Cent. Eur. J. Math. • 9(5) • 2011 • 1185-1191
DOI: 10.2478/s11533-011-0063-6

VERSITA

Central European Journal of **Mathematics**

Algebraic axiomatization of tense intuitionistic logic

Research Article

Ivan Chajda[1]*

1 *Department of Algebra and Geometry, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic*

Received 24 February 2011; accepted 24 June 2011

Abstract: We introduce two unary operators G and H on a relatively pseudocomplemented lattice which form an algebraic axiomatization of the tense quantifiers "it is always going to be the case that" and "it has always been the case that". Their axiomatization is an extended version for the classical logic and it is in accordance with these operators on many-valued Łukasiewicz logic. Finally, we get a general construction of these tense operators on complete relatively pseudocomplemented lattice which is a power lattice via the so-called frame.

MSC: 06D20, 03B44, 03G25, 03G10

Keywords: Brouwerian lattice • Heyting algebra • Complete lattice • Relative pseudocomplementation • Tense operators • Intuitionistic logic

Intuitionistic logic was introduced by L. E. J. Brouwer [3] and his collaborator A. Heyting [11]. The semantic of intuitionistic logic is algebraically axiomatized by the so-called Brouwerian lattice, which is a relatively pseudocomplemented lattice, or by the so-called Heyting algebra, which is a bounded Brouwerian lattice. For our goals, we need work with a complete lattice which is Brouwerian and hence also a Heyting algebra, thus we will not make distinction between these two algebras.

It is known that propositional logic (classic or non-classic) does not incorporate the dimension of time. To obtain a tense logic we enrich the given propositional logic by new unary operators which are usually denoted by G, H, F and P, see e.g. [4, 7–9]. We usually define F and P via G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x.

It is worth noticing that the operators G and H can be considered as certain kind of modal operators which were already studied for intuitionistic calculus by D. Wijesekera [14] and in a general setting by W. B. Ewald [10].

Consider a pair $(T; \leq)$ where T is a non-empty set and $\leq$ is a reflexive and transitive binary relation on T. Let $t \in T$ and $f(x)$ be a formula of a given propositional logic $\mathcal{L}$. We say that $G(f(t))$ is valid if for every $s \geq t$ the formula $f(s)$

* *E-mail: chajda@inf.upol.cz*

is valid in $\mathcal{L}$. Analogously, $H(f(t))$ is valid if $f(s)$ is valid in $\mathcal{L}$ for each $s \leq t$. Hence, $P(f(t))$ is valid if there exists $s \leq t$ such that $f(s)$ is valid in $\mathcal{L}$ and $F(f(t))$ is valid if there exists $s \geq t$ such that $f(s)$ is valid in $\mathcal{L}$. Thus the unary operator G constitutes an algebraic counterpart of the tense operator "it is always going to be the case that" and H constitutes an algebraic counterpart of the tense operator "it has always been the case that". In other words, G and H can be recognized as tense "for all" quantifiers and P and F as tense existential quantifiers.

Tense operators were introduced for the classical propositional calculus in [4] as operators on the corresponding Boolean algebra satisfying the axioms

$$\begin{aligned} G(1) &= 1, & H(1) &= 1,\\ G(x \wedge y) &= G(x) \wedge G(y), & H(x \wedge y) &= H(x) \wedge H(y),\\ x &\leq GP(x), & x &\leq HF(x). \end{aligned}$$

However, studying non-classical logics, the list of axioms for tense operators has been enlarged. For example, for MV-algebras there were inserted axioms concerning the logical connectives implication and disjunction (for MV-algebras, the disjunction $\oplus$ is distinct from the lattice join $\vee$ in the induced lattice) as follows:

$$\begin{aligned} G(x \to y) &\leq G(x) \to G(y), & H(x \to y) &\leq H(x) \to H(y),\\ G(x) \oplus G(y) &\leq G(x \oplus y), & H(x) \oplus H(y) &\leq H(x \oplus y). \end{aligned}$$

Similarly, for Moisil propositional logic and for Łukasiewicz–Moisil algebras the tense operators were introduced by C. Chiriţă [7, 8]. In what follows, we use the approach involved by C. Chiriţă but we restrict the list of axioms only for operations of relatively pseudocomplemented lattice.

Let us note that this approach was generalized for the so-called basic algebras (see e.g. [5]) in [2] and it was already applied in the logic of quantum mechanics by means of the so-called dynamic effect algebras in [6].

The aim of the paper is to get an algebraic axiomatization of tense operators G, H, P and F in intuitionistic logic. As mentioned above, an algebraic axiomatization of intuitionistic logic is usually done via relatively pseudocomplemented lattices. Let us recall that a lattice $\mathcal{L} = (L; \vee, \wedge)$ is *relatively pseudocomplemented* if for every $a, b \in L$ there exists a *relative pseudocomplement* $a \to b$, i.e. the greatest element with the property (the so-called residuation property):

$$x \leq a \to b \qquad \text{if and only if} \qquad x \wedge a \leq b.$$

For the properties of relatively pseudocomplemented lattices, the reader is referred to [1, 12] and, in more details, in [5]. Since several constructions in these lattices will use an arbitrary infima, we assume that the lattice $\mathcal{L}$ is *complete*. Hence, it has also a least and a greatest element and so it is a Heyting algebra. The fact that the lattice is relatively pseudocomplemented will be expressed by notation $\mathcal{L} = (L; \vee, \wedge, \to)$, i.e., the relative pseudocomplementation is considered as a fundamental binary operation in $\mathcal{L}$.

Let us mention two well-known results on relatively pseudocomplemented lattices:

- Every relatively pseudocomplemented lattice is distributive.
- A complete lattice is relatively pseudocomplemented if and only if it is infinitely join-distributive, i.e., if it satisfies $a \wedge \bigvee\{b_i : i \in I\} = \bigvee\{a \wedge b_i : i \in I\}$.

We are going to introduce tense operators on a complete relatively pseudocomplemented lattice. Since the logical connective implication in intuitionistic logic is algebraically axiomatized by means of relative pseudocomplement and disjunction by the lattice operation join, we can apply the axiomatization for MV-algebras and modify it as follows.

Definition 1.

Let $\mathcal{L} = (L; \vee, \wedge, \rightarrow, 0, 1)$ be a bounded relatively pseudocomplemented lattice. Denote by $x^* = x \rightarrow 0$ (the so-called pseudocomplement of x). Unary operators G, H on L are called *tense operators* if the following conditions hold:

(1) $G(1) = 1$ and $H(1) = 1$,

(2) $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$ and $H(x \rightarrow y) \leq H(x) \rightarrow H(y)$,

(3) $G(x) \vee G(y) \leq G(x \vee y)$ and $H(x) \vee H(y) \leq H(x \vee y)$,

(4) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,

(5) $x \leq GP(x)$ and $x \leq HF(x)$, where $P = H(x^*)^*$ and $F(x) = G(x^*)^*$.

Remark 2.

It is immediately clear that if $x \rightarrow y = x' \vee y$ where x' is a complement of x in $\mathcal{L}$, then $\mathcal{L}$ is a Boolean algebra and the axioms of tense operators in Boolean algebra are satisfied. Hence, our definition can be considered as an extension of tense operators from the classical propositional calculus to the intuitionistic one.

Example 3.

Let $\mathcal{L} = (\{0, a, 1\}, \vee, \wedge, \rightarrow)$ where $0 < a < 1$ is a chain and hence $\rightarrow$ is defined as follows:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then we can put $G(0) = G(a) = H(0) = H(a) = 0$ and $G(1) = H(1) = 1$. It is an easy exercise to verify the axioms (1)–(5), thus G, H are tense operators on $\mathcal{L}$.

Remark 4.

Let $\mathcal{L} = (L; \vee, \wedge, \rightarrow)$ be an arbitrary relatively pseudocomplemented lattice. One can easily check that there exists two sorts of "extremal" tense operators:

(a) $G(1) = 1 = H(1)$ and $G(x) = 0 = H(x)$ for each $x \neq 1$,

(b) $G(x) = x$ and $H(x) = x$ for each $x \in L$.

It is evident that this is an uninteresting case. A bit more interesting case can be given as follows:

Example 5.

Consider a lattice $\mathcal{L}$ visualized in Figure 1.

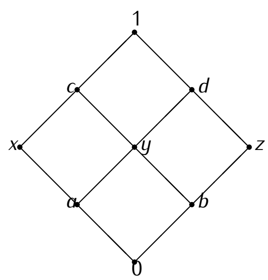

Figure 1.

Then $\mathcal{L}$ is a relatively pseudocomplemented lattice which can be equipped with a couple of tense operators G, H which are not "extremal" as follows:

x	0	a	b	c	d	x	y	z	1
$G(x)$	0	a	0	c	y	x	y	0	1
$H(x)$	0	0	b	y	d	0	y	z	1

Lemma 6.

Let G, H be tense operators on a relative pseudocomplemented bounded lattice $\mathcal{L} = (L; \vee, \wedge, \rightarrow, 0, 1)$. Let $x^* = x \rightarrow 0$, $P(x) = H(x^*)^*$ and $F(x) = G(x^*)^*$. Then

(i) *$x \leq y$ implies $G(x) \leq G(y)$ and $H(x) \leq H(y)$,*

(ii) *$x \leq y$ implies $P(x) \leq P(y)$ and $F(x) \leq F(y)$,*

(iii) *$x^{**} \geq FH(x^{**})$ and $x^{**} \geq PG(x^{**})$.*

Proof. The assertion (i) follows immediately by (4) since $x \leq y$ implies $G(x) = G(x \wedge y) = G(x) \wedge G(y)$, thus $G(x) \leq G(y)$; analogously for the operator H.

To prove (ii), we know [5] that $x \leq y$ implies $y^* \leq x^*$ and $x \leq x^{**}$. Then, using (i), we have $H(x^*) \geq H(y^*)$, thus $P(x) = H(x^*)^* \leq H(y^*)^* = P(y)$. Analogously it can be shown for the operator F.

To prove (iii), by (5) we obtain $x^* \leq HF(x^*) = H(G(x^{**})^*)$. Hence $x^{**} \geq H(G(x^{**})^*)^* = PG(x^{**})$. Analogously we can reach the second inequality. □

It is a natural question if the tense operators G, H satisfy also the equality $G(0) = 0 = H(0)$ which is dual to (1). The answer is as follows.

Lemma 7.

Let G, H be tense operators on a bounded relative pseudo-complemented lattice $\mathcal{L} = (L; \vee, \wedge, \rightarrow, 0, 1)$. Then $G(0) = 0$ if and only if $G(x^*) \leq G(x)^*$ and $H(0) = 0$ if and only if $H(x^*) \leq H(x)^*$.

Proof. Due to symmetry, we will prove the assertion only for the operator G. Assume $G(0) = 0$. Then $G(x^*) = G(x \rightarrow 0) \leq G(x) \rightarrow G(0) = G(x) \rightarrow 0 = G(x)^*$. Conversely, if $G(x^*) \leq G(x)^*$ then $G(0) = G(x \wedge x^*) = G(x) \wedge G(x^*) \leq G(x) \wedge G(x)^* = 0$. □

Remark 8.

The assumption $G(0) = 0 = H(0)$ is very natural. Due to Lemma 7, it yields immediately $F(x) = G(x^*)^* \geq G(x)^{**} \geq G(x)$ and, analogously, $P(x) \geq H(x)$. Since G and H are considered as general quantifiers and F and P as the corresponding existential quantifiers, these relations are generally accepted in every logic.

In what follows, we will derive a procedure for finding tense operators in every complete relatively pseudocomplemented lattices. For this we need two lemmas which are well-known.

Lemma 9 ([13, (1.64)]).

Let $\mathcal{L} = (L; \vee, \wedge, \rightarrow)$ be a complete relatively pseudocomplemented lattice and $a, b_i \in L$ for $i \in I$. Then

$$a \rightarrow \bigwedge\{b_i : i \in I\} \;=\; \bigwedge\{a \rightarrow b_i : i \in I\}.$$

Lemma 10 ([12, (6), p. 136]).

Let $\mathcal{L} = (L; \vee, \wedge, \rightarrow)$ be a complete relatively pseudocomplemented lattice and $a_i, b_i \in L$ for $i \in I$. Then

$$\bigwedge\{a_i : i \in I\} \rightarrow \bigwedge\{b_i : i \in I\} \;\geq\; \bigwedge\{a_i \rightarrow b_i : i \in I\}.$$

By a *frame* (see e.g. [9] or [3, 6]) we mean a couple $(T, \leq)$ where T is a non-empty set and $\leq$ is a quasiorder on T, i.e., a reflexive and transitive binary relation on T. A frame $(T, \leq)$ can be considered as a time scale, i.e., if $r, s \in T$ and $r \leq s$ then we say that r is "before" s and s is "after" r. Given an element $t \in T$, all $s \leq t$ form past tense and all $r \geq t$ form future in the time scale. Hence, having a relatively pseudocomplemented lattice $\mathcal{L} = (L; \vee, \wedge, \rightarrow)$ which is considered as an algebraic counterpart of intuitionistic logic, we can form a power structure $\mathcal{L}^T$. Then the time scaling by means of $(T, \leq)$ enables us to define past or future of $\mathcal{L}$ and hence also of the true-values of formulas or propositions from $\mathcal{L}$. Thus we have in hand an algebraic tool evaluating what has been said in the introduction: for a formula $f(x)$ of $\mathcal{L}$ and $t \in T$, $G(f(t))$ is valid if for each $s \geq t$ the formula $f(s)$ is valid in $\mathcal{L}$. In other words, validity of $G(f(t))$ cannot exceed validity of $f(s)$ for each $s \geq t$. Similarly for $H(f(t))$ we take $s \leq t$ and formulas $f(s)$ to express the validity in past. The following theorem shows a general construction of such tense operators G, H which satisfy the needs given above.

Theorem 11.

Let $\mathcal{L} = (L; \vee, \wedge, \rightarrow)$ be a complete relatively pseudocomplemented lattice and $(T, \leq)$ be a frame. For $p \in \mathcal{L}^T$ we define coordinatewise

$$G(p)(x) = \bigwedge\{p(y) : x \leq y\} \qquad \text{and} \qquad H(p)(x) = \bigwedge\{p(y) : y \leq x\}.$$

Then G, H are tense operators on $\mathcal{L}^T$ such that $G(0) = 0 = H(0)$.

Proof. It is plain to check the axiom (1). Due to the fact that $\mathcal{L}$ is a complete lattice and the operation of infimum is associative, we get (4) immediately. By the use of the distributive inequality, one can easily check (3) as follows:

$$G(p \vee q)(x) = \bigwedge\{p(y) \vee q(y) : x \leq y\} \geq \left(\bigwedge\{p(y); x \leq y\}\right) \vee \left(\bigwedge\{q(y) : x \leq y\}\right) = G(p)(x) \vee G(q)(x).$$

Analogously we can prove this condition for the operator H.

Prove (2): $G(p)(x) \rightarrow G(q)(x) = \left(\bigwedge\{p(y) : x \leq y\}\right) \rightarrow \left(\bigwedge\{q(y) : x \leq y\}\right) \geq \bigwedge\{p(y) \rightarrow q(y) : x \leq y\} = G(p \rightarrow q)(x)$ directly by Lemma 10. Analogously it can be shown for H. It remains to prove (5). At first we mention that

$$P(p)(x) = H(p^*)^*(x) = \left(\bigwedge\{p^*(y) : y \leq x\}\right)^*$$

However, $\left(\bigwedge\{b_i^* : i \in I\}\right)^* \geq b_i^{**} \geq b_i$ thus

$$\left(\bigwedge\{b_i^* : i \in I\}\right)^* \geq \bigvee\{b_i : i \in I\}. \tag{C}$$

Using (C) in the previous computation, we obtain

$$\left(\bigwedge\{p^*(y) : y \leq x\}\right)^* \geq \bigvee\{p(y) : y \leq x\} \geq p(x).$$

Thus

$$GP(p)(x) = \bigwedge\left\{\left(\bigwedge\{p^*(y) : y \leq x\}\right)^* : x \leq y\right\} \geq \bigwedge\left\{\left(\bigvee\{p(y); y \leq x\}\right) : x \leq y\right\} \geq p(x).$$

Analogously we can show $HF(p)(x) \geq p(x)$. Hence, G, H are tense operators on $\mathcal{L}^T$. It is almost evident that $G(0) = 0 = H(0)$. □

Example 12.

Let $T = \{1, 2\}$, $1 \leq 2$, $1 \leq 1$, $2 \leq 2$ and let $\mathcal{L}$ be the relatively pseudocomplemented three-element chain of Example 3, i.e., $\mathcal{L} = (\{0, a, 1\}, \vee, \wedge, \rightarrow)$. Consider the lattice $\mathcal{L}^T$ which has just 9 elements and whose diagram is visualized in Figure 1. Then we have identified the elements as follows:

0	a	x	b	y	c	z	d	1
$(0,0)$	$(0,a)$	$(0,1)$	$(a,0)$	(a,a)	$(a,1)$	$(1,0)$	$(1,a)$	$(1,1)$

Since $T = \{1, 2\}$, we derive by Theorem 11:

$$G(p)(1) = p(1) \wedge p(2), \qquad G(p)(2) = p(2), \qquad H(p)(1) = p(1), \qquad H(p)(2) = p(1) \wedge p(2).$$

Then G, H are just the operators of Example 5 and we can visualize them as follows:

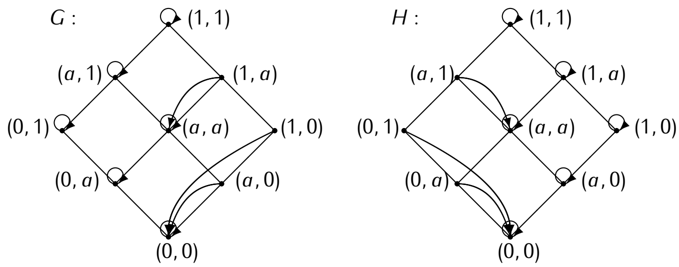

Figure 2.

One can mention that the operator G which expresses future is really essentially influenced by $t = 2$, namely if $x \in L$ has the second coordinate equal to 1, then $G(x) = x$. On the contrary, if the second coordinate is equal to 0 then $G(x) = 0$. Dually we have it for the operator H where it is essentially influenced by the first coordinate (i.e. the past tense). If the first coordinate of x equals 0 then $H(x) = 0$, if it equals 1 then $H(x) = x$. The whole table is as follows:

x	$(0,0)$	$(0,a)$	$(0,1)$	$(a,0)$	(a,a)	$(a,1)$	$(1,0)$	$(1,a)$	$(1,1)$
$G(x)$	$(0,0)$	$(0,a)$	$(0,1)$	$(0,0)$	(a,a)	$(a,1)$	$(0,0)$	(a,a)	$(1,1)$
$H(x)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	(a,a)	(a,a)	$(1,0)$	$(1,a)$	$(1,1)$

Acknowledgements

This work is supported by the Research and Development Council of the Czech Government via the project MSM6198959214.

References

[1] Birkhoff G., Lattice Theory, Amer. Math. Soc. Colloq. Publ., 3rd ed., 25, American Mathematical Society, Providence, 1967

[2] Botur M., Chajda I., Halaš R., Kolařík M., Tense operators on basic algebras, Internat. J. Theoret. Phys. (in press), DOI: 10.1007/s10773-011-0748-4

[3] Brouwer L.E.J., Intuitionism and formalism, Bull. Amer. Math. Soc., 1913, 20(2), 81–96

[4] Burgess J.P., Basic tense logic, In: Handbook of Philosophical Logic II, Synthese Lib., 165, Reidel, Dordrecht, 1984, 89–133

[5] Chajda I., Halaš R., Kühr J., Semilattice Structures, Res. Exp. Math., 30, Heldermann, Lemgo, 2007

[6] Chajda I., Kolařík M., Dynamic effect algebras, Math. Slovaca (in press)

[7] Chiriţă C., Tense θ-valued Moisil propositional logic, International Journal of Computers, Communications & Control, 2010, 5(5), 642–653
[8] Chiriţă C., Tense θ-valued Łukasiewicz–Moisil algebras, J. Mult.-Valued Logic Soft Comput., 2011, 17(1), 1–24
[9] Diaconescu D., Georgescu G., Tense operators on MV-algebras and Łukasiewicz–Moisil algebras, Fund. Inform., 2007, 81(4), 379–408
[10] Ewald W.B., Intuitionistic tense and modal logic, J. Symbolic Logic, 1986, 51(1), 166–179
[11] Heyting A., Intuitionism. An Introduction, North-Holland, Amsterdam, 1956
[12] Rasiowa H., Sikorski R., The Mathematics of Metamathematics, Monogr. Mat., 41, PWN, Warszawa, 1963
[13] Turunen E., Mathematics Behind Fuzzy Logic, Adv. Soft Comput., Physica-Verlag, Heidelberg, 1999
[14] Wijesekera D., Constructive modal logics. I, Ann. Pure Appl. Logic, 1990, 50(3), 271–301