

$$PL(\gamma)P^{-1} = \begin{pmatrix} 1 & x & -|x|^2/2 \\ 0 & I & -{}^t x \\ 0 & 0 & 1 \end{pmatrix}.$$

Then A of (21) has two eigenvalues 1 so $[\gamma, \mathbb{Z}^2] = (A-I)\mathbb{Z}^2$ has rank less than 2. Hence there is an element $m \in \mathbb{Z}^2$ such that $[\gamma, m] = 1$. As $\phi(\gamma_0) = 1$, $\gamma_0 m \gamma_0^{-1} = m$. Hence m is a central element of $\Gamma \cap \mathbb{R}^4$.

Case 2. Suppose that $\mathbb{Z}^3 = \Gamma \cap \mathbb{R}^4$. There is an induced affine action $\rho\colon L(\Gamma) \to \mathrm{A}(1)$ in this case so that $\rho(L(\Gamma))$ consists of a translation group up to finite index. As above we obtain

$$\gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}\right),$$

where $A \in \mathrm{GL}(3,\mathbb{Z})$. Since $L(\gamma)$ has the eigenvalue 1, in view of (24), it follows either $T = I$ or $\lambda = 1$. If $T = I$, A has at least one eigenvalue 1. As $\Gamma = \mathbb{Z}^3 \rtimes \mathbb{Z}$, it follows $\operatorname{Rank}[\gamma, \mathbb{Z}^3] < 3$. Again there exists an element $n \in \mathbb{Z}^3$ such that $\gamma n \gamma^{-1} = n$. Hence n is a central element in Γ. $\square$

Let $\mathbb{Z}$ be a central translation subgroup of Γ. Put $Q = \Gamma/\mathbb{Z}$. As every element $\gamma \in \Gamma$ has the form

$$\gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} 1 & C \\ 0 & B \end{pmatrix}\right),$$

where $B \in \mathrm{GL}(3, \mathbb{R})$, there is an induced action

$$\varphi\colon Q \to \mathrm{A}(3), \qquad \varphi(\bar{\gamma}) = (b, B).$$

Although $\mathbb{Z}$ is not necessarily equal to $\Gamma \cap \mathbb{R}^4$, it can be easily checked that $\varphi\colon Q \to \mathrm{A}(3)$ is a properly discontinuous action such that $\mathbb{R}^3/\varphi(Q)$ is compact and $\operatorname{Ker}\varphi$ is finite as in Lemma 3.1. If $\mathbb{R}$ is the span of $\mathbb{Z}$ in $\mathbb{R}^4$, then $\mathbb{R}$ is a causal action on $\mathbb{R}^4$.

Proposition 3.5.

Every compact complete Lorentzian flat 4-manifold admits a causal circle bundle M in its finite cover.

(i) *S^1 is a timelike circle. $M = T^4 = S^1 \times T^3$, where T^3 is a Riemannian flat torus.*

(ii) *S^1 is a spacelike circle.*

- (ii.1) *$M = S^1 \times T^3$.*
- (ii.2) *$M = S^1 \times \mathcal{N}^3/\Delta$.*
- (ii.3) *$M = S^1 \times \mathcal{R}/\pi$. Each 3-dimensional factor is a Lorentzian flat manifold.*

(iii) *S^1 is a lightlike circle. $M = \mathrm{S}^1 \times \mathcal{N}^3/\Delta$, where $S^1 \to M \to \mathrm{S}^1 \times T^2$ is a nontrivial principal bundle over the affine torus with Euler number $k \in \mathbb{Z}$. Moreover, S^1 is spacelike so M coincides with (ii.2).*

Proof. According to whether $\mathbb{R}$ is timelike or spacelike, we see that the induced action is Euclidean $\varphi\colon Q \to \mathrm{E}(3)$ or Lorenztian $\varphi\colon Q \to \mathrm{E}(2,1)$, respectively. Moreover, we have a decomposition $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ with respect to the Lorentz inner product. Then the formula of (21) becomes

$$\gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}\right).$$

For $\varphi(Q) \le \mathrm{E}(3)$, it follows $\varphi(Q) \le \mathbb{R}^3$ up to finite index by the Bieberbach theorem and hence

$$\gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, I\right).$$

As a consequence, $\Gamma \le \mathbb{R}^4$. This shows (i).

For $\varphi(Q) \le \mathrm{E}(2,1)$, we assume $\varphi(Q)$ is torsion-free. It is known that a compact Lorentzian flat 3-manifold $\mathbb{R}^3/\varphi(Q)$ is T^3, a Heisenberg nilmanifold $\mathcal{N}/\Delta$ or a solvmanifold $\mathcal{R}/\pi$. See, for example, [11, 18]. When $\mathbb{R}^3/\varphi(Q) = \mathcal{N}/\Delta$, the center $\mathbb{R}$ of $\mathcal{N}$ is the translation subgroup consisting of

$$\left\{\begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

The corresponding subgroup Δ in Γ belongs to the translation subgroup

$$\left\{\left(\begin{bmatrix} a \\ b_1 \\ 0 \\ 0 \end{bmatrix}, I\right)\right\}.$$

It is easy to see that Δ is a central subgroup of rank 2.

On the other hand, there are two isomorphism classes of 4-dimensional (compact) nilmanifolds which are Nil^4/Γ or $S^1\times\mathcal{N}/\Delta$. They are characterized as whether the center $C(\mathrm{Nil}^4)=\mathbb{R}$ or $C(\mathbb{R}\times\mathcal{N})=\mathbb{R}^2$. (See [26] for the classification of 4-dimensional Riemannian geometric manifolds in the sense of Thurston and Kulkarni.) By this classification, $\mathbb{R}^4/\Gamma=S^1\times\mathcal{N}/\Delta$.

When $\mathbb{R}^3/\varphi(Q)=\mathcal{R}/\pi$, it follows that $[\pi,\pi]=\mathbb{Z}^2$. As $\mathbb{Z}\leq\Gamma$ is central, it implies $[\Gamma,\Gamma]=\mathbb{Z}^2$. By the classification [26] of 4-dimensional solvmanifolds, the universal covering group G is either one of solvable Lie groups of Inoue type: $\mathrm{Sol}^4_1=\mathcal{N}\rtimes\mathbb{R}$, $\mathrm{Sol}^4_0=\mathbb{R}^3\rtimes\mathbb{R}$, or $\mathrm{Sol}^4_{m,n}=\mathbb{R}^3\rtimes\mathbb{R}$, $m\neq n$, $\mathbb{R}\times\mathcal{R}$, $m=n$. Therefore $[G,G]=\mathcal{N}$ or $\mathbb{R}^3$ except for $\mathbb{R}\times\mathcal{R}$. As $[G,G]=[\mathcal{R},\mathcal{R}]=\mathbb{R}^2$ for $\mathbb{R}\times\mathcal{R}$, we obtain $\mathbb{R}^4/\Gamma=S^1\times\mathcal{R}/\pi$.

We treat the last case of $\mathbb{R}$ being lightlike. By an ad-hoc argument or using the result of [10], it is shown that Γ is nilpotent with $\mathrm{Rank}\,C(\Gamma)=2$. So $\mathbb{R}^4/\Gamma=\mathrm{S}^1\times\mathcal{N}/\Delta$ again. The universal cover $\mathrm{R}\times\mathcal{N}$ is isomorphic to the semidirect product of the translation subgroup $\mathbb{R}^3$ with $\mathbb{R}$;

$$\mathbb{R}^3=\left(\begin{bmatrix}a\\b\\c\\0\end{bmatrix},I\right),\qquad \mathbb{R}=\left(\begin{bmatrix}-t^3/6\\-t^2/2\\0\\t\end{bmatrix},\begin{pmatrix}1&t&0&-t^2/2\\0&1&0&-t\\0&0&1&0\\0&0&0&1\end{pmatrix}\right).$$

Hence the lightlike action

$$\mathbb{R}=\begin{bmatrix}a\\0\\0\\0\end{bmatrix}$$

lies in $\mathcal{N}$ and there is another central group

$$\mathrm{R}=\begin{bmatrix}0\\0\\c\\0\end{bmatrix}$$

which constitutes a principal bundle and its quotient:

$$\mathbb{R}\to\mathrm{R}\times\mathcal{N}\to\mathrm{R}\times\mathbb{R}^2,\qquad S^1\to\mathbb{R}^4/\Gamma\to\mathrm{S}^1\times T^2.$$

As $[\Delta,\Delta]=k\mathbb{Z}$, $k\in\mathbb{Z}$, $S^1\to\mathcal{N}/\Delta\to T^2$ is a circle bundle with Euler number k. □

Remark 3.6.
For the last case, the translation group is the same $\mathbb{R}^3=\mathbb{R}^3\times 0$, but for $\mathbb{R}=\{\varphi_t\}_{t\in\mathbb{R}}$ there are other possibilities. Namely, φ_t has the following form:

$$\left(\begin{bmatrix}-t^3/6\\0\\-t^2/2\\t\end{bmatrix},\begin{pmatrix}1&0&t&-t^2/2\\0&1&0&0\\0&0&1&-t\\0&0&0&1\end{pmatrix}\right),\qquad \left(\begin{bmatrix}-t^3/6\\-t^2/2\\-t^2/2\\t\end{bmatrix},\begin{pmatrix}1&t&t&-t^2/2\\0&1&0&-t\\0&0&1&-t\\0&0&0&1\end{pmatrix}\right).$$

4. Fefferman–Lorentz parabolic structure

Let $\mathbb{Z}_2$ be the subgroup of the center S^1 in $\mathrm{U}(n+1,1)$. Put $\widehat{\mathrm{U}}(n+1,1)=\mathrm{U}(n+1,1)/\mathbb{Z}_2$. The inclusion $\mathrm{U}(n+1,1)\to\mathrm{O}(2n+2,2)$ defines a natural embedding $\widehat{\mathrm{U}}(n+1,1)\to\mathrm{PO}(2n+2,2)$. Then $\widehat{\mathrm{U}}(n+1,1)$ acts transitively on $S^{2n+1,1}$ so that $\left(\widehat{\mathrm{U}}(n+1,1),S^{2n+1,1}\right)$ is a subgeometry of $\left(\mathrm{PO}(2n+2,2),S^{2n+1,1}\right)$.

As in Introduction, a conformally flat *Fefferman–Lorentz parabolic* manifold M is a $(2n+2)$-dimensional smooth manifold locally modeled on the geometry $\big(\mathrm{U}(n+1,1), S^1\times S^{2n+1}\big)$. See [18] for details. We first observe which subgroup in $\mathrm{Sim}_L(\mathbb{R}^{2n+2})$ corresponds to *conformally flat Fefferman–Lorentz parabolic structure*. Let $q\colon S^{2n+1,1}\to S^{2n+1}$ be the projection and $\{\hat\infty\}\in S^{2n+1,1}$ as before. Put $q(\hat\infty)=\{\infty\}\in S^{2n+1}$ which is a point at infinity. As a spherical CR-manifold, $S^{2n+1}-\{\infty\}$ is identified with the Heisenberg Lie group $\mathcal{N}$. Since the stabilizer is

$$\mathrm{PO}(2n+2,2)_{\hat\infty} = \mathbb{R}^{2n+2}\rtimes(\mathrm{O}(2n+1,1)\times\mathbb{R}^+) = \mathrm{Sim}_L(\mathbb{R}^{2n+2}),$$

the intersection $\widehat{\mathrm{U}}(n+1,1)\cap \mathrm{PO}(2n+2,2)_{\hat\infty}$ becomes

$$\widehat{\mathrm{U}}(n+1,1)_{\hat\infty} = \mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+).$$

Noting that $\mathrm{Sim}^*(\mathbb{R}^{2n}) = \mathbb{R}^{2n}\rtimes(\mathrm{O}(2n)\times\mathbb{R}^*)\le \mathrm{O}(2n+1,1)$, it follows

$$\mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+) \le \mathbb{R}^{2n+2}\rtimes(\mathrm{Sim}^*(\mathbb{R}^{2n})\times\mathbb{R}^+) = (\mathbb{R}^{2n+2}\rtimes\mathbb{R}^{2n})\rtimes(\mathrm{O}(2n)\times\mathbb{R}^*)\times\mathbb{R}^+,$$

where $\mathbb{R}^{2n+2}\rtimes\mathbb{R}^{2n}$ is a nilpotent Lie group such that $\mathcal{N}\le\mathbb{R}^{2n+2}\rtimes\mathbb{R}^{2n}$. We have shown in [18], cf. [8], that

Theorem 4.1.
A Fefferman–Lorentz manifold $S^1\times N$ is conformally flat if and only if N is a spherical CR-manifold.

Note that S^1 acts as lightlike isometries on Fefferman–Lorentz manifolds $S^1\times N$ so does its lift $\mathbb{R}$ on $\mathbb{R}\times N$. If $\big(\mathrm{U}(n+1,1)^\sim, \mathbb{R}\times S^{2n+1}\big)$ is an infinite covering of $\big(\widehat{\mathrm{U}}(n+1,1), S^{2n+1,1}\big)$, then the subgroup $\mathbb{R}\times(\mathcal{N}\rtimes\mathrm{U}(n))$ of $\mathrm{U}(n+1,1)^\sim$ acts transitively on the complement $\mathbb{R}\times S^{2n+1}-\mathbb{R}\cdot\infty = \mathbb{R}\times\mathcal{N}$. If $\mathbb{Z}\times\Delta$ is a discrete cocompact subgroup of $\mathbb{R}\times(\mathcal{N}\rtimes\mathrm{U}(n))$, then we obtain, cf. [18],

Proposition 4.2.
$S^1\times\mathcal{N}/\Delta$ is a conformally flat Lorentzian parabolic manifold on which S^1 acts as lightlike isometries.

Remark 4.3.
In (iii) of Proposition 3.5, we saw that a finite cover of a compact (complete) Lorentzian flat 4-manifold admitting a lightlike circle S^1 is the nilmanifold $\mathrm{S}^1\times\mathcal{N}^3/\Delta$ with nontrivial circle bundle $S^1\to \mathrm{S}^1\times\mathcal{N}^3/\Delta\to \mathrm{S}^1\times T^2$. The circle S^1 acts as spacelike isometries. Therefore, the 4-nilmanifold $S^1\times\mathcal{N}^3/\Delta$ of Proposition 4.2 is not conformal to a Lorentzian flat manifold. In fact, if it admits a Lorentzian flat structure within the conformal class, S^1 would be spacelike as above. But S^1 is still lightlike under the conformal change of the Lorentzian metric, which is a contradiction.

5. Developing maps

Suppose that M is a $(2n+2)$-dimensional conformally flat Fefferman–Lorentz parabolic manifold. There is a developing pair

$$(\rho,\mathrm{dev})\colon (\pi,\widetilde{M}) \to \big(\mathrm{U}(n+1,1)^\sim, \widetilde{S}^{2n+1,1}\big).$$

We have the following equivariant projections:

$$\mathbb{Z}\to\big(\mathrm{U}(n+1,1)^\sim, \widetilde{S}^{2n+1,1}\big)\xrightarrow{(Q,q)}\big(\mathrm{U}(n+1,1), S^1\times S^{2n+1}\big),$$
$$S^1\to\big(\mathrm{U}(n+1,1), S^1\times S^{2n+1}\big)\xrightarrow{(P,p)}\big(\mathrm{PU}(n+1,1), S^{2n+1}\big).$$

We call the immersion $q\circ\mathrm{dev}\colon\widetilde{M}\to S^1\times S^{2n+1}$ also a developing map. Let $\Gamma=\rho(\pi)$ be the holonomy group of M in $\mathrm{U}(n+1,1)^\sim$ as before.

Theorem 5.1.

Let M be a compact conformally flat Fefferman–Lorentz parabolic manifold in dimension $2n+2$. Suppose that the holonomy image $Q(\Gamma)$ is discrete in $\mathrm{U}(n+1,1)$. If the developing map $q \circ \mathrm{dev}\colon \widetilde{M} \to S^1 \times S^{2n+1}$ is not surjective and such that the complement $\Lambda = S^1 \times S^{2n+1} - q \circ \mathrm{dev}(\widetilde{M})$ is S^1-invariant, then $q \circ \mathrm{dev}$ is a covering map onto the image.

Proof. As $S^1 \to S^1 \times S^{2n+1} \xrightarrow{p} S^{2n+1}$ is a principal bundle, $p(\Lambda)$ is a closed subset in S^{2n+1}. Put $P(Q(\Gamma)) = G \leq \mathrm{PU}(n+1,1)$ and let $\mathrm{L}(G)$ be the *limit set* for a hyperbolic group G, cf. [7].

I. Suppose that $p(\Lambda)$ contains more than one point in S^{2n+1}. *Minimality* of *limit set* implies that $\mathrm{L}(G) \subset p(\Lambda)$, cf. [7, Lemma 4.3.3]. Since Λ is S^1-invariant, $p^{-1}(\mathrm{L}(G)) \subset \Lambda$. The developing map reduces to the following:

$$q \circ \mathrm{dev}\colon \widetilde{M} \to S^1 \times S^{2n+1} - \Lambda \subset S^1 \times S^{2n+1} - p^{-1}(\mathrm{L}(G)). \tag{26}$$

(i) If G is discrete, then G acts properly discontinuously on *the domain of discontinuity* $S^{2n+1} - \mathrm{L}(G)$, cf. [18, 20]. It is easy to see that $Q(\Gamma)$ acts properly discontinuously on $S^1 \times S^{2n+1} - p^{-1}(\mathrm{L}(G))$ so there exists a $Q(\Gamma)$-invariant Riemannian metric on $S^1 \times S^{2n+1} - p^{-1}(\mathrm{L}(G))$; cf., e.g., [19]. As usual, $q \circ \mathrm{dev}\colon \widetilde{M} \to S^1 \times S^{2n+1,1} - \Lambda$ is a covering map.

We have a commutative diagram of group extensions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S^1 & \longrightarrow & \mathrm{U}(n+1,1) & \xrightarrow{P} & \mathrm{PU}(n+1,1) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & & & \\
1 & \longrightarrow & S^1 & \longrightarrow & S^1 \cdot Q(\Gamma) & \xrightarrow{P} & G & \longrightarrow & 1
\end{array} \tag{27}$$

Here $S^1 \cdot Q(\Gamma)$ is the pushout.

(ii) Suppose that G is not discrete. As the identify component of the closed subgroup $S^1 \cdot Q(\Gamma)$ is S^1 and $P(S^1 \cdot Q(\Gamma)) = G$, the identity component of the closure $\overline{G}^0$ is solvable by Auslander's theorem [25, 8.24 Theorem]. We may assume that $\overline{G}^0$ is noncompact, so it follows up to conjugacy that

$$\overline{G}^0 \leq \mathrm{PU}(n+1,1)_\infty = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+).$$

As the normalizer of $\overline{G}^0$ is also contained in $\mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ up to finite index, we have $G \leq \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. Hence (27) shows that $Q(\Gamma) \leq S^1 \cdot \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. Recall that $\mathbb{R}^+$ acts as the multiplication

$$\lambda(a,z) = (\lambda^2 \cdot a, \lambda \cdot z)$$

for all $\lambda \in \mathbb{R}^+$, $(a,z) \in \mathcal{N}$, cf. [17]. Since $Q(\Gamma)$ is discrete by the hypothesis, it is easy to check that

$$Q(\Gamma) \leq S^1 \times (\mathrm{U}(n) \times \mathbb{R}^+), \quad \text{when } \Gamma \text{ is nontrivial in } \mathbb{R}^+, \tag{28a}$$

$$Q(\Gamma) \leq S^1 \cdot \mathcal{N} \rtimes \mathrm{U}(n), \quad \text{otherwise.} \tag{28b}$$

Then it follows respectively that

$$\begin{array}{ll}
\mathrm{L}(G) \subset \mathrm{L}(\mathrm{U}(n) \times \mathbb{R}^+) = \{0,\infty\}, & p^{-1}(\mathrm{L}(G)) = S^1 \cdot \{0,\infty\}, \\
\mathrm{L}(G) \subset \mathrm{L}(\mathcal{N} \rtimes \mathrm{U}(n)) = \{\infty\}, & p^{-1}(\mathrm{L}(G)) = S^1 \cdot \{\infty\}.
\end{array}$$

Case (28a). The developing map of the first case reduces to $q \circ \mathrm{dev}\colon \widetilde{M} \to S^1 \times S^{2n+1} - S^1 \cdot \{0,\infty\} = S^1 \times (S^{2n} \times \mathbb{R}^+)$ by (26). Since $S^1 \times (\mathrm{U}(n) \times \mathbb{R}^+)$ is a Riemannian isometry group of $S^1 \times (S^{2n} \times \mathbb{R}^+)$, $\widetilde{M}$ admits a π-invariant Riemannian metric by (28) such that $q \circ \mathrm{dev}$ is a local isometry. As $M = \widetilde{M}/\pi$ is compact, $\widetilde{M}$ is complete. Hence $q \circ \mathrm{dev}: \widetilde{M} \to S^1 \times (S^{2n} \times \mathbb{R}^+)$ is a covering map. This proves the case (28a). In this case, M is diffeomorphic to $\mathbb{R} \times (S^{2n} \times \mathbb{R}^+)/\Gamma$ and so M is finitely covered by $S^1 \times S^{2n} \times S^1$.

Case (28b). Similarly as above $q \circ \mathrm{dev}: \widetilde{M} \to S^1 \times S^{2n+1} - S^1 \cdot \{\infty\} = S^1 \times (S^{2n+1} - \{\infty\})$ is a covering map so that $\widetilde{M}$ is diffeomorphic to $\mathbb{R} \times \mathcal{N}$. Then M is diffeomorphic to $\mathbb{R} \times \mathcal{N}/\Gamma$ for which M is finitely covered by a nilmanifold $S^1 \times \mathcal{N}/\Delta$. However in this case, $p(\Lambda) = \{\infty\}$ which is excluded by the hypothesis of **I**.

II. Suppose that $p(\Lambda)$ consists of a single point, say $\{\infty\} \in S^{2n+1}$. It follows $\Lambda = S^1 \cdot \infty$. As Λ is the complement of $q \circ \mathrm{dev}$, we have

$$q \circ \mathrm{dev}(\widetilde{M}) = \widetilde{S}^1 \times S^{2n+1} - S^1 \cdot \{\infty\} = S^1 \times \mathcal{N}. \tag{29}$$

Since G fixes $\{\infty\}$, similarly as in the argument of (**ii**), the discreteness of $Q(\Gamma)$ shows

$$Q(\Gamma) \le S^1 \cdot \mathcal{N} \rtimes \mathrm{U}(n), \quad \text{or} \tag{30a}$$

$$Q(\Gamma) \le S^1 \times (\mathrm{U}(n) \times \mathbb{R}^+). \tag{30b}$$

Case (30a). $S^1 \times \mathcal{N}$ admits an $S^1 \cdot \mathcal{N} \rtimes \mathrm{U}(n)$-invariant Riemannian metric so $q \circ \mathrm{dev}: \widetilde{M} \to S^1 \times \mathcal{N}$ is a covering map. Then M is diffeomorphic to $\mathbb{R} \times \mathcal{N}/\Gamma$. A finite cover of M is a conformally flat Lorentzian parabolic manifold $S^1 \times \mathcal{N}/\Delta$ with nilpotent fundamental group.

Case (30b). Let $Q(\Gamma) \le S^1 \times (\mathrm{U}(n) \times \mathbb{R}^+)$. We consider the *set of points of normality* N for the action $(Q(\Gamma), S^1 \times \mathcal{N})$ from [21]. First note that if $x \in S^1 \times \mathcal{N}$ is a point of normality, see [21, (3.3)], then so is the orbit $t \cdot x$ for $t \in S^1$ because S^1 centralizes $Q(\Gamma)$. Let U_x be a neighborhood for the point x of normality (with respect to $Q(\Gamma)$). For each $\gamma \in Q(\Gamma)$, there is a commutative diagram

$$\begin{array}{ccc} S^1 \cdot U_x & \xrightarrow{\ \gamma\ } & S^1 \times \mathcal{N} \\ \big\downarrow p & & \big\downarrow p \\ p(U_x) & \xrightarrow{\ P(\gamma)\ } & \mathcal{N}. \end{array} \tag{31}$$

If $\lim_{i\to\infty} \gamma_i = g \in C(S^1 \cdot U_x, S^1 \times \mathcal{N})$ in the mapping space, then g commutes with every $t \in S^1$ and so g induces a map $P(g): p(U_x) \to \mathcal{N}$ such that $\lim_{i\to\infty} P(\gamma_i) = P(g)$ for $P(\gamma_i) \in G$. In particular, when $Q(\Gamma)|S^1 \cdot U_x$ is relatively compact in $C(S^1 \cdot U_x, S^1 \times \mathcal{N})$, $G|p(U_x)$ is relatively compact in $C(p(U_x), \mathcal{N})$. Since the action $(G, \mathcal{N})$ is a restriction of the spherical CR-action $(\mathrm{U}(n) \times \mathbb{R}^+, \mathcal{N})$, the set of points of normality for $(G, \mathcal{N})$ is exactly $\mathcal{N} - \{0\}$ on which $\mathrm{U}(n) \times \mathbb{R}^+$ acts properly, cf. [21, (5.8)]. Then $S^1 \times (\mathcal{N} - \{0\})$ is the set of points of normality for the action $(Q(\Gamma), S^1 \times \mathcal{N})$. Noting $q \circ \mathrm{dev}(\widetilde{M}) = S^1 \times \mathcal{N}$ from (29), it follows from [21, Theorem (1.4.1)] that the restriction map

$$q \circ \mathrm{dev}: (q \circ \mathrm{dev})^{-1}(S^1 \times (\mathcal{N} - \{0\})) \to S^1 \times (\mathcal{N} - \{0\}) \tag{32}$$

is a covering map. Since $(\mathrm{dev})^{-1}\big(q^{-1}(S^1 \times (\mathcal{N} - \{0\}))\big) = (\mathrm{dev})^{-1}(\mathbb{R} \times (\mathcal{N} - \{0\})) = \widetilde{M} - \mathrm{dev}^{-1}(\mathbb{R} \times \{0\})$ which is connected, $q \circ \mathrm{dev}: \widetilde{M} - \mathrm{dev}^{-1}(\mathbb{R} \times \{0\}) \to S^1 \times (\mathcal{N} - \{0\})$ is a covering map by (32) so that $\mathrm{dev}: \widetilde{M} - \mathrm{dev}^{-1}(\mathbb{R} \times \{0\}) \to \mathbb{R} \times (\mathcal{N} - \{0\})$ is a diffeomorphism. As above, $\mathrm{dev}(\widetilde{M}) = \mathbb{R} \times \mathcal{N}$, hence $\mathrm{dev}: \widetilde{M} \to \mathbb{R} \times \mathcal{N}$ is a diffeomorphism. However, this cannot occur since $\Gamma \le \mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+)$ is a discrete subgroup with *cohomological dimension* $\mathrm{cd}\,\Gamma \le 2$. This finishes the proof of the theorem. □

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