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Positive characteristic analogs of closed polynomials

Research Article

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Abstract: The notion of a closed polynomial over a field of zero characteristic was introduced by Nowicki and Nagata. In this paper we discuss possible ways to define an analog of this notion over fields of positive characteristic. We are mostly interested in conditions of maximality of the algebra generated by a polynomial in a respective family of rings. We also present a modification of the condition of integral closure and discuss a condition involving partial derivatives.

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Introduction

If k is a field, then by $k[x_1,\dots,x_n]$ we denote the k-algebra of polynomials in n variables. A polynomial $f\in k[x_1,\dots,x_n]\setminus k$, where k is a field of zero characteristic, is called *closed* if the ring $k[f]$ is integrally closed in $k[x_1,\dots,x_n]$. Nowicki [6, 7] proved that a polynomial f is closed if and only if $k[f]$ is a ring of constants of a k-derivation.

The definition of a closed polynomial may be literally applied to the case of positive characteristic, but then we lose the connection with rings of constants. We want to preserve this connection, and we ask about single generators of rings of constants. If $\operatorname{char} k = p > 0$, then such a ring is a $k\left[x_1^p,\dots,x_n^p\right]$-algebra, so it is natural to consider the following condition:

$$\text{"}k\left[x_1^p,\dots,x_n^p,f\right]\text{ is a ring of constants of a }k\text{-derivation"}$$

for a polynomial $f\in k[x_1,\dots,x_n]\setminus k\left[x_1^p,\dots,x_n^p\right]$.

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Nowicki and Nagata showed in [8] that for $f \in k[x_1, \ldots, x_n] \setminus k$, $\operatorname{char} k = 0$, the ring $k[f]$ is integrally closed in $k[x_1, \ldots, x_n]$ if and only if it is a maximal element of the family $\{k[g] : g \in k[x_1, \ldots, x_n]\}$. Ayad observed in [2] that this holds also in the case of positive characteristic. We present in Proposition 3.1 a characterization of rings of constants of derivations in terms of maximality in a suitable family of rings. In Proposition 3.2 we obtain implications between maximality of respective rings in the three families we consider; this shows how far it is from the integral closedness of $k[f]$.

In Theorem 4.1 we present an analog of the condition of integral closedness for rings of constants in the positive characteristic case. In the last section we discuss the condition $\gcd\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = 1$, which is too strong if $\operatorname{char} k = 0$, but may be both sufficient and necessary if $\operatorname{char} k > 0$.

1. Preliminaries

Let A be a ring. An additive map $d\colon A \to A$ is called a *derivation*, if $d(fg) = d(f)g + d(g)f$ for every $f, g \in A$. The set

$$A^d = \{f \in A : d(f) = 0\}$$

is called a *ring of constants of* d. If A is a K-algebra, where K is a ring, and a derivation $d\colon A \to A$ is K-linear, then d is called a K-*derivation*.

If $d\colon A \to A$ is a derivation, then A^d is a subring of A. If A is a K-algebra and d is a K-derivation of A, then A^d is a K-subalgebra of A. If A is a domain (that is, a commutative ring with unity, without zero divisors) of characteristic $p > 0$ and $d\colon A \to A$ is a derivation, then $A^p \subset A^d$ where $A^p = \{a^p : a \in A\}$.

If k is a field of characteristic $p > 0$ and d is a k-derivation of $k[x_1, \ldots, x_n]$, then $k[x_1, \ldots, x_n]^d$ is a $k\left[x_1^p, \ldots, x_n^p\right]$-algebra. If, moreover, $d(f) = 0$ for some $f \in k[x_1, \ldots, x_n]$, then $k\left[x_1^p, \ldots, x_n^p, f\right] \subset k[x_1, \ldots, x_n]^d$.

Note that, for an arbitrary polynomial $f \in k[x_1, \ldots, x_n]$, there exists the smallest (with respect to inclusion) ring of constants of a k-derivation, containing f. It is:

- the integral closure of the ring $k[f]$ in $k[x_1, \ldots, x_n]$, if $\operatorname{char} k = 0$ ([7, Corollary 7.2.1], [8, Proposition 3.3]),
- the ring $k\left(x_1^p, \ldots, x_n^p, f\right) \cap k[x_1, \ldots, x_n]$, if $\operatorname{char} k = p > 0$ [3].

The following theorem of Nowicki and Nagata explains why rings of constants of the form $k[f]$ are so important.

Theorem 1.1 ([7, 7.1.4, 7.1.5], [8, 2.8]).
If k is a field of zero characteristic, and d is a k-derivation of $k[x_1, \ldots, x_n]$ such that $\operatorname{tr\,deg}_k k[x_1, \ldots, x_n]^d \leqslant 1$, then $k[x_1, \ldots, x_n]^d = k[f]$ for some $f \in k[x_1, \ldots, x_n]$.

In particular, if d is a nonzero k-derivation of $k[x, y]$, then $k[x, y]^d = k[f]$ for some $f \in k[x, y]$.

The following theorem presents a variety of equivalent conditions defining a closed polynomial.

Theorem 1.2 ([1], [2], [6], [8]).
Let k be a field and let $f \in k[x_1, \ldots, x_n] \setminus k$. Denote by $\overline{k}$ the algebraic closure of k. Consider the following conditions:

(1) *$k[f]$ is a ring of constants of some k-derivation of $k[x_1, \ldots, x_n]$;*

(2) *the ring $k[f]$ is integrally closed in $k[x_1, \ldots, x_n]$;*

(3) *the ring $k[f]$ is a maximal element (with respect to inclusion) of the family $\{k[g] : g \in k[x_1, \ldots, x_n]\}$;*

(4) *for some $c \in \overline{k}$ the polynomial $f + c$ is irreducible over $\overline{k}$;*

(5) *for all but finitely many $c \in \overline{k}$ the polynomial $f + c$ is irreducible over $\overline{k}$.*

We have:

a) *if* char $k = 0$, *then the conditions* (1)–(5) *are equivalent;*

b) *if k is a perfect field, then the conditions* (2)–(5) *are equivalent;*

c) *for an arbitrary field the conditions* (2) *and* (3) *are equivalent.*

The equivalence of (1), (2) and (3) in the case of char $k = 0$ was proved by Nowicki and Nagata ([6, Theorem 2.1], [7, Proposition 5.2.1], [8, Lemma 3.1]). The condition (4) in the case of char $k = 0$ comes from Ayad [2, Théorème 8, Remarque] and is based on the theorem of Płoski in the case of $k = \mathbb{C}$ ([9], [10, 3.3, Corollary 1, p. 220]). Ayad observed that the equivalence of conditions (2) and (3) holds also in the case of positive characteristic. The case of a perfect field was considered by Arzhantsev and Petravchuk [1, Theorem 1].

The above theorem gives the most natural way to define an analog of a closed polynomial in the case of positive characteristic. However, we will see in the next section that in this case there is no implication between the condition (3) and an analog of the condition (1) from Theorem 1.2.

2. Some examples

Consider the family of rings introduced in the condition (3) of Theorem 1.2:

$$\mathcal{A} = \{k[g] : g \in k[x_1, \dots, x_n]\},$$

where k is a field of arbitrary characteristic. Note that the ring $k[f]$ is a maximal element of $\mathcal{A}$ if and only if the polynomial f is non-composite, that is, it cannot be presented in the form $f = w(g)$, where $g \in k[x_1, \dots, x_n]$, and $w \in k[T]$, $\deg w > 1$.

If we consider rings of constants of k-derivations in the case of char $k = p > 0$, then it is more natural to consider a family

$$\mathcal{B} = \{k[x_1^p, \dots, x_n^p, g] : g \in k[x_1, \dots, x_n]\}.$$

It is clear that the maximality of $k[f]$ in $\mathcal{A}$ does not imply, in general, the maximality of $k[x_1^p, \dots, x_n^p, f]$ in $\mathcal{B}$.

Example 2.1.

Let char $k = p > 0$. The ring $k[x_1^p x_2]$ is a maximal element of $\mathcal{A}$, but the ring $k[x_1^p, \dots, x_n^p, x_1^p x_2]$ is not a maximal element of $\mathcal{B}$.

If $x_1^p x_2 = w(g)$ for some polynomial $w(T) \in k[T]$, then, comparing degrees with respect to x_2, we obtain that $1 = \deg w \cdot \deg_{x_2} g$, so $\deg w = 1$. On the other hand, $k[x_1^p, \dots, x_n^p, x_1^p x_2] \subset k[x_1^p, \dots, x_n^p, x_2]$ and $k[x_1^p, \dots, x_n^p, x_1^p x_2] \neq k[x_1^p, \dots, x_n^p, x_2]$, because $x_2 \notin k[x_1^p, \dots, x_n^p, x_1^p x_2]$.

Observe that the maximality of $k[x_1^p, \dots, x_n^p, f]$ in $\mathcal{B}$ also does not imply, in general, the maximality of $k[f]$ in $\mathcal{A}$.

Example 2.2.

Let char $k = p > 0$. The ring $k[x_1^p, \dots, x_n^p, x_1 + x_1^p] = k[x_1, x_2^p, \dots, x_n^p]$ is a maximal element of $\mathcal{B}$, but the ring $k[x_1 + x_1^p]$ is not a maximal element of $\mathcal{A}$.

If $k[x_1, x_2^p \dots, x_n^p] \subset k[x_1^p, \dots, x_n^p, g]$ for some $g \in k[x_1, \dots, x_n]$, then $x_1 = b_0 + b_1 g + \dots + b_r g^r$, where $b_0, \dots, b_r \in k[x_1^p, \dots, x_n^p]$ and $r \geqslant 1$ is minimal. Applying the partial derivative $\frac{\partial}{\partial x_i}$ we obtain $1 = (b_1 + \dots + r b_r g^{r-1}) \frac{\partial g}{\partial x_i}$, so $b_1 + \dots + r b_r g^{r-1} \in k$. By the minimality of r we have $r = 1$ and $b_1 \in k$, so $k[x_1^p, \dots, x_n^p, g] = k[x_1, x_2^p \dots, x_n^p]$. The ring $k[x_1 + x_1^p]$ is not a maximal element of $\mathcal{A}$, because $k[x_1 + x_1^p] \subset k[x_1]$ and $k[x_1 + x_1^p] \neq k[x_1]$.

Note that the family $\mathcal{B}$ has too many maximal elements, not all of them are rings of constants of derivations.

Example 2.3.
Let $\operatorname{char} k = p > 0$. If $p > 2$, then $k\big[x_1^p, \dots, x_n^p, x_1^{p-1}x_2\big]$ is a maximal element of $\mathcal{B}$, and it is not a ring of constants of any k-derivation of $k[x_1, \dots, x_n]$.

Assume that $k\big[x_1^p, \dots, x_n^p, x_1^{p-1}x_2\big] \subset k\big[x_1^p, \dots, x_n^p, g\big]$, where $g \in k[x_1, \dots, x_n]$. Then $x_1^{p-1}x_2 = w(g)$ for some polynomial $w(T) \in K[T]$, where $K = k\big[x_1^p, \dots, x_n^p\big]$. Since $g^p \in K$, we may assume that $\deg w < p$. Taking the partial derivatives with respect to x_1 and x_2 we obtain that $-x_1^{p-2}x_2 = w'(g) \cdot \frac{\partial g}{\partial x_1}$ and $x_1^{p-1} = w'(g) \cdot \frac{\partial g}{\partial x_2}$, so $w'(g) = cx_1^i$, where $c \in k \setminus \{0\}$ and $0 \leqslant i \leqslant p-2$.

Thus $\frac{\partial g}{\partial x_1} = -\frac{1}{c}\, x_1^{p-2-i}x_2$ and $\frac{\partial g}{\partial x_2} = \frac{1}{c}\, x_1^{p-1-i}$, so $-\frac{1}{c}\, x_1^{p-2-i} = \frac{\partial}{\partial x_2}\big(\frac{\partial g}{\partial x_1}\big) = \frac{\partial}{\partial x_1}\big(\frac{\partial g}{\partial x_2}\big) = (p-1-i)\,\frac{1}{c}\, x_1^{p-2-i}$. We see that $i \equiv 0 \pmod p$, so $i = 0$. Hence $w'(g) = c$.

Now, observe that the field extension $k\big(x_1^p, \dots, x_n^p\big) \subset k(x_1, \dots, x_n)$ is of degree p^n, so the field extension $k\big(x_1^p, \dots, x_n^p\big) \subset k\big(x_1^p, \dots, x_n^p, g\big)$ is of degree p^r for some $r \in \{0, \dots, n\}$. Since $g \in k\big[x_1^p, \dots, x_n^p\big]$ and, by the assumption, $g \notin k\big[x_1^p, \dots, x_n^p\big]$, we see that $r = 1$. Thus the polynomials $1, g, \dots, g^{p-1}$ are linearly independent over $k\big(x_1^p, \dots, x_n^p\big)$. This means that $w'(T) = c$, so $w(T) = c \cdot T + b$ for some $b \in k\big[x_1^p, \dots, x_n^p\big]$. Then $x_1^{p-1}x_2 = w(g) = cg + b$, and we obtain that $k\big[x_1^p, \dots, x_n^p, x_1^{p-1}x_2\big] = k\big[x_1^p, \dots, x_n^p, g\big]$.

Finally, suppose that the ring $k\big[x_1^p, \dots, x_n^p, x_1^{p-1}x_2\big]$ is the ring of constants of a k-derivation d of $k[x_1, \dots, x_n]$. Since $x_1^{p-1}x_2\, d\big(x_1x_2^{p-1}\big) = d\big(x_1^{p-1}x_2x_1x_2^{p-1}\big) = d\big(x_1^px_2^p\big) = 0$, we obtain that $d\big(x_1x_2^{p-1}\big) = 0$. On the other hand, $x_1x_2^{p-1}$, as a polynomial of degree 1 with respect to x_1, does not belong to $k\big[x_1^p, \dots, x_n^p, x_1^{p-1}x_2\big]$, since $p-1 > 1$.

3. Rings of constants as maximal subalgebras

If we are interested in rings of constants of the form $k\big[x_1^p, \dots, x_n^p, f\big]$, then we should use another family of rings:

$$\mathcal{C} = \Big\{R \subset k[x_1, \dots, x_n] \,:\, k\big[x_1^p, \dots, x_n^p\big] \subset R,\ \big(R_0 \,{:}\, k\big(x_1^p, \dots, x_n^p\big)\big) = p\Big\},$$

where R_0 denotes the field of fractions of a domain R and $(L\,{:}\,K)$ denotes the degree of a field extension $K \subset L$.

Proposition 3.1.
A ring $R \in \mathcal{C}$ is a ring of constants of some k-derivation of $k[x_1, \dots, x_n]$ if and only if it is a maximal element of $\mathcal{C}$.

Proof. Recall from [3, Theorem 1.1] that a ring $R \subset k[x_1, \dots, x_n]$ is a ring of constants of some k-derivation of $k[x_1, \dots, x_n]$ if and only if $k\big[x_1^p, \dots, x_n^p\big] \subset R$ and $R_0 \cap k[x_1, \dots, x_n] = R$.

Assume that a ring $R \in \mathcal{C}$ is a ring of constants of some k-derivation and consider a ring $T \in \mathcal{C}$ such that $R \subset T$. We have $R_0 \subset T_0$ and $\big(R_0 : k\big(x_1^p, \dots, x_n^p\big)\big) = \big(T_0 : k\big(x_1^p, \dots, x_n^p\big)\big) = p$, so $R_0 = T_0$. Hence $T \subset T_0 \cap k[x_1, \dots, x_n] = R_0 \cap k[x_1, \dots, x_n] = R$, by the assumption on R.

Assume that a ring R is a maximal element of $\mathcal{C}$. Let $R' = R_0 \cap k[x_1, \dots, x_n]$. Observe that $R \subset R'$ and $R_0 = (R')_0$, so $R' \in \mathcal{C}$. Thus $R = R'$, that is, R is a ring of constants of some k-derivation. □

We see that the ring from Example 2.3 is maximal in $\mathcal{B}$, and is not maximal in $\mathcal{C}$. Observe also that we can correct Example 2.2 in the following way. If we replace $x_1 + x_1^p$ by x_1, then we do not change the ring $k\big[x_1^p, \dots, x_n^p, f\big]$ being maximal in $\mathcal{B}$, and the ring $k[f]$ becomes maximal in $\mathcal{A}$. This leads us to introduce the condition (1) in Proposition 3.2.

We obtain the following relations between maximality of respective subalgebras in the three given families.

Proposition 3.2.
Let $\operatorname{char} k = p > 0$, $f \in k[x_1, \dots, x_n]$. *Consider the following conditions:*

(0) *$k[f]$ is a maximal element of $\mathcal{A}$;*

(1) *$k[f+h]$ is a maximal element of $\mathcal{A}$ for some $h \in k\big[x_1^p, \dots, x_n^p\big]$;*

(2) *$k\big[x_1^p, \dots, x_n^p, f\big]$ is a maximal element of $\mathcal{B}$;*

(3) *$k\big[x_1^p, \dots, x_n^p, f\big]$ is a maximal element of $\mathcal{C}$.*

Then the following implications hold:

$$\begin{array}{ccc} (3) \Rightarrow (2) \Rightarrow & (1) \\ & \Uparrow \\ & (0). \end{array}$$

Proof. The implication (3)⇒(2) follows from the inclusion $\mathcal{B} \subset \mathcal{C} \cup \big\{k\big[x_1^p, \dots, x_n^p\big]\big\}$. The implication (0)⇒(1) is obvious. We will prove the implication (2)⇒(1).

Assume that $k\big[x_1^p, \dots, x_n^p, f\big]$ is a maximal element of $\mathcal{B}$.

Observe that, if $k[g_1] \subsetneq k[g_2]$ for some $g_1, g_2 \in k[x_1, \dots, x_n] \setminus k$, then $\deg g_1 > \deg g_2$, so each increasing chain (with respect to inclusion) in $\mathcal{A}$ is finite. This yields that every non-empty subfamily of $\mathcal{A}$ has a maximal element. Take $h_0 \in k\big[x_1^p, \dots, x_n^p\big]$ such that $k[f+h_0]$ is a maximal element of the subfamily $\big\{k[f+h] : h \in k\big[x_1^p, \dots, x_n^p\big]\big\}$ of $\mathcal{A}$.

Consider a polynomial $g \in k[x_1, \dots, x_n]$ such that $k[f+h_0] \subset k[g]$. Then $f + h_0 \in k[g]$, so $f \in k\big[x_1^p, \dots, x_n^p, g\big]$ and $k\big[x_1^p, \dots, x_n^p, f\big] \subset k\big[x_1^p, \dots, x_n^p, g\big]$. Therefore $k\big[x_1^p, \dots, x_n^p, f\big] = k\big[x_1^p, \dots, x_n^p, g\big]$, by the maximality of $k\big[x_1^p, \dots, x_n^p, f\big]$ in $\mathcal{B}$, so $f - ag \in k\big[x_1^p, \dots, x_n^p\big]$ for some $a \in k \setminus \{0\}$, by [3, Proposition 2.7]. This means that $ag = f + h_1$ for some $h_1 \in k\big[x_1^p, \dots, x_n^p\big]$, and then $k[g] = k[f+h_1]$. By the choice of h_0, the inclusion $k[f+h_0] \subset k[f+h_1]$ implies the equality $k[f+h_0] = k[f+h_1]$, that is, $k[f+h_0] = k[g]$. Hence $k[f+h_0]$ is a maximal element of $\mathcal{A}$. □

As we have just observed, the reverse implications, in general, do not hold.

Recall that in case of positive characteristic, $k[f]$ is maximal in $\mathcal{A}$ if and only if $k[f]$ is integrally closed in $k[x_1, \dots, x_n]$ (Theorem 1.2). So if the definition of a closed polynomial is literally applied in the case of positive characteristic, from the propositions above it follows that the connection between integral closedness and the rings of constants is no longer valid. Note that the condition (0) is, in general, independent of the conditions (2) and (3). Example 2.1 shows that it does not imply (2), and Example 2.2 shows that it is not implied by (3).

4. Another conditions

When considering rings of constants of derivations of the form $k\big[x_1^p, \dots, x_n^p, f\big]$, it is natural to ask about integral closedness. However, if $n > 1$, then this ring is not integrally closed in $k[x_1, \dots, x_n]$, because for a polynomial $g \in k[x_1, \dots, x_n] \setminus k\big[x_1^p, \dots, x_n^p, f\big]$ we have $g^p \in k\big[x_1^p, \dots, x_n^p, f\big]$. Nevertheless, we can modify the condition of integral closedness.

Theorem 4.1.

Let A be a domain of characteristic $p > 0$ and let B be a subring of A, containing A^p, and such that A is finitely generated as a B-algebra. Let R be a subring of A such that $B \subset R$. The following conditions are equivalent:

(1) *the ring R is a ring of constants of some B-derivation of A,*

(2) *for every $g \in A$ and $a_0, a_1, \dots, a_{p-1} \in R$, such that $a_i \neq 0$ for some i, if $a_{p-1}g^{p-1} + \ldots + a_1 g + a_0 = 0$, then $g \in R$,*

(3) *for every $a \in R \setminus \{0\}$, $b \in R$, and $g \in A$, if $ag + b = 0$, then $g \in R$,*

(4) *for every $a \in B \setminus \{0\}$, $b \in R$, and $g \in A$, if $ag + b = 0$, then $g \in R$.*

Proof. First, note that the condition (3) means that $R_0 \cap A = R$.

(1)⇔(3) We have already assumed that $B \subset R$, so the ring R is a ring of constants of some B-derivation of A if and only if $R_0 \cap A = R$ ([4, Theorem 2.5] – a generalization of [3, Theorem 1.1]).

The implications (2)⇒(3) and (3)⇒(4) are obvious.

(3)⇒(2) Assume that $R_0 \cap A = R$. Consider $g \in A$ and $a_0, a_1, \dots, a_{p-1} \in R$, such that $a_i \neq 0$ for some i and

$$a_{p-1}g^{p-1} + \ldots + a_1 g + a_0 = 0.$$

The elements $1, g, \dots, g^{p-1}$ are linearly dependent over R_0, but the field extension $R_0 \subset R_0(g)$ is purely inseparable, so $g \in R_0$ (see [5, Lemma 1.1] for details), that is, $g \in R$.

(4)⇒(3) Assume that the condition (4) holds and consider $a \in R \setminus \{0\}$, $b \in R$, and $g \in A$ such that $ag + b = 0$. We have $a^p g + a^{p-1} b = 0$ and $a^p \in B \setminus \{0\}$, so, by the assumption, $g \in R$. □

The condition (2) is a good analog of integral closedness for subrings $R \subset A$ such that $A^p \subset R$. On the other hand, however, we see that it reduces to a quite simple form (4).

Note that, in the case of zero characteristic, some conditions involving partial derivatives imply that the polynomial is closed. The following proposition was proved by Ayad in [2, Proposition 14], in the case of two variables, but his proof may be easily generalized to the case of n variables.

Proposition 4.2 (Ayad).

If $\operatorname{char} k = 0$, $f \in k[x_1, \dots, x_n] \setminus k$ *and*

$$\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = 1,$$

then the ring $k[f]$ *is integrally closed in* $k[x_1, \dots, x_n]$.

There is no reverse implication in the proposition above, in general.

Example 4.3.

Let $\operatorname{char} k = 0$, $n \geqslant 2$. Put $f = x_1^r x_2^s \in k[x_1, \dots, x_n]$, where $r, s \geqslant 1$, $\gcd(r, s) = 1$ and $(r, s) \neq (1, 1)$. Then f is a closed polynomial, but $\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \neq 1$.

Consider the k-derivation $d = s x_1 \frac{\partial}{\partial x_1} - r x_2 \frac{\partial}{\partial x_2}$. Observe that $d(x_1^i x_2^j) = (si - rj)\, x_1^i x_2^j$ for every i, j, so $k[x_1, \dots, x_n]^d$ is k-linearly spanned by monomials $x_1^i x_2^j$ such that $si - rj = 0$, so $k[x_1, \dots, x_n]^d = k\big[x_1^r x_2^s\big]$. On the other hand, $\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = x_1^{r-1} x_2^{s-1}$.

In the case of positive characteristic we have the following [5, Theorem 2.3].

Proposition 4.4.

If $\operatorname{char} k = p > 0$, $f \in k[x_1, \dots, x_n] \setminus k\big[x_1^p, \dots, x_n^p\big]$ *and* $\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = 1$, *then* $k\big[x_1^p, \dots, x_n^p, f\big]$ *is a ring of constants of some* k*-derivation of* $k[x_1, \dots, x_n]$.

In contrary to the case of zero characteristic, the condition

$$\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = 1$$

is of use here. In some special cases (see [5] for details) there is equivalence in the proposition above, perhaps it holds for an arbitrary f.

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References

[1] Arzhantsev I.V., Petravchuk A.P., Closed polynomials and saturated subalgebras of polynomial algebras, Ukrainian Math. J., 2007, 59(12), 1783–1790
[2] Ayad M., Sur les polynômes $f(X,Y)$ tels que $K[f]$ est intégralement fermé dans $K[X,Y]$, Acta Arith., 2002, 105(1), 9–28
[3] Jędrzejewicz P., Rings of constants of p-homogeneous polynomial derivations, Comm. Algebra, 2003, 31(11), 5501–5511
[4] Jędrzejewicz P., Eigenvector p-bases of rings of constants of derivations, Comm. Algebra, 2008, 36(4), 1500–1508
[5] Jędrzejewicz P., One-element p-bases of rings of constants of derivations, Osaka J. Math., 2009, 46(1), 223–234
[6] Nowicki A., On the jacobian equation $J(f,g)=0$ for polynomials in $k[x,y]$, Nagoya Math. J., 1988, 109, 151–157
[7] Nowicki A., Polynomial Derivations and their Rings of Constants, UMK, Toruń, 1994
[8] Nowicki A., Nagata M., Rings of constants for k-derivations in $k[x_1,\dots,x_n]$, J. Math. Kyoto Univ., 1988, 28(1), 111–118
[9] Płoski A., On the irreducibility of polynomials in several complex variables, Bull. Polish Acad. Sci. Math., 1991, 39(3-4), 241–247
[10] Schinzel A., Polynomials with Special Regard to Reducibility, Encyclopedia Math. Appl., 77, Cambridge University Press, Cambridge, 2000