Cent. Eur. J. Math. • 9(5) • 2011 • 1164-1170
DOI: 10.2478/s11533-011-0056-5

Central European Journal of **Mathematics**

On some finite difference schemes for solution of hyperbolic heat conduction problems

Research Article

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Received 9 December 2010; accepted 1 June 2011

Abstract: We consider the accuracy of two finite difference schemes proposed recently in [Roy S., Vasudeva Murthy A.S., Kudenatti R.B., A numerical method for the hyperbolic-heat conduction equation based on multiple scale technique, Appl. Numer. Math., 2009, 59(6), 1419–1430], and [Mickens R.E., Jordan P.M., A positivity-preserving nonstandard finite difference scheme for the damped wave equation, Numer. Methods Partial Differential Equations, 2004, 20(5), 639–649] to solve an initial-boundary value problem for hyperbolic heat transfer equation. New stability and approximation error estimates are proved and it is noted that some statements given in the above papers should be modified and improved. Finally, two robust finite difference schemes are proposed, that can be used for both, the hyperbolic and parabolic heat transfer equations. Results of numerical experiments are presented.

MSC: 65M06, 65M12

Keywords: Hyperbolic heat conduction • Finite difference method • Stability analysis • Convergence analysis

1. Introduction

We consider the hyperbolic-heat conduction initial boundary value problem

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t > 0, \tag{2}$$

$$u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = \mu(x), \qquad 0 \le x \le 1. \tag{3}$$

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Numerical methods for solution of the problem (1)–(3) and some applications are investigated in many papers, see, e.g. [1–4, 9, 10]. This note deals with two finite difference schemes constructed and studied in two recent papers [5, 6]. A positivity preserving finite difference scheme is proposed in [5] and numerical algorithms based on multiple scale technique are constructed in [6]. In both papers, the authors were interested to obtain very robust numerical algorithms that can be used for a broad spectrum of the parameter ε.

In this note, we want to discuss and improve some statements presented in [5, 6]. It is arranged as follows. In Section 2, we investigate the positivity preserving the finite difference scheme from [5]. We show that this scheme coincides with the classical explicit Euler scheme for the parabolic heat transfer problem. This fact explains most of conclusions done in [5]. We investigate the stability of one three level finite difference scheme which was used in numerical experiments in both papers [5, 6]. In Section 3, we study the finite difference scheme based on multiple scale technique [6]. We derive an improved stability estimate of this finite difference scheme and prove that the approximation error is of order $\mathcal{O}(\varepsilon^3/\tau^2)$, where τ is a discrete time step. Results of numerical experiments are presented. In Section 4, we formulate the three-level explicit and implicit Euler type finite difference schemes to solve the hyperbolic heat transfer equation. It is proved that these schemes can be used even when $\varepsilon \to 0$. Finally, in Section 5, some conclusions are made.

2. Positivity preserving finite difference scheme

Let us define in the domain $(0,1) \times (0,T]$ a uniform grid $\omega_{xt} = \omega_x \times \omega_t$, where

$$\omega_x = \left\{x_j \,:\, x_j = jh,\ 0 < j < J,\ h = \frac{1}{J}\right\}, \qquad \omega_t = \left\{t^n = n\tau \,:\, n = 1,\dots,N,\ t^N = t_f\right\}.$$

Discrete function $U_j^n = U(x_j, t^n)$ is defined on the grid points. In this paper we use standard notations of finite difference operators, that approximate time and space derivatives:

$$U_t^n = \frac{U_j^{n+1} - U_j^n}{\tau}, \quad U_{\bar t}^n = \frac{U_j^n - U_j^{n-1}}{\tau}, \quad U_{\overset{\circ}{t}}^n = \frac{U_j^{n+1} - U_j^{n-1}}{2\tau}, \quad \partial_x U = \frac{U_{j+1} - U_j}{h}, \quad \partial_{\bar x} U = \frac{U_j - U_{j-1}}{h}.$$

In Mickens and Jordan [5], the following finite difference scheme is considered for the approximation of (1):

$$\varepsilon U_{t\bar t}^n + (1 - 2\nu)\, U_t^n + 2\nu U_{\overset{\circ}{t}}^n = \partial_x \partial_{\bar x} U^n. \tag{4}$$

Here, the first-oder time derivative of the solution is approximated by a weighted sum of forward- and central-difference formulas.

In order to get a finite difference scheme, which satisfies the positivity condition

$$U_j^n \geq 0 \quad \text{for all } x_j \in \bar\omega_x \quad \Longrightarrow \quad U_j^{n+1} \geq 0 \quad \text{for all } x_j \in \bar\omega_x, \tag{5}$$

the authors of [5] propose to take $\nu = \varepsilon/\tau$. After simple computations, we get that for $\nu = \varepsilon/\tau$ the finite difference scheme (4) coincides with the classical explicit Euler scheme for the parabolic heat conduction problem:

$$U_t = \partial_x \partial_{\bar x} U^n. \tag{6}$$

Thus, in [5] it is proposed to approximate the damped wave problem (1)–(3) by the parabolic initial-boundary value problem

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \\ &v(0,t) = 0, \quad v(1,t) = 0, \qquad t > 0, \\ &v(x,0) = g(x), \qquad 0 \leq x \leq 1. \end{aligned} \tag{7}$$

Now, the initial condition for $\partial u/\partial t$ is not required; and, therefore, this condition is fully ignored in the scheme (6).

From the modelling point of view, approximation of the hyperbolic heat conduction problem (1)–(3) by the finite difference scheme (6) means tacit assumption that for a considered application the modification of the parabolic heat conduction model (7) to the hyperbolic model (1) is not required.

It is noted in [5] that the positivity condition for the scheme (6) is satisfied if $\tau \le h^2/2$ and that this condition is identical to the usual von Neumann stability criteria for the linear heat equation. The last statement is trivial, since the scheme (6) coincides with the explicit Euler scheme for the linear parabolic equation.

Further, as an interesting case of (6), the authors of [5] select $\tau = h^2/4$:

$$U_j^{n+1} = \frac{1}{4}\left(U_{j+1}^n + 2U_j^n + U_{j-1}^n\right), \qquad x_j \in \omega_x, \tag{8}$$

and recommend to use this scheme for any $\tau \le h^2/4$. In fact, the finite difference scheme (8) does not depend on τ and its solution satisfies the maximum principle, hence, this scheme is stable for any τ. If $\tau = qh^2/4$, then the scheme (8) can be written as

$$q\, U_t = \partial_x \partial_{\bar{x}} U^n.$$

Therefore, it approximates the perturbed parabolic equation

$$q\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

In [5], the numerical solution of the finite difference scheme (6) is compared with the solution of the three level finite difference scheme

$$\varepsilon U_{t\bar{t}} + U^n_{\overset{0}{t}} = \partial_x \partial_{\bar{x}} U^n, \tag{9}$$

which can be obtained from (4) for $\nu = 1/2$.

First, we investigate the stability of the explicit scheme (9), since this result will be used also in the following sections. The well-known stability result is valid for three-level finite difference schemes written in a canonical form [7, 8]:

$$\mathcal{B}U^n_{\overset{0}{t}} + \tau^2 \mathcal{R} U^n_{\bar{t}t} + \mathcal{A}U^n = 0. \tag{10}$$

Theorem 2.1.

Let operators $\mathcal{R}$ and $\mathcal{A}$ be symmetric, positively definite, i.e., $\mathcal{A} = \mathcal{A}^* > 0$, $\mathcal{R} = \mathcal{R}^* > 0$, and do not depend on t. Then the conditions $\mathcal{B}(t) \ge 0$, $\mathcal{R} > \mathcal{A}/4$ are necessary and sufficient for the stability of (10) with respect to the initial conditions

$$\left\|Y^{n+1}\right\|_1 \le \left\|Y^1\right\|_1,$$

where the norm is defined by

$$\left\|Y^{n+1}\right\|_1^2 = \frac{1}{4}\left(\mathcal{A}(y^{n+1} + y^n),\, y^{n+1} + y^n\right) + h_t^2\left(\left(\mathcal{R} - \frac{\mathcal{A}}{4}\right) y_t,\, y_t\right).$$

The finite difference scheme (9) can be written in the canonical form by taking

$$\mathcal{B} = \mathcal{I}, \qquad \mathcal{R} = \frac{\varepsilon}{\tau^2}\mathcal{I}, \qquad \mathcal{A}U^n = -\partial_x \partial_{\bar{x}} U^n.$$

Since $\mathcal{A} < 4\mathcal{I}/h^2$, the stability condition is satisfied if the estimate $\varepsilon/\tau^2 \ge 1/h^2$ is valid, which gives the relation $\tau \le \sqrt{\varepsilon}h$. From this relation it follows that the discrete scheme (9) is unstable for the parabolic problem when $\varepsilon = 0$.

The main conclusions done in [5] are quite trivial and they can be explained by properties of the parabolic problem (7) and discrete scheme (6):

(a) it is obvious that the solution of (6) is monotone even for $\tau < \varepsilon$, since the discrete solution does not depend on the parameter ε;

(b) the difference between solutions of hyperbolic problem (1) and difference scheme (6) goes to zero as $\varepsilon \to 0$, since then the scheme (6) approximates the parabolic equation (7).

3. Analysis of a finite difference scheme based on the multiple scale technique

In this section, we consider a new finite difference scheme proposed by Roy, Murthy and Kudenatti [6]. It is based on the multiple scale technique. They have started from the finite difference scheme (9) and solved numerically the problem (1)–(3) when the initial conditions are defined as (see also [5])

$$g(x) = \sin \pi x, \quad \mu(x) = 0, \qquad 0 \le x \le 1. \tag{11}$$

As a conclusion from the results of numerical experiments, it is stated in [6] that there arises a spurious stability problem due to the damping term. No theoretical stability analysis of this scheme is done in [6]. We note that in Section 2 it is proved that the scheme (9) is stable if condition $\tau \le \sqrt{\varepsilon} h$ is satisfied.

The authors of [6] have proposed a new and more stable scheme based on multi-scale method (we rewrite this scheme in a standard three-level form):

$$\varepsilon \sqrt{1+\varepsilon^2}\, U_{t\bar{t}} + \sqrt{1+\varepsilon^2}\, U^n_{\overset{\circ}{t}} + 2\varepsilon \frac{\sqrt{1+\varepsilon^2}-1}{\tau^2}\, U = \partial_x \partial_{\bar{x}} U^n . \tag{12}$$

A spectral stability analysis of the scheme (12) is done in [6], and it is stated that this scheme is stable in the L_2 norm if the condition

$$\tau < \sqrt{\varepsilon \sqrt{1+\varepsilon^2}}\, h \tag{13}$$

is satisfied. We apply Theorem 2.1 to prove the stability of (12) in the energy norm.

Theorem 3.1.
The finite difference scheme (12) is stable if

$$\tau \le h \sqrt{\varepsilon \left(\sqrt{1+\varepsilon^2}+1\right)/2} . \tag{14}$$

Proof. Let us write the finite difference scheme (12) in the canonical form (10) using

$$\mathcal{B} = \sqrt{1+\varepsilon^2}\, \mathcal{I}, \qquad \mathcal{R} = \frac{\varepsilon \sqrt{1+\varepsilon^2}}{\tau^2}\, \mathcal{I}, \qquad \mathcal{A} U^n = -\partial_x \partial_{\bar{x}} U^n + 2\varepsilon \frac{\sqrt{1+\varepsilon^2}-1}{\tau^2}\, U^n .$$

It is easy to check that operators $\mathcal{R}$ and $\mathcal{A}$ are symmetric and positively definite. The inequality $\mathcal{B} > 0$ is also satisfied unconditionally. It remains to verify the last condition $\mathcal{R} > \mathcal{A}/4$. Since

$$\mathcal{A} < \left(\frac{4}{h^2} + 2\varepsilon \frac{\sqrt{1+\varepsilon^2}-1}{\tau^2} \right) \mathcal{I},$$

the stability condition is satisfied if the estimate

$$\frac{\varepsilon \sqrt{1+\varepsilon^2}}{\tau^2} \ge \frac{1}{h^2} + \varepsilon \frac{\sqrt{1+\varepsilon^2}-1}{2\tau^2}$$

holds, which gives the relation (14). □

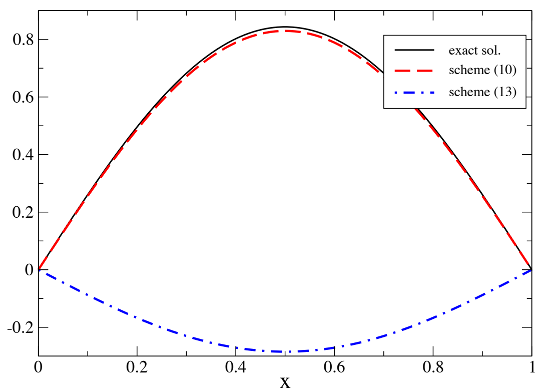
We see that the obtained stability condition (14) is more restrictive than the condition (13) which was stated in [6]. Results of numerical experiments confirm that the scheme (12) is unstable if the stability condition (14) is violated.

The scheme (12) is developed with the aim to improve the stability of the classical scheme (9). According to the Lax convergence theorem, in order to guarantee the convergence of the numerical solution, we should investigate the approximation error of (12). It is easy to check that this approximation error is given by

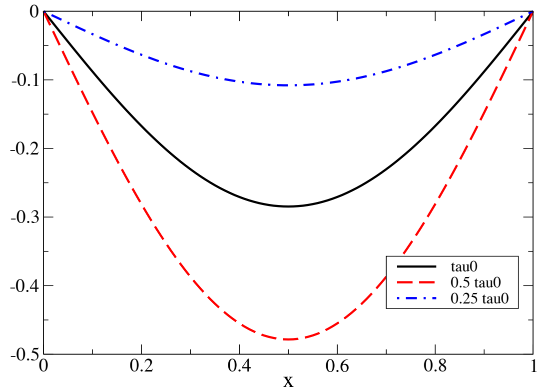
$$\psi = \left(\sqrt{1+\varepsilon^2}-1\right)\left(\varepsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\frac{2\varepsilon}{\tau^2}u\right)+\mathcal{O}\left(\tau^2+h^2\right)=C_1\varepsilon^2+C_2\frac{\varepsilon^3}{\tau^2}+\mathcal{O}\left(\tau^2+h^2\right). \tag{15}$$

We see that $|\psi| \to \infty$ when $\tau \to 0$. Therefore, results of computational experiments presented in [6] are not correct. It is obvious that numerical solution of the scheme (9) converges to the solution of (1)–(3) if $\tau \le \sqrt{\varepsilon}h$ and the solution of (12) diverges when $\tau \to 0$.

Next, we present results of numerical experiments that confirm our theoretical conclusions. We have solved the same test problem as in [5, 6]: the problem (1)–(3) with initial conditions (11). Results of computations for $\varepsilon = 1/10$ are presented in Figure 1. In all computations, we choose $h = 1/100$ and $T = 20\,\tau_0$, where $\tau_0 = h/\sqrt{10}$. In Figure 1 a), we present the exact solution and numerical solutions obtained for $\tau = \tau_0$ by the classical scheme (9) and the new scheme (12). We see a good agreement between the analytical solution and the solution of (9), but the numerical solution based on multiscale technique (12) deviates essentially from the analytical solution (note, that exactly opposite conclusions are done in [6]). Results in Figure 1 b) confirm the validity of estimate (15), i.e., the numerical solution of scheme (12) diverges when $\tau \to 0$.



a)



b)

Figure 1. Numerical results for difference schemes (9) and (12): a) analytical solution (black solid line), numerical solutions of (9) (red dashed line) and (12) (blue dashed and doted line) for $\tau = \tau_0$, b) plots of numerical solutions of (12) for $\tau = \tau_0/j$, $j = 1, 2, 4$.

4. Unconditionally stable scheme

The scheme (10) approximates the differential equation (1) with the second order of accuracy, but it is stable only when $\tau \le \sqrt{\varepsilon}h$. This requirement is very restrictive for small values of parameter ε and the scheme (9) is unconditionally unstable for the parabolic problem (7) when $\varepsilon = 0$.

In order to construct finite difference schemes that are stable for any parameter ε, we change the approximation of the first order damping term by the backward difference Euler formula. Let us consider the following two schemes, see [1]. The first one approximates diffusion operator by the explicit Euler formula:

$$\varepsilon U^n_{t\bar{t}} + U^{n+1}_{\bar{t}} = \partial_x \partial_{\bar{x}} U^n, \tag{16}$$

and the second one is based on the backward Euler formula:

$$\varepsilon U^n_{\bar{t}t} + U^{n+1}_{\bar{t}} = \partial_x \partial_{\bar{x}} U^{n+1}. \tag{17}$$

These schemes belong to a family of finite-difference schemes defined in [8] for multidimensional hyperbolic heat conduction equations.

The approximation error of both schemes (16) and (17) is of order $\mathcal{O}\big(\tau + h^2\big)$.

Theorem 4.1.
The finite difference scheme (16) is stable if

$$\tau \le \frac{h^2}{4} + \sqrt{\frac{h^4}{16} + \varepsilon h^2} \tag{18}$$

and the finite difference scheme (17) is stable unconditionally.

Proof. Using a simple equality

$$U^{n+1}_{\bar{t}} = U^n_{\overset{0}{t}} + \frac{\tau}{2} U^n_{\bar{t}t},$$

we obtain the canonical form (10) of the difference scheme (16):

$$\mathcal{B} = \mathcal{I}, \qquad \mathcal{R} = \frac{\varepsilon + \tau/2}{\tau^2}\mathcal{I}, \qquad \mathcal{A}U^n = -\partial_x \partial_{\bar{x}} U^n.$$

In order to use Theorem 2.1, it is sufficient to prove the estimate $\mathcal{R} > \mathcal{A}/4$. Since $0 < \mathcal{A} < 4\mathcal{I}/h^2$, the stability condition is satisfied if $\tau^2 \le (\varepsilon + \tau/2)\,h^2$. By solving this quadratical inequality we get (18).

Next, we investigate the stability of the implicit Euler scheme (17). The canonical form (10) of this scheme is given by

$$\mathcal{B} = \mathcal{I} + \tau\mathcal{A}, \qquad \mathcal{R} = \frac{\varepsilon\mathcal{I} + \tau(\mathcal{I} + \tau\mathcal{A})/2}{\tau^2}, \qquad \mathcal{A}U^n = -\partial_x \partial_{\bar{x}} U^n.$$

After simple computations, we conclude that all estimates of Theorem 2.1 are satisfied unconditionally. □

5. Conclusions

In order to solve the hyperbolic heat transfer problem, we have constructed two robust finite difference schemes. These schemes have the important property that they can be used even when $\varepsilon \to 0$ (in fact, the implicit scheme (17) is unconditionally stable).

It should be noted that the classical approximation estimates and unconditional stability of the scheme are still not sufficient properties to obtain accurate numerical solutions if the problem (1)–(3) has discontinuous initial-boundary conditions. Due to the Gibbs phenomenon, strong oscillations of the numerical solution are developed; see, e.g., [9]. In order to regularize implicit Euler scheme (17), a simple linear relation between time and space steps is proposed in [1]. Then a controlled amount of numerical viscosity is introduced and numerical oscillations are not appearing. Thus on the basis of implicit Euler scheme, robust solvers can be constructed not only for the parabolic problem but also for more general modified heat conduction problems.

Acknowledgements

We thank the anonymous referees for constructive remarks which allowed us to improve the paper.

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