Karhunen–Loève expansions of α-Wiener bridges

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Abstract: We study Karhunen–Loève expansions of the process $\left(X_t^{(\alpha)}\right)_{t \in [0,T)}$ given by the stochastic differential equation $dX_t^{(\alpha)} = -\frac{\alpha}{T-t} X_t^{(\alpha)} dt + dB_t$, $t \in [0, T)$, with the initial condition $X_0^{(\alpha)} = 0$, where $\alpha > 0$, $T \in (0, \infty)$, and $(B_t)_{t \geq 0}$ is a standard Wiener process. This process is called an α-Wiener bridge or a scaled Brownian bridge, and in the special case of $\alpha = 1$ the usual Wiener bridge. We present weighted and unweighted Karhunen–Loève expansions of $X^{(\alpha)}$. As applications, we calculate the Laplace transform and the distribution function of the $L^2[0, T]$-norm square of $X^{(\alpha)}$ studying also its asymptotic behavior (large and small deviation).

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1. Introduction

There are few stochastic processes of interest, even among Gaussian ones, for which the Karhunen–Loève (KL) expansion is explicitly known. Some examples are the Wiener process, the Ornstein–Uhlenbeck process and the Wiener bridge, see, e.g., Ash and Gardner [3, Example 1.4.4], Papoulis [32, Problem 12.7], Liu and Ulukus [22, Section III], Corlay and Pagès [10, Section 5.4 B], and Deheuvels [12, Remark 1.1]. Recently, there aroused a renewed interest in this field: some KL expansions were provided for weighted Wiener processes and weighted Wiener bridges with weighting function having the form t^β (these expansions make use of Bessel functions), see Deheuvels and Martynov [14]. The most recent results on this topic are those of Deheuvels, Peccati and Yor [15], Deheuvels [12], [13], Luschgy and Pagès [24], Nazarov

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and Nikitin [29], and Nazarov and Pusev [30] (the latter two give exact small deviation asymptotics for weighted L^2-norm of some Gaussian processes).

Let $0 < S < T < \infty$ and $0 < \alpha < \infty$ be arbitrarily fixed and let $(B_t)_{t \geq 0}$ be a standard Wiener process on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. Let us consider the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}X_t^{(\alpha)} = -\frac{\alpha}{T-t} X_t^{(\alpha)} \mathrm{d}t + \mathrm{d}B_t, & t \in [0, S], \\ X_0^{(\alpha)} = 0. \end{cases}$$

The drift and diffusion terms satisfy the usual Lipschitz and linear growth conditions, so, by Øksendal [31, Theorem 5.2.1] or Jacod and Shiryaev [18, Chapter III, Theorem 2.32], the above SDE has a strong solution which is pathwise unique (i.e., it has a unique strong solution). Since $S \in (0, T)$ is chosen arbitrarily, we obtain by successive extension that also the SDE

$$\begin{cases} \mathrm{d}X_t^{(\alpha)} = -\frac{\alpha}{T-t} X_t^{(\alpha)} \mathrm{d}t + \mathrm{d}B_t, & t \in [0, T), \\ X_0^{(\alpha)} = 0, \end{cases} \tag{1}$$

has a unique strong solution. Namely, it is

$$X_t^{(\alpha)} = \int_0^t \left(\frac{T-t}{T-s} \right)^\alpha \mathrm{d}B_s, \qquad t \in [0, T), \tag{2}$$

as it can be checked by Itô's formula. The process $\left(X_t^{(\alpha)} \right)_{t \in [0,T)}$ given by (2) is called an α-*Wiener bridge* (from 0 to 0 on the time interval $[0, T]$). To our knowledge, such kinds of processes have been first considered by Brennan and Schwartz [9], see also Mansuy [25]. In Brennan and Schwartz [9] α-Wiener bridges are used to model the arbitrage profit associated with a given futures contract in the absence of transaction costs. Sondermann, Trede and Wilfling [34], and Trede and Wilfling [35] use the SDE (1) to describe the fundamental component of an exchange rate process and they call the process $X^{(\alpha)}$ a scaled Brownian bridge. The essence of these models is that the coefficient of $X_t^{(\alpha)}$ in the drift term in (1) represents some kind of mean reversion, a stabilizing force that keeps pulling the process towards its mean (zero in this reduced form), and the absolute value of this force is increasing proportionally to the inverse of the remaining time $T - t$, with the rate constant α.

This process has been studied also by Barczy and Pap [5], [6, Section 4] from several points of view, e.g., singularity of probability measures induced by the process $X^{(\alpha)}$ with different values of α, sample path properties, Laplace transforms of certain functionals of $X^{(\alpha)}$ and maximum likelihood estimation of α. The process $\left(X_t^{(\alpha)} \right)_{t \in [0,T)}$ is Gaussian, $\mathrm{E}\, X_t^{(\alpha)} = 0$ for all $t \in [0, T)$, and the covariance function of $X^{(\alpha)}$ given in Barczy and Pap [5, Lemma 2.1] is

$$R^{(\alpha)}(s, t) = \mathrm{Cov}\left(X_s^{(\alpha)}, X_t^{(\alpha)} \right) = \begin{cases} \frac{(T-s)^\alpha (T-t)^\alpha}{1-2\alpha} \left(T^{1-2\alpha} - (T - (s \wedge t))^{1-2\alpha} \right) & \text{if } \alpha \neq \frac{1}{2}, \\ \sqrt{(T-s)(T-t)} \ln \frac{T}{T-(s \wedge t)} & \text{if } \alpha = \frac{1}{2}, \end{cases} \tag{3}$$

for all $s, t \in [0, T)$, where $s \wedge t = \min(s, t)$. By Barczy and Pap [5, Lemma 3.1], the α-Wiener bridge $\left(X_t^{(\alpha)} \right)_{t \in [0,T)}$ has an almost surely continuous extension $\left(X_t^{(\alpha)} \right)_{t \in [0,T]}$ onto the time interval $[0, T]$ such that $X_T^{(\alpha)} = 0$ with probability one. The possibility of such an extension is based on the fact that the parameter α is positive and on a strong law of large numbers for square integrable local martingales. We note here also that (1)–(2) continue to hold for $\alpha \leq 0$ as well. However, there does not exist an almost surely continuous extension of the process $\left(X_t^{(\alpha)} \right)_{t \in [0,T)}$ onto $[0, T]$ which would take some constant at time T with probability one (i.e., which would be a bridge), and that is why the range of the parameter α is restricted to positive values. Indeed, for $\alpha = 0$ we obtain the Wiener process, and in case of $\alpha < 0$ the second moment of the solution $X_t^{(\alpha)}$ given by (2) converges to infinity, as (3) (with $s = t$) shows. Hence the assumption of the existence of an almost surely continuous extension to $[0, T]$ such that this extension takes some constant at time T with probability one (i.e., that there is a bridge) would result in a contradiction. We note that another proof of the impossibility of such an extension in the case of $\alpha < 0$ can be found in Barczy and Pap [5, Remark 3.5]. Finally, we remark that Mansuy [25, Proposition 4] studied the question whether it is possible to derive the α-Wiener bridge from a (single) Gaussian process by taking a bridge.

Next we check that the α-Wiener bridge $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ is L^2-continuous. By Theorem 1.3.4 in Ash and Gardner [3], it is enough to show that the covariance function $R^{(\alpha)}$ can be extended continuously onto $[0, T]^2 = [0, T] \times [0, T]$ so that this extension (which will be also denoted by $R^{(\alpha)}$) is zero on the set $\{(s, T) : s \in [0, T]\} \cup \{(T, t) : t \in [0, T]\}$. This follows by

$$\lim_{(s,t)\to(s_0,T)} R^{(\alpha)}(s, t) = \lim_{(s,t)\to(T,t_0)} R^{(\alpha)}(s, t) = 0, \qquad s_0, t_0 \in [0, T].$$

For a detailed discussion, see Barczy and Iglói [4].

We also have $R^{(\alpha)} \in L^2[0, T]^2$. So, the integral operator associated to the kernel function $R^{(\alpha)}$, i.e., the operator $A_{R^{(\alpha)}} : L^2[0, T] \to L^2[0, T]$,

$$\left(A_{R^{(\alpha)}}(\phi)\right)(t) = \int_0^T R^{(\alpha)}(t, s)\, \phi(s)\, ds, \qquad t \in [0, T], \quad \phi \in L^2[0, T],$$

is of the Hilbert–Schmidt type, thus $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ has a Karhunen–Loève (KL) expansion based on $[0, T]$:

$$X_t^{(\alpha)} = \sum_{k=1}^\infty \sqrt{\lambda_k^{(\alpha)}}\, \xi_k\, e_k^{(\alpha)}(t), \qquad t \in [0, T], \tag{4}$$

where ξ_k, $k \in \mathbb{N}$, are independent standard normally distributed random variables, $\lambda_k^{(\alpha)}$, $k \in \mathbb{N}$, are the non-zero eigenvalues of the integral operator $A_{R^{(\alpha)}}$ and $e_k^{(\alpha)}(t)$, $t \in [0, T]$, $k \in \mathbb{N}$, are the corresponding normed eigenfunctions, which are pairwise orthogonal in $L^2[0, T]$, see, e.g., Ash and Gardner [3, Theorem 1.4.1]. Observe that (4) has infinitely many terms. Indeed, if it had a finite number of terms, i.e., if there were only a finite number of eigenfunctions, say N, then with the help of (1) (considering it as an integral equation) we would obtain that the Wiener process $(B_t)_{t\in[0,T]}$ is concentrated in an N-dimensional subspace of $L^2[0, T]$, and so even of $C[0, T]$, with probability one. This results in a contradiction, since the integral operator associated to the covariance function (as a kernel function) of a standard Wiener process has infinitely many eigenvalues and eigenfunctions. We also note that the normed eigenfunctions are unique only up to sign. The series in (4) converges in $L^2(\Omega, \mathcal{A}, P)$ to $X_t^{(\alpha)}$, uniformly on $[0, T]$, i.e.,

$$\sup_{t\in[0,T]} \mathsf{E}\left(\left|X_t^{(\alpha)} - \sum_{k=1}^n \sqrt{\lambda_k^{(\alpha)}}\, \xi_k e_k^{(\alpha)}(t)\right|^2\right) \to 0 \qquad \text{as } n \to \infty.$$

Moreover, since $R^{(\alpha)}$ is continuous on $[0, T]^2$, the eigenfunctions corresponding to non-zero eigenvalues are also continuous on $[0, T]$, see, e.g. Ash and Gardner [3, p. 38] (this will be important in the proof of Theorem 2.1, too). Since the terms on the right-hand side of (4) are independent normally distributed random variables and $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ has continuous sample paths with probability one, the series converges uniformly on $[0, T]$ even with probability one (see, e.g., Adler [2, Theorem 3.8]).

The rest of the paper is organized as follows. In Section 2 we make the KL representation (4) of the α-Wiener bridge $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ as explicit as it is possible, see Theorem 2.1. We also consider two special cases of the KL representation (4): first we study the case $\alpha \downarrow 0$ (standard Wiener process) and then the case $\alpha = 1$ (Wiener bridge), see Remark 2.2 and Remark 2.3, respectively. In Remark 2.4 we present a so-called space-time transformed Wiener process representation of the α-Wiener bridge. Using this representation and a result of Gutiérrez and Valderrama [16, Theorem 1], we derive a weighted KL representation of the α-Wiener bridge $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ based on $[0, S]$, where $0 < S < T$, see Theorem 2.5. We also consider two special cases of this weighted KL representation: first we study the case $\alpha \downarrow 0$ and then the case $\alpha = 1$, see Remark 2.6 and Remark 2.7, respectively. Further, we give an infinite series representation of $\int_0^S \left(X_u^{(1/2)}\right)^2/(T-u)^2\, du$ where $0 < S < T$, see Remark 2.8. Section 3 is devoted to the applications. In Proposition 3.1 we determine the Laplace transform of the $L^2[0, T]$-norm square of $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ and of the $L^2[0, S]$-norm square of $\left(X_t^{(1/2)}/(T - t)\right)_{t\in[0,S]}$, where $0 < S < T$. In Corollary 3.2 we give a new probabilistic proof for the well-known result of Rayleigh, namely, for the sum of the square of the reciprocals of the positive zeros of Bessel functions of the first kind (with order greater than $-1/2$ in our case). Based on the Smirnov formula (see, e.g., Smirnov [33, formula (97)]), we write the survival function

of the $L^2[0, T]$-norm square of $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ in an infinite series form, see Proposition 3.3. We also consider two special cases of Proposition 3.3: the case $\alpha \downarrow 0$ and the case $\alpha = 1$, see Remark 3.4 and Remark 3.5. In Corollary 3.7, based on a result of Zolotarev [38], we study large deviation probabilities for the $L^2[0, T]$-norm square of the α-Wiener bridge. Finally, based on a result of Li [21, Theorem 2], we describe the behavior at zero of the distribution function (small deviation probabilities) of the $L^2[0, T]$-norm square of the α-Wiener bridge, see Corollary 3.8. In the appendix we list some important properties of Bessel functions of the first and second kind, respectively.

We remark that our results for α-Wiener bridges may have some generalizations for random fields. Namely, for all $S > 0$, $T > 0$ and $\alpha \geq 0$, $\beta \geq 0$ one can consider a zero-mean Gaussian random field $\left(X_{s,t}^{(\alpha,\beta)}\right)_{(s,t)\in[0,S]\times[0,T]}$ with the covariances

$$\mathsf{E}\left(X_{s_1,t_1}^{(\alpha,\beta)} X_{s_2,t_2}^{(\alpha,\beta)}\right) = R^{(\alpha)}(s_1, s_2)\, R^{(\beta)}(t_1, t_2), \qquad (s_1, t_1), (s_2, t_2) \in [0, S] \times [0, T].$$

Such a random field exists, since $\left(X_s^{(\alpha)} X_t^{(\beta)}\right)_{(s,t)\in[0,S]\times[0,T]}$ admits the above covariances, where $X^{(\alpha)}$ and $X^{(\beta)}$ are independent, and Kolmogorov's consistency theorem comes into play. This class of Gaussian processes may deserve more attention since it would generalize some well-known limit processes in mathematical statistics such as the Kiefer process (known also a tied down Brownian sheet), see, e.g., Csörgő and Révész [11, Section 1.15]. Indeed, with $S = 1$, $T = \infty$, $\alpha = 1$ and $\beta = 0$ the process $X^{(\alpha,\beta)}$ is nothing else but the Kiefer process having covariance function $(s_1 \wedge s_2 - s_1 s_2)(t_1 \wedge t_2)$, $(s_1, t_1), (s_2, t_2) \in [0, 1] \times [0, \infty)$.

In all what follows $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{Z}$ denote the set of natural numbers, nonnegative integers and integers, respectively.

2. Karhunen–Loève expansions of α-Wiener bridges

First we recall the notion of Bessel functions of the first kind which plays a key role in the KL expansions we will obtain, and also the Bessel functions of the second kind (also called Neumann functions) which will appear in the proofs. They can be defined, respectively, as

$$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \qquad x \in (0, \infty), \quad \nu \in \mathbb{R},$$

$$Y_\nu(x) = \begin{cases} \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}, & x \in (0, \infty), \quad \nu \in \mathbb{R} \setminus \mathbb{Z}, \\ Y_n(x) = \lim_{\mu\to n} Y_\mu(x), & x \in (0, \infty), \quad \nu = n \in \mathbb{Z}, \end{cases}$$

where $\Gamma(z)$ for $z < 0$, $z \notin \mathbb{Z}$, is defined by a recursive application of the rule $\Gamma(z) = \Gamma(z+1)/z$, $z < 0$, $z \notin \mathbb{Z}$, and we use the convention that $1/\Gamma(-k) = 0$, $k \in \mathbb{Z}_+$, yielding that the first n terms in the series of $J_\nu(x)$ vanish if $\nu = -n$, $n \in \mathbb{N}$, see, e.g., Watson [36, pp. 40, 64].

In all what follows we will put $\nu = \alpha - 1/2$ where $\alpha > 0$. Now we present our main theorem.

Theorem 2.1.
Let $\alpha > 0$, $\nu = \alpha - 1/2$, and $z_k^{(\nu)}$, $k \in \mathbb{N}$, be the (positive) zeros of J_ν. Then in the KL expansion (4) *of the α-Wiener bridge* $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$ *the eigenvalues and the corresponding normed eigenfunctions are*

$$\lambda_k^{(\alpha)} = \frac{T^2}{\left(z_k^{(\nu)}\right)^2}, \qquad k \in \mathbb{N}, \tag{5}$$

$$e_k^{(\alpha)}(t) = \sqrt{\frac{2}{T}\left(1 - \frac{t}{T}\right)}\, \frac{J_\nu\left(z_k^{(\nu)}(1 - t/T)\right)}{\left|J_{\nu+1}\left(z_k^{(\nu)}\right)\right|} = \sqrt{\frac{2}{T}\left(1 - \frac{t}{T}\right)}\, \frac{J_\nu\left(z_k^{(\nu)}(1 - t/T)\right)}{\left|J_{\nu-1}\left(z_k^{(\nu)}\right)\right|}, \qquad t \in [0, T],$$

where we take the continuous extension of $e_k^{(\alpha)}$ at $t = T$ for $-1/2 < \nu < 0$, i.e., $e_k^{(\alpha)}(T) = 0$ for $\alpha < 1/2$ (see part (i) *of Proposition 4.1).*

Proof. For simplicity we first assume that $T = 1$, and return to the interval $[0, T]$ at the end of the proof. We consider the cases $\alpha \neq 1/2$ and $\alpha = 1/2$ separately, since the covariance function $R^{(\alpha)}$ has different forms in these two cases.

If $\alpha \neq 1/2$, i.e., $\nu \neq 0$, then the covariance function of $\left(X_t^{(\alpha)}\right)_{t\in[0,1]}$ takes the form

$$R^{(\alpha)}(t, s) = \begin{cases} \frac{(1-s)^\alpha(1-t)^\alpha}{1-2\alpha}\left(1 - (1 - (s \wedge t))^{1-2\alpha}\right) & \text{if } s, t \in [0, 1), \\ 0 & \text{if } s \in \{0, 1\} \text{ or } t \in \{0, 1\}. \end{cases}$$

Let λ be a non-zero (and hence positive) eigenvalue of the integral operator $A_{R^{(\alpha)}}$. Then the eigenvalue equation is

$$\int_0^1 R^{(\alpha)}(t, s)\, e(s)\, ds = \lambda e(t), \qquad t \in [0, 1]. \tag{6}$$

Choosing $t = 0$ and $t = 1$ we have the boundary conditions $e(0) = e(1) = 0$. By (6), we have

$$\int_0^t R^{(\alpha)}(t, s)e(s)\, ds + \int_t^1 R^{(\alpha)}(t, s)\, e(s)\, ds = \lambda e(t), \qquad t \in (0, 1),$$

and twice differentiation with respect to t (which can be done since the integrands are continuously differentiable with respect to t) gives

$$\lambda e'(t) = -\frac{\alpha\lambda}{1 - t}\, e(t) + (1 - t)^{-\alpha} \int_t^1 (1 - s)^\alpha e(s)\, ds, \tag{7}$$

$$\lambda e''(t) = -\frac{\alpha\lambda}{(1 - t)^2}\, e(t) - \frac{\alpha\lambda}{1 - t}\, e'(t) + \alpha(1 - t)^{-1-\alpha} \int_t^1 (1 - s)^\alpha e(s)\, ds - e(t) \tag{8}$$

for all $t \in (0, 1)$. For a detailed discussion, see Barczy and Iglói [4]. Combining the equations (7) and (8) we have

$$e''(t) = -\left(\frac{1}{\lambda} + \frac{\alpha(1 - \alpha)}{(1 - t)^2}\right) e(t), \qquad t \in (0, 1). \tag{9}$$

Let us introduce the function $f : [0, 1/\sqrt{\lambda}] \to \mathbb{R}$,

$$f(x) = e\left(1 - \sqrt{\lambda}x\right), \qquad x \in [0, 1/\sqrt{\lambda}].$$

By (9), we obtain the Riccati–Bessel differential equation (first studied by Plana, see Watson [36, p. 95])

$$x^2 f''(x) + \left(x^2 - (\alpha - 1)\alpha\right) f(x) = 0, \qquad x \in \left(0, 1/\sqrt{\lambda}\right) \tag{10}$$

with the boundary conditions $f(0) = f(1/\sqrt{\lambda}) = 0$.

In the other case, when $\alpha = 1/2$, i.e., $\nu = 0$, the covariance function of $\left(X_t^{(\alpha)}\right)_{t\in[0,1]}$ takes the form

$$R^{(\alpha)}(t, s) = \begin{cases} -\sqrt{(1 - s)(1 - t)} \ln\left(1 - (s \wedge t)\right) & \text{if } s, t \in [0, 1), \\ 0 & \text{if } s \in \{0, 1\} \text{ or } t \in \{0, 1\}. \end{cases}$$

Performing the same steps as above, we obtain the differential equations (7), (8) and (9) with $\alpha = 1/2$.

According to Watson [36, pp. 76, 83, 95], the functions $\sqrt{x}\, J_\nu(x)$ and $\sqrt{x}\, Y_\nu(x)$, $x \in \left(0, 1/\sqrt{\lambda}\right)$, form a fundamental system of solutions for (10), and the general solution is of the form

$$f(x) = a_\nu \sqrt{x}\, J_\nu(x) + b_\nu \sqrt{x}\, Y_\nu(x), \qquad x \in \left(0, 1/\sqrt{\lambda}\right),$$

where a_ν and b_ν are arbitrary periodic functions of ν with period 1.

Let us assume temporarily that $\alpha > 1$, i.e., $\nu > 1/2$. Then the boundary condition $f(0) = 0$ and $\lim_{x\downarrow 0} \left(\sqrt{x}\, J_\nu(x)\right) = 0$ (see part (i) of Proposition 4.1) imply that $b_\nu \lim_{x\downarrow 0} \left(\sqrt{x}\, Y_\nu(x)\right) = 0$. However, since $\nu > 1/2$, by (44), we get

$$\lim_{x\downarrow 0}\left(\sqrt{x}\, Y_\nu(x)\right) = \lim_{x\downarrow 0}\left(x^{1/2-\nu} x^\nu\, Y_\nu(x)\right) = \frac{-2^\nu\, \Gamma(\nu)}{\pi} \lim_{x\downarrow 0} x^{1/2-\nu} = -\infty.$$

Therefore b_ν must be zero and because of the periodicity, $b_\nu = 0$ even for $\nu \in (-1/2, 1/2]$, i.e., for $\alpha \in (0, 1]$. Hence we can drop the assumption $\alpha > 1$ and obtain that $b_\nu = 0$ for all $\nu > -1/2$, i.e., for all $\alpha > 0$.

So, the other boundary condition $f\left(1/\sqrt{\lambda}\right) = 0$ implies $a_\nu J_\nu\left(1/\sqrt{\lambda}\right) = 0$. If $a_\nu = 0$, then $f(x) = 0$, $x \in \left(0, 1/\sqrt{\lambda}\right)$, which yields that $e(t) = 0$, $t \in (0, 1)$. However, since e has to be an eigenfunction corresponding to the eigenvalue λ, it cannot be identically zero and hence we have $a_\nu \neq 0$. This yields that $J_\nu\left(1/\sqrt{\lambda}\right) = 0$. Hence the eigenvalues $\lambda_k^{(\alpha)} = 1/\left(z_k^{(\nu)}\right)^2$, $k \in \mathbb{N}$, are the reciprocals of the squares of the (positive) zeros of J_ν. The corresponding normed eigenfunctions are

$$e_k^{(\alpha)}(t) = z_k^{(\nu)}\left(\int_0^{z_k^{(\nu)}} s\, J_\nu^2(s)\, ds\right)^{-1/2}\sqrt{1-t}\, J_\nu\left(z_k^{(\nu)}(1-t)\right), \qquad t \in [0, 1],$$

and using (47) we obtain

$$e_k^{(\alpha)}(t) = \sqrt{2(1-t)}\, \frac{J_\nu\left(z_k^{(\nu)}(1-t)\right)}{\left|J_{\nu+1}\left(z_k^{(\nu)}\right)\right|} = \sqrt{2(1-t)}\, \frac{J_\nu\left(z_k^{(\nu)}(1-t)\right)}{\left|J_{\nu-1}\left(z_k^{(\nu)}\right)\right|}, \qquad t \in [0, 1],$$

where we take the continuous extension of $e_k^{(\alpha)}$ at $t = 1$ for $0 < \alpha < 1/2$, i.e.,

$$e_k^{(\alpha)}(1) = \frac{1}{\left|J_{\nu+1}\left(z_k^{(\nu)}\right)\right|} \lim_{t\uparrow 1}\left(\sqrt{2(1-t)}\, J_\nu\left(z_k^{(\nu)}(1-t)\right)\right) = 0$$

for $0 < \alpha < 1/2$ (see part (i) of Proposition 4.1). We also note that $J_{\nu+1}\left(z_k^{(\nu)}\right) \neq 0$ and $J_{\nu-1}\left(z_k^{(\nu)}\right) \neq 0$, by part (iii) of Proposition 4.1.

Now we return to the interval $[0, T]$. Using (3) (and recalling that we have already extended the covariance function $R^{(\alpha)}$ continuously onto $[0, T]^2$) we obtain the scaling law

$$R^{(\alpha,T)}(s, t) = T R^{(\alpha,1)}\left(\frac{s}{T}, \frac{t}{T}\right), \qquad s, t \in [0, T], \tag{11}$$

where $R^{(\alpha,T)}$ denotes the covariance function on $[0, T]^2$ of the α-Wiener bridge on the time interval $[0, T]$. Let $\lambda^{(\alpha,T)}$ be a non-zero (and hence positive) eigenvalue of the integral operator $A_{R^{(\alpha,T)}}$, and $e^{(\alpha,T)}$ be a normed eigenfunction. Using (11), writing down the eigenvalue equations for $R^{(\alpha,T)}$ and $R^{(\alpha,1)}$ and taking into account part (iii) of Proposition 4.1, one can obtain

$$\lambda_k^{(\alpha,T)} = T^2 \lambda_k^{(\alpha,1)}, \qquad e_k^{(\alpha,T)}(t) = \frac{1}{\sqrt{T}}\, e_k^{(\alpha,1)}\left(\frac{t}{T}\right), \qquad t \in [0, T], \quad k \in \mathbb{N}.$$

For a detailed discussion see Barczy and Iglói [4]. Hence the statement of the theorem follows. □

We note that 0 is not an eigenvalue of the integral operator $A_{R^{(\alpha)}}$, see Barczy and Iglói [4, Remark 2.2].

In the next remark we will study the convergence of the coefficients of the random variables in the terms on the right-hand side of the KL representation (4) as $\alpha \downarrow 0$.

Remark 2.2.

First we recall that in the case of $\alpha = 0$ the process $\left(X_t^{(0)}\right)_{t \in [0,T)}$ is the standard Wiener process on $[0,T]$. If $\alpha \downarrow 0$ then the left–hand side of the KL representation (4) converges in $L^2(\Omega, \mathcal{A}, P)$ to the standard Wiener process $(B_t)_{t \in [0,T]}$ uniformly in $t \in [0,S]$ on every interval $[0,S] \subset [0,T)$. Hereafter we show that the coefficients of the random variables in the terms on the right–hand side of (4) also converge uniformly in $t \in [0,S]$ to those of the corresponding terms of the KL expansion of $(B_t)_{t \in [0,T]}$, based on $[0,T]$. For the latter KL representation see, e.g., Papoulis [32, Example 12.10] (which unfortunately contains a misprint). Indeed, exploiting the fact that the eigenfunctions are unique only up to sign, the KL expansion of $(B_t)_{t \in [0,T]}$, based on $[0,T]$, can be written in the form

$$B_t = \sum_{k=1}^{\infty} \eta_k (-1)^{k-1} \sqrt{2T} \, \frac{\sin\left((k-1/2)\pi t/T\right)}{(k-1/2)\pi} \tag{12}$$

for all $t \in [0,T]$, where η_k, $k \in \mathbb{N}$, are independent, standard normally distributed random variables. Moreover, using Theorem 2.1, parts (ii) and (vii) of Proposition 4.1, and identities $z_k^{(-1/2)} = (k-1/2)\pi$, $k \in \mathbb{N}$, we obtain

$$\lim_{\alpha \downarrow 0} \left(\sqrt{\lambda_k^{(\alpha)}} e_k^{(\alpha)}(t) \right) = (-1)^{k-1} \sqrt{2T} \, \frac{\sin\left((k-1/2)\pi t/T\right)}{(k-1/2)\pi}$$

for each $k \in \mathbb{N}$, with the convergence being uniform in $t \in [0,S]$. For a detailed discussion see Barczy and Iglói [4, Remark 2.3].

In the next remark we consider the special case $\alpha = 1$ of Theorem 2.1.

Remark 2.3.

If $\alpha = 1$, i.e., $\nu = 1/2$, then by Theorem 2.1, part (vii) of Proposition 4.1 and the identities $z_k^{(1/2)} = k\pi$, $k \in \mathbb{N}$, we obtain that the eigenvalue–normed eigenfunction pairs are $\lambda_k^{(1)} = T^2/(k\pi)^2$ and

$$e_k^{(1)}(t) = (-1)^k \sqrt{\frac{2}{T}} \, \sin \frac{k\pi t}{T}, \qquad t \in [0,T], \quad k \in \mathbb{N}.$$

Further,

$$X_t^{(1)} = \sqrt{2T} \sum_{k=1}^{\infty} \xi_k (-1)^k \frac{\sin\left(k\pi t/T\right)}{k\pi} \overset{\mathcal{L}}{=} \sqrt{2T} \sum_{k=1}^{\infty} \xi_k \frac{\sin\left(k\pi t/T\right)}{k\pi}$$

for all $t \in [0,T]$, where $\overset{\mathcal{L}}{=}$ denotes equality in distribution. As we expected, this is the KL expansion of the Wiener bridge, see, e.g., Deheuvels [12, Remark 1.1] or Gutiérrez and Valderrama [16, formula (10)].

In the next remark we present a space–time transformed Wiener process representation of the α-Wiener bridge, needed further on. The idea comes from the similar representation of the Wiener bridge, see, e.g., Csörgő and Révész [11, Proposition 1.4.2], and from Barczy and Pap [5, proof of Lemma 3.1]. For historical fidelity we note that our representation (for the α-Wiener bridge) is an analogue of the formula (17) in Brennan and Schwartz [9].

Remark 2.4.

Let $(W_u)_{u \geq 0}$ be a standard Wiener process,

$$\tau_T^{(\alpha)}(t) = \int_0^t \frac{1}{(T-s)^{2\alpha}} \, ds = \frac{R^{(\alpha)}(t,t)}{(T-t)^{2\alpha}}, \qquad t \in [0,T), \tag{13}$$

and

$$Z_t^{(\alpha)} = (T-t)^\alpha W_{\tau_T^{(\alpha)}(t)}, \qquad t \in [0,T). \tag{14}$$

Since $\tau_T^{(\alpha)}(0) = 0$ and $\tau_T^{(\alpha)}$ is strictly increasing and continuous, $\left(Z_t^{(\alpha)}\right)_{t\in[0,T)}$ can be called a space–time transformed Wiener process. One can see at once that this is a Gaussian process with almost surely continuous sample paths, zero mean and the covariance function (3). Therefore the process $\left(Z_t^{(\alpha)}\right)_{t\in[0,T)}$ is a weak solution of the SDE (1). Since the SDE (1) has a strong solution which is pathwise unique, we get $\left(Z_t^{(\alpha)}\right)_{t\in[0,T)}$ is an α-Wiener bridge (there exists some appropriate standard Wiener process for which (2) holds).

In the following we deal with the weighted KL expansion of the α-Wiener bridge. The series expansion which we call the weighted KL expansion of a space–time transformed centered process with continuous covariance function was introduced by Gutiérrez and Valderrama [16]. Let $S \in (0, T)$ and $\mu_T^{(\alpha)}$ be a (weight) measure defined on (the Borel sets of) $[0, S]$ with the help of the space–time transform in (14) as

$$\mathrm{d}\mu_T^{(\alpha)}(s) = (T - s)^{-2\alpha}\,\mathrm{d}\tau_T^{(\alpha)}(s) = (T - s)^{-4\alpha}\mathrm{d}s,$$

and let us denote by $L^2\big([0, S], \mu_T^{(\alpha)}\big)$ the Hilbert space of measurable functions on $[0, S]$ which are square integrable with respect to $\mu_T^{(\alpha)}$. Furthermore, let

$$W_u = \sum_{k=1}^{\infty} \sqrt{\kappa_k^{(\alpha)}}\, \xi_k\, d_k^{(\alpha)}(u), \qquad u \in \big[0, \tau_T^{(\alpha)}(S)\big], \tag{15}$$

be the (unweighted) KL expansion of the standard Wiener process $(W_u)_{u\in[0,\tau_T^{(\alpha)}(S)]}$, based on $\big[0, \tau_T^{(\alpha)}(S)\big]$, i.e., $\big(\kappa_k^{(\alpha)}, d_k^{(\alpha)}\big)$, $k \in \mathbb{N}$, are the eigenvalue–normed eigenfunction pairs of the integral operator associated to the covariance function of the standard Wiener process (for explicit formulae see (18) and (19) later on) and ξ_k, $k \in \mathbb{N}$, are independent standard normally distributed random variables. Finally, let

$$f_k^{(\alpha)}(t) = (T - t)^{\alpha}\, d_k^{(\alpha)}\big(\tau_T^{(\alpha)}(t)\big), \qquad t \in [0, S], \quad k \in \mathbb{N}, \tag{16}$$

i.e., we apply the same time change and rescaling to the normed eigenfunction $d_k^{(\alpha)}$ in order to define $f_k^{(\alpha)}$; we apply it to a standard Wiener process in order to get an α-Wiener bridge, see (14). Using (14) and Gutiérrez and Valderrama [16, Theorem 1], the weighted KL expansion of the α-Wiener bridge $\big(X_t^{(\alpha)}\big)_{t\in[0,T]}$, based on $[0, S]$, with respect to the weight measure $\mu_T^{(\alpha)}$ is

$$X_t^{(\alpha)} = \sum_{k=1}^{\infty} \sqrt{\kappa_k^{(\alpha)}}\, \xi_k\, f_k^{(\alpha)}(t), \qquad t \in [0, S]. \tag{17}$$

It also follows that the properties of the weighted KL expansion (17) and the weighted normed eigenfunctions (16) are completely analogous to those of (15) and the unweighted normed eigenfunctions therein. The difference is in the measure with respect to which we integrate. Namely, in the weighted case we integrate with respect to a Lebesgue–Stieltjes measure and there is the $L^2\big([0, S], \mu_T^{(\alpha)}\big)$ space in the background, instead of the Lebesgue measure and the $L^2\big([0, \tau_T^{(\alpha)}(S)]\big)$ space in the unweighted case. So, the series in (17) is convergent in $L^2(\Omega, \mathcal{A}, P)$ uniformly in $t \in [0, S]$ and

$$\int_0^S R^{(\alpha)}(t, s)\, f_k^{(\alpha)}(s)\,\mathrm{d}\mu_T^{(\alpha)}(s) = \kappa_k^{(\alpha)} f_k^{(\alpha)}(t), \qquad t \in [0, S], \quad k \in \mathbb{N},$$

$$\int_0^S \big(f_k^{(\alpha)}(s)\big)^2\,\mathrm{d}\mu_T^{(\alpha)}(s) = 1, \qquad \int_0^S f_k^{(\alpha)}(s)\, f_\ell^{(\alpha)}(s)\,\mathrm{d}\mu_T^{(\alpha)}(s) = 0, \qquad k \neq \ell, \quad k, \ell \in \mathbb{N}.$$

By Papoulis [32, Example 12.10] (watch a misprint there), we have

$$\kappa_k^{(\alpha)} = \left(\frac{\tau_T^{(\alpha)}(S)}{(k - 1/2)\,\pi}\right)^2, \qquad k \in \mathbb{N}, \tag{18}$$

$$d_k^{(\alpha)}(u) = \sqrt{\frac{2}{\tau_T^{(\alpha)}(S)}}\, \sin\frac{(k - 1/2)\,\pi u}{\tau_T^{(\alpha)}(S)}, \qquad u \in \big[0, \tau_T^{(\alpha)}(S)\big], \quad k \in \mathbb{N}, \tag{19}$$

and then using (16)–(17) we obtain the following theorem.

Theorem 2.5.
In the weighted KL expansion (17) *of the α-Wiener bridge the weighted eigenvalues $\kappa_k^{(\alpha)}$, $k \in \mathbb{N}$, are given by* (18) *and the corresponding weighted normed eigenfunctions*

$$f_k^{(\alpha)}(t) = \sqrt{\frac{2}{\tau_T^{(\alpha)}(S)}} (T-t)^\alpha \sin \frac{(k-1/2)\,\pi\,\tau_T^{(\alpha)}(t)}{\tau_T^{(\alpha)}(S)}, \qquad t \in [0,S], \quad k \in \mathbb{N}. \tag{20}$$

Hence

$$X_t^{(\alpha)} = \sum_{k=1}^{\infty} \sqrt{2\tau_T^{(\alpha)}(S)}\,\frac{(T-t)^\alpha}{(k-1/2)\,\pi}\,\sin \frac{(k-1/2)\,\pi\,\tau_T^{(\alpha)}(t)}{\tau_T^{(\alpha)}(S)}\,\xi_k, \qquad t \in [0,S].$$

In the next remark we will study the convergence of the coefficients of the random variables in the terms on the right–hand side of (17) as $\alpha \downarrow 0$.

Remark 2.6.
If $\alpha \downarrow 0$, then the left–hand side of the weighted KL representation (17) converges in $L^2(\Omega, \mathcal{A}, P)$ to B_t uniformly in $t \in [0,S]$, see the beginning of Remark 2.2. One can show that the coefficients of the random variables in the terms on the right–hand side of (17) (given with the help of (13), (18) and (20)) converge uniformly in $t \in [0,S]$ to the coefficients of the corresponding terms of the (unweighted) KL expansion of the standard Wiener process, based on $[0,S]$. For a detailed discussion see Barczy and Iglói [4, Remark 2.7].

In the next remark we consider the special case $\alpha = 1$ of Theorem 2.5.

Remark 2.7.
If $\alpha = 1$, then $\tau_T^{(\alpha)}$ (given by (13)) takes the form $\tau_T^{(1)}(t) = t/(T(T-t))$, $t \in [0,T)$, so (18) takes the form

$$\kappa_k^{(1)} = \left(\frac{S}{T(T-S)}\right)^2 \frac{1}{(k-1/2)^2\pi^2}, \qquad k \in \mathbb{N},$$

and (20) becomes

$$f_k^{(1)}(t) = \sqrt{\frac{2T\,(T-S)}{S}}\,(T-t)\sin \frac{(k-1/2)\,\pi t\,(T-S)}{S\,(T-t)}, \qquad t \in [0,S], \quad k \in \mathbb{N}.$$

In particular, for $T = 1$ and $S = 1/2$ we reobtain the weighted KL expansion

$$X_t^{(1)} = \sqrt{2}\,(1-t)\sum_{k=1}^{\infty} \xi_k\,\frac{\sin\left((k-1/2)\,\pi t/(1-t)\right)}{(k-1/2)\,\pi}, \qquad t \in [0,1/2],$$

given by Gutiérrez and Valderrama [16, formula (12)].

In the next remark we formulate a corollary of Theorem 2.5 in the case $\alpha = 1/2$.

Remark 2.8.
For all $0 < S < T$, we have

$$\int_0^S \frac{\left(X_u^{(1/2)}\right)^2}{(T-u)^2}\,du = \left(\ln\left(\frac{T}{T-S}\right)\right)^2 \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2\pi^2}\,\xi_k^2, \tag{21}$$

where ξ_k, $k \in \mathbb{N}$, are independent standard normally distributed random variables. For a detailed discussion see Barczy and Iglói [4, Remark 2.9].

3. Applications

In this section we present some applications of the KL expansion (4) given in Theorem 2.1. First we calculate the Laplace transform of the $L^2[0, T]$-norm square of $\left(X_t^{(\alpha)}\right)_{t\in[0,T]}$.

Proposition 3.1.
Let $T > 0$, $\alpha > 0$ and $\nu = \alpha - 1/2$. Then

$$\mathsf{E}\exp\left\{-c\int_0^T \left(X_u^{(\alpha)}\right)^2 du\right\} = \prod_{k=1}^{\infty} \frac{1}{\sqrt{1 + 2c\, T^2/\left(z_k^{(\nu)}\right)^2}}, \qquad c \geq 0, \tag{22}$$

$$\mathsf{E}\exp\left\{-c\int_0^1 \left(X_u^{(0)}\right)^2 du\right\} = \frac{1}{\sqrt{\cosh\sqrt{2c}}}, \qquad c \geq 0, \tag{23}$$

$$\mathsf{E}\exp\left\{-c\int_0^1 \left(X_u^{(1)}\right)^2 du\right\} = \sqrt{\frac{\sqrt{2c}}{\sinh\sqrt{2c}}}, \qquad c > 0. \tag{24}$$

Further, for all $0 < S < T$,

$$\mathsf{E}\exp\left\{-c\int_0^S \frac{\left(X_u^{(1/2)}\right)^2}{(T-u)^2} du\right\} = \frac{1}{\sqrt{\cosh\left(\sqrt{2c}\ln\left(T/(T-S)\right)\right)}}, \qquad c \geq 0. \tag{25}$$

Proof. By (4), we have

$$\left(X_t^{(\alpha)}\right)^2 = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \sqrt{\lambda_k^{(\alpha)}\lambda_\ell^{(\alpha)}}\, \xi_k\, \xi_\ell\, e_k^{(\alpha)}(t)\, e_\ell^{(\alpha)}(t), \qquad t \in [0, T],$$

and hence using the fact that $e_k^{(\alpha)}$, $k \in \mathbb{N}$, is an orthonormal system in $L^2[0, T]$ we get

$$\int_0^T \left(X_u^{(\alpha)}\right)^2 du = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \sqrt{\lambda_k^{(\alpha)}\lambda_\ell^{(\alpha)}}\, \xi_k\, \xi_\ell \int_0^T e_k^{(\alpha)}(u)\, e_\ell^{(\alpha)}(u)\, du = \sum_{k=1}^{\infty} \lambda_k^{(\alpha)}\xi_k^2, \tag{26}$$

which is nothing else but the Parseval identity in $L^2[0, T]$. Since ξ_k, $k \in \mathbb{N}$, are independent standard normally distributed random variables, for all $c \geq 0$ we get

$$\mathsf{E}\exp\left\{-c\int_0^T \left(X_u^{(\alpha)}\right)^2 du\right\} = \prod_{k=1}^{\infty}\mathsf{E}\, e^{-c\lambda_k^{(\alpha)}\xi_k^2} = \prod_{k=1}^{\infty}\frac{1}{\sqrt{1 + 2c\lambda_k^{(\alpha)}}},$$

which yields (22).

Now we turn to prove (23). First we show that for all $c \geq 0$,

$$\lim_{\alpha\downarrow 0}\mathsf{E}\exp\left\{-c\int_0^1 \left(X_u^{(\alpha)}\right)^2 du\right\} = \mathsf{E}\exp\left\{-c\int_0^1 \left(X_u^{(0)}\right)^2 du\right\}, \tag{27}$$

$$\lim_{\alpha\downarrow 0}\prod_{k=1}^{\infty}\frac{1}{\sqrt{1 + 2c\, T^2/\left(z_k^{(\nu)}\right)^2}} = \prod_{k=1}^{\infty}\frac{1}{\sqrt{1 + 2c\, T^2/\left(z_k^{(-1/2)}\right)^2}}. \tag{28}$$

First we check (27). By Cauchy–Schwarz's inequality and Minkowski's inequality we get

$$
\begin{aligned}
\mathsf{E}\left|\left(X_u^{(\alpha)}\right)^2 - \left(X_u^{(0)}\right)^2\right| &= \mathsf{E}\left(\left|X_u^{(\alpha)} - X_u^{(0)}\right| \cdot \left|X_u^{(\alpha)} + X_u^{(0)}\right|\right) \\
&\leq \left\|X_u^{(\alpha)} - X_u^{(0)}\right\|_2 \cdot \left(\left\|X_u^{(\alpha)}\right\|_2 + \left\|X_u^{(0)}\right\|_2\right), \qquad u \in [0,1),
\end{aligned}
\tag{29}
$$

where $\|\cdot\|_2$ denotes the $L^2(\Omega, \mathcal{A}, \mathsf{P})$-norm. According to the beginning of Remark 2.2,

$$
\lim_{\alpha \downarrow 0} \left\|X_u^{(\alpha)} - X_u^{(0)}\right\|_2 = \lim_{\alpha \downarrow 0} \left\|X_u^{(\alpha)} - B_u\right\|_2 = 0
$$

uniformly in u from any subinterval of $[0,1)$, where $(B_t)_{t \in [0,1]}$ is a standard Wiener process. Obviously, $\left\|X_u^{(0)}\right\|_2^2 = \mathsf{E}\,B_u^2 = u \leq 1$ if $u \in [0,1]$, and using (3) we get for all $\alpha \in (0, 1/4)$ and $u \in [0,1]$,

$$
\left\|X_u^{(\alpha)}\right\|_2^2 = \mathsf{E}\left(X_u^{(\alpha)}\right)^2 = \frac{(1-u)^{2\alpha}}{1-2\alpha}\left(1 - (1-u)^{1-2\alpha}\right) = \frac{(1-u)^{2\alpha} - (1-u)}{1-2\alpha} \leq 2.
$$

Hence, by (29),

$$
\mathsf{E}\left|\left(X_u^{(\alpha)}\right)^2 - \left(X_u^{(0)}\right)^2\right| \leq 3\left\|X_u^{(\alpha)} - X_u^{(0)}\right\|_2 \leq 3\left\|X_u^{(\alpha)}\right\|_2 + 3\left\|X_u^{(0)}\right\|_2 \leq 9, \qquad u \in [0,1].
$$

Now we can use Lebesgue's dominated convergence theorem to obtain

$$
\lim_{\alpha \downarrow 0} \mathsf{E}\left|\int_0^1 \left(X_u^{(\alpha)}\right)^2 \mathrm{d}u - \int_0^1 \left(X_u^{(0)}\right)^2 \mathrm{d}u\right| \leq \lim_{\alpha \downarrow 0} \int_0^1 \mathsf{E}\left|\left(X_u^{(\alpha)}\right)^2 - \left(X_u^{(0)}\right)^2\right| \mathrm{d}u \leq 3 \lim_{\alpha \downarrow 0} \int_0^1 \left\|X_u^{(\alpha)} - X_u^{(0)}\right\|_2 \mathrm{d}u = 0.
$$

This is the L^1-convergence of the integrals which yields their convergence in distribution and so, the convergence (27) of their Laplace transforms.

To prove (28) we rewrite the infinite product on the left hand side of (28) in the following form

$$
\prod_{k=1}^{\infty} \frac{1}{\sqrt{1 + 2c\, T^2/\left(z_k^{(\nu)}\right)^2}} = \exp\left\{-\frac{1}{2} \sum_{k=1}^{\infty} \ln\left(1 + 2c\, \frac{T^2}{\left(z_k^{(\nu)}\right)^2}\right)\right\}.
$$

Each zero $z_k^{(\nu)}$ of J_ν is a strictly increasing and continuous function of $\nu \in (-1, \infty)$ (see part (iii) of Proposition 4.1), whence

$$
\lim_{\alpha \downarrow 0} \ln\left(1 + 2c\, \frac{T^2}{\left(z_k^{(\nu)}\right)^2}\right) = \ln\left(1 + 2c\, \frac{T^2}{\left(z_k^{(-1/2)}\right)^2}\right), \qquad k \in \mathbb{N}.
$$

For each $k \in \mathbb{N}$ the function $(-1, \infty) \ni \nu \mapsto \ln\left(1 + 2c\, T^2/\left(z_k^{(\nu)}\right)^2\right)$ is decreasing, therefore the monotone convergence theorem can be applied to change the order of the limit and the infinite sum, and after all, the order of the limit and the infinite product on the left-hand side of (28).

The proof of (23) can be finished using (22) and Abramowitz and Stegun [1, formula 4.5.69]. Indeed, we get

$$
\mathsf{E}\exp\left\{-c\int_0^1 \left(X_u^{(0)}\right)^2 \mathrm{d}u\right\} = \prod_{k=1}^{\infty} \frac{1}{\sqrt{1 + 2c/\left((k-1/2)^2 \pi^2\right)}} = \frac{1}{\sqrt{\cosh\sqrt{2c}}}.
$$

This is a well-known formula due to Lévy [20], see also Liptser and Shiryaev [23, formula (7.147)].

Now we turn to prove (24). If $T = 1$ and $\alpha = 1$ (i.e., in the case of a Wiener bridge from 0 to 0 on the time interval $[0, 1]$), by (22), part (vii) of Proposition 4.1 and Abramowitz and Stegun [1, formula 4.5.68], for all $c > 0$,

$$\mathrm{E} \exp \left\{ -c \int_0^1 \left(X_u^{(1)} \right)^2 \mathrm{d}u \right\} = \prod_{k=1}^\infty \frac{1}{\sqrt{1 + 2c/(k^2 \pi^2)}} = \sqrt{\frac{\sqrt{2c}}{\sinh \sqrt{2c}}}.$$

This is also a well-known formula, see, e.g., Borodin and Salminen [7, formula 1.9.7].

Now we turn to prove (25). By (21) we get for all $c \geq 0$

$$\mathrm{E} \exp \left\{ -c \int_0^S \frac{\left(X_u^{(1/2)} \right)^2}{(T - u)^2} \mathrm{d}u \right\} = \mathrm{E} \prod_{k=1}^\infty \exp \left\{ -c \left(\ln \left(\frac{T}{T - S} \right) \right)^2 \frac{\xi_k^2}{(k - 1/2)^2 \pi^2} \right\}$$

$$= \prod_{k=1}^\infty \frac{1}{\sqrt{1 + 2c \left(\ln \left(\frac{T}{T-S} \right) \right)^2 / \left((k - 1/2)^2 \pi^2 \right)}} = \frac{1}{\sqrt{\cosh \left(\sqrt{2c} \ln \left(\frac{T}{T-S} \right) \right)}},$$

where the last equality follows again by Abramowitz and Stegun [1, formula 4.5.69]. We note that (25) yields

$$\mathrm{P} \left(\lim_{S \uparrow T} \int_0^S \frac{\left(X_u^{(1/2)} \right)^2}{(T - u)^2} \mathrm{d}u = \infty \right) = 1,$$

which was also obtained by Barczy and Pap [5, (4.6)] for general α-Wiener bridges. Indeed,

$$\lim_{S \uparrow T} \mathrm{E} \exp \left\{ -\int_0^S \frac{\left(X_u^{(1/2)} \right)^2}{(T - u)^2} \mathrm{d}u \right\} = 0,$$

and hence, by the Markov inequality, we get $\int_0^S \left(X_u^{(1/2)} \right)^2 / (T - u)^2 \, \mathrm{d}u$ converges in probability to infinity as $S \uparrow T$. Using Riesz's theorem and the fact that the (random) function

$$[0, T] \ni S \mapsto \int_0^S \frac{\left(X_u^{(1/2)} \right)^2}{(T - u)^2} \mathrm{d}u$$

is monotone increasing with probability one, we get the desired property. $\qquad \square$

We remark that a corresponding version of (25) for general α-Wiener bridges can be proved by a different technique, see Barczy and Pap [6, Theorem 4.1].

Next we give a simple probabilistic proof for the sum of the square of the reciprocals of the positive zeros of J_ν with $\nu > -1/2$. For $\nu > -1$ this is a well-known result due to Rayleigh, see, e.g., Watson [36, Section 15.51, p. 502]. We note that Yor [37, (11.47)–(11.49)] and Deheuvels and Martynov [14, Corollary 1.3] also gave probabilistic proofs of Rayleigh's results; we show that the proof of Deheuvels and Martynov can be carried through starting from the Karhunen–Loève expansion of the α-Wiener bridge as well.

Corollary 3.2.
Let $\alpha > 0$, $\nu = \alpha - 1/2$ and $z_k^{(\nu)}$, $k \in \mathbb{N}$, be the positive zeros of J_ν. Then

$$\sum_{k=1}^\infty \frac{1}{\left(z_k^{(\nu)} \right)^2} = \frac{1}{4 (\nu + 1)}.$$

Proof. Taking expectation of (26) we get

$$\int_0^T E\left(X_u^{(\alpha)}\right)^2 du = \sum_{k=1}^\infty \lambda_k^{(\alpha)} = T^2 \sum_{k=1}^\infty \frac{1}{\left(z_k^{(v)}\right)^2}.$$

By (3), if $\alpha \neq 1/2$ then

$$\int_0^T E\left(X_u^{(\alpha)}\right)^2 du = \int_0^T \left(\frac{T^{1-2\alpha}}{1-2\alpha}(T-u)^{2\alpha} - \frac{T-u}{1-2\alpha}\right) du = \frac{T^2}{2(1+2\alpha)},$$

and if $\alpha = 1/2$ then

$$\int_0^T E\left(X_u^{(\alpha)}\right)^2 du = \frac{T^2 \ln T}{2} - \left(\frac{T^2 \ln T}{2} - \frac{1}{2}\int_0^T (T-u) du\right) = \frac{T^2}{4}.$$

Hence

$$\sum_{k=1}^\infty \frac{1}{\left(z_k^{(\alpha-1/2)}\right)^2} = \frac{1}{2(1+2\alpha)} \qquad \text{for all} \quad \alpha > 0,$$

which implies the statement. □

A consequence of (22) is the following form of the survival function (complementary distribution function) of the $L^2[0, T]$-norm square of the α-Wiener bridge.

Proposition 3.3.
Let $\alpha > 0$ and $v = \alpha - 1/2$. Then

$$P\left(\int_0^T \left(X_t^{(\alpha)}\right)^2 dt > x\right) = \frac{2^{1-v/2}}{\pi\sqrt{\Gamma(v+1)}} \sum_{k=1}^\infty (-1)^{k+1} \int_{z_{2k-1}^{(v)}}^{z_{2k}^{(v)}} u^{v/2-1} \frac{e^{-xu^2/(2T^2)}}{\sqrt{|J_v(u)|}} du \tag{30}$$

for all $x > 0$.

Proof. Corollary 3.2 ensures that $\sum_{k=1}^\infty \lambda_k^{(\alpha)} < \infty$, so, (5) and the Smirnov formula (see, e.g., Smirnov [33, formula (97)] or Martynov [26, formula (4)], [27, formula (2)]) imply that

$$P\left(\int_0^T \left(X_t^{(\alpha)}\right)^2 dt > x\right) = \frac{1}{\pi} \sum_{k=1}^\infty (-1)^{k+1} \int_{(z_{2k-1}^{(v)})^2/T^2}^{(z_{2k}^{(v)})^2/T^2} \frac{e^{-ux/2}}{u\sqrt{|F(u)|}} du, \qquad x > 0, \tag{31}$$

where

$$F(u) = \prod_{k=1}^\infty \left(1 - \lambda_k^{(\alpha)} u\right) = \prod_{k=1}^\infty \left(1 - \frac{T^2}{\left(z_k^{(v)}\right)^2} u\right), \qquad u \geq 0, \tag{32}$$

is the so-called Fredholm determinant. Expressing this product by the help of the Euler formula (46) we obtain

$$F(u) = J_v(\sqrt{u}\, T) \frac{\Gamma(v+1)}{\left(\sqrt{u}\, T/2\right)^v}, \qquad u > 0. \tag{33}$$

With this replacement (31) becomes

$$P\left(\int_0^T \left(X_t^{(\alpha)}\right)^2 dt > x\right) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int_{(z_{2k-1}^{(\nu)})^2/T^2}^{(z_{2k}^{(\nu)})^2/T^2}\frac{e^{-ux/2}\left(\sqrt{u}\,T/2\right)^{\nu/2}}{u\,\sqrt{\Gamma(\nu+1)}\left|J_\nu(\sqrt{u}\,T)\right|}\,du$$

$$= \frac{2^{1-\nu/2}}{\pi\sqrt{\Gamma(\nu+1)}}\sum_{k=1}^{\infty}(-1)^{k+1}\int_{z_{2k-1}^{(\nu)}}^{z_{2k}^{(\nu)}}u^{\nu/2-1}\frac{e^{-xu^2/(2T^2)}}{\sqrt{\left|J_\nu(u)\right|}}\,du$$

for all $x > 0$. $\qquad\qquad\square$

The next remark says that the formula for the survival function of the $L^2[0,1]$-norm square of a standard Wiener process can be derived by taking the limit of (30) with $T = 1$ as $\alpha \downarrow 0$.

Remark 3.4.
The survival function of the $L^2[0,1]$-norm square of a standard Wiener process is

$$P\left(\int_0^1 B_t^2\,dt > x\right) = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int_{(2k-3/2)\pi}^{(2k-1/2)\pi}\frac{e^{-xu^2/2}}{u\,\sqrt{-\cos u}}\,du \qquad (34)$$

for all $x > 0$, see, e.g., Deheuvels and Martynov [14, formula (1.50)]. One can check that the right-hand side of (30) converges to the right-hand side of (34) for every $x > 0$, i.e., the survival function of the $L^2[0,1]$-norm square of a standard Wiener process is the limit of the survival function of the $L^2[0,1]$-norm square of the α-Wiener bridge (on the time interval $[0,1]$) as $\alpha \downarrow 0$. For a detailed discussion, see Barczy and Iglói [4, 3.4 Remark].

In the next remark we consider the Proposition 3.3 for particular values $\alpha = 1$ and $T = 1$.

Remark 3.5.
For $\alpha = 1$, i.e. $\nu = 1/2$, and $T = 1$ in Proposition 3.3 we have for all $x > 0$,

$$P\left(\int_0^1 \left(X_t^{(1)}\right)^2 dt > x\right) = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\int_{(2k-1)\pi}^{2k\pi}\frac{e^{-xu^2/2}}{\sqrt{-u\sin u}}\,du,$$

where we used part (vii) of Proposition 4.1. We reobtained the survival function of the $L^2[0,1]$-norm square of the Wiener bridge, see Deheuvels and Martynov [14, formula (1.51)].

Remark 3.6.
We note that Deheuvels and Martynov [14, formula (1.43)] gave an expression for the survival function of the $L^2[0,1]$-norm square of a particularly weighted time transformed Wiener bridge, namely $\left(t^{1/2-\nu}X_{t^{2\nu}}^{(1)}\right)_{t\in[0,1]}$, where $\left(X_t^{(1)}\right)_{t\in[0,1]}$ is a Wiener bridge on the time interval $[0,1]$ and $\nu > 0$. We can notice that the only difference between that formula and our formula (30) is the denominator of the fraction in the argument of the exponential function, namely instead of 4ν we have $2T^2$. This means that the distribution of the $L^2[0,1]$-norm square of the above mentioned particularly weighted time transformed Wiener bridge is the same as the distribution of the $L^2[0,T]$-norm square of an appropriate α-Wiener bridge. Namely, in case of $\alpha > 1/2$, i.e., $\nu > 0$, the random variables

$$\int_0^1 t^{1-2\nu}\left(X_{t^{2\nu}}^{(1)}\right)^2 dt = \int_0^1 t^{2(1-\alpha)}\left(X_{t^{2\alpha-1}}^{(1)}\right)^2 dt \qquad\text{and}\qquad \int_0^{\sqrt{2\alpha-1}}\left(X_t^{(\alpha)}\right)^2 dt$$

have the same distribution, where $\left(X_t^{(\alpha)}\right)_{t\in[0,\sqrt{2\alpha-1}]}$ is an α-Wiener bridge on the time interval $\left[0,\sqrt{2\alpha-1}\right]$.

Zolotarev [38, formula (6)] gives the tail behaviour of the distribution function of $\sum_{k=1}^{\infty} \lambda_k \xi_k^2$, where ξ_k, $k \in \mathbb{N}$, are independent standard normally distributed random variables and $(\lambda_k)_{k \in \mathbb{N}}$ is a sequence of positive real numbers such that $\lambda_1 > \lambda_2 > \ldots > 0$ and $\sum_{k=1}^{\infty} \lambda_k < \infty$ (see also Hwang [17, Theorem 1] or Deheuvels and Martynov [14, Lemma 1.1 and Remark 1.2]). This result can be directly applied together with Theorem 2.1 to obtain the following corollary about the large deviation probabilities for the $L^2[0, T]$-norm square of the α-Wiener bridge.

Corollary 3.7.
Let $\alpha > 0$, $\nu = \alpha - 1/2$ and $z_k^{(\nu)}$, $k \in \mathbb{N}$, be the positive zeros of J_ν. Then

$$\mathsf{P}\left(\int_0^T \left(X_t^{(\alpha)} \right)^2 dt > x \right) = (1 + o(1)) \frac{2^{1-\nu/2} \, T \left(z_1^{(\nu)} \right)^{(\nu-3)/2}}{\sqrt{\pi \, \Gamma(\nu+1) \, J_{\nu+1}\left(z_1^{(\nu)} \right)}} \, x^{-1/2} \exp\frac{-\left(z_1^{(\nu)} \right)^2 x}{2T^2} \tag{35}$$

$$= (1 + o(1)) \sqrt{\frac{2}{\pi}} \, \frac{T}{z_1^{(\nu)}} \prod_{k=2}^{\infty} \left(1 - \frac{\left(z_1^{(\nu)} \right)^2}{\left(z_k^{(\nu)} \right)^2} \right)^{-1/2} x^{-1/2} \exp\frac{-\left(z_1^{(\nu)} \right)^2 x}{2T^2} \tag{36}$$

as $x \to \infty$.

Proof. Theorem 2.1 and Zolotarev [38, formula (6)] (see alternatively Hwang [17, Theorem 1] or Deheuvels and Martynov [14, Lemma 1.1 and Remark 1.2]) yield that for all $x > 0$,

$$\mathsf{P}\left(\int_0^T \left(X_t^{(\alpha)} \right)^2 dt > x \right) = (1 + o(1)) \sqrt{\frac{2}{\pi}} \, \frac{T^2}{\left(z_1^{(\nu)} \right)^2 \sqrt{-F'\left(\left(z_1^{(\nu)} \right)^2 / T^2 \right)}} \, x^{-1/2} \exp\frac{-\left(z_1^{(\nu)} \right)^2 x}{2T^2}, \tag{37}$$

where F is defined in (32). Now we check that

$$F'\left(\frac{\left(z_1^{(\nu)} \right)^2}{T^2} \right) = -\Gamma(\nu+1) \, 2^{\nu-1} \, T^2 \left(z_1^{(\nu)} \right)^{-\nu-1} J_{\nu+1}\left(z_1^{(\nu)} \right). \tag{38}$$

By (33) we get for all $u > 0$,

$$F'(u) = \Gamma(\nu+1) \, 2^{\nu-1} \, T^{-\nu} u^{-1-\nu/2} \left(-\nu J_\nu\left(\sqrt{u} \, T \right) + T\sqrt{u} \, J_\nu'\left(\sqrt{u} \, T \right) \right),$$

and so,

$$F'\left(\frac{\left(z_1^{(\nu)} \right)^2}{T^2} \right) = \Gamma(\nu+1) \, 2^{\nu-1} \, T^{-\nu} \left(\frac{\left(z_1^{(\nu)} \right)^2}{T^2} \right)^{-1-\nu/2} \left(-\nu J_\nu\left(z_1^{(\nu)} \right) + z_1^{(\nu)} J_\nu'\left(z_1^{(\nu)} \right) \right) = \Gamma(\nu+1) \, 2^{\nu-1} \, T^2 \left(z_1^{(\nu)} \right)^{-\nu-1} J_\nu'\left(z_1^{(\nu)} \right).$$

By Watson [36, p. 45, (3)–(4)], the derivative of J_ν can be expressed as $J_\nu'(x) = \nu J_\nu(x)/x - J_{\nu+1}(x)$. This yields (38) and hence (35).

Now we turn to prove (36). For this it is enough to check that $F'\left(\left(z_1^{(\nu)} \right)^2 / T^2 \right)$ can be also written in the form

$$F'\left(\frac{\left(z_1^{(\nu)} \right)^2}{T^2} \right) = -\frac{T^2}{\left(z_1^{(\nu)} \right)^2} \prod_{k=2}^{\infty} \left(1 - \frac{\left(z_1^{(\nu)} \right)^2}{\left(z_k^{(\nu)} \right)^2} \right). \tag{39}$$

By (32), we have

$$F(u) = \left(1 - u\lambda_1^{(\alpha)} \right) \prod_{k=2}^{\infty} \left(1 - u\lambda_k^{(\alpha)} \right), \qquad u > 0,$$

and hence

$$F'(u) = -\lambda_1^{(\alpha)} \prod_{k=2}^{\infty} \left(1 - u\lambda_k^{(\alpha)}\right) + \left(1 - u\lambda_1^{(\alpha)}\right) \frac{\mathrm{d}}{\mathrm{d}u} \left(\prod_{k=2}^{\infty} \left(1 - u\lambda_k^{(\alpha)}\right)\right), \qquad u > 0.$$

This yields that

$$F'\left(\frac{1}{\lambda_1^{(\alpha)}}\right) = -\lambda_1^{(\alpha)} \prod_{k=2}^{\infty} \left(1 - \frac{\lambda_k^{(\alpha)}}{\lambda_1^{(\alpha)}}\right),$$

which implies (39) using (5). $\qquad\qquad\square$

The next corollary describes the behaviour at zero of the distribution function (small deviation probabilities) of the $L^2[0, T]$-norm square of the α-Wiener bridge.

Corollary 3.8.
Let $\alpha > 0$ and $v = \alpha - 1/2$. Then there exists some constant $c > 0$ such that

$$P\left(\int_0^T \left(X_t^{(\alpha)}\right)^2 \mathrm{d}t < \varepsilon\right) = (c + o(1))\, \varepsilon^{1/4 - v/2} \exp\frac{-T^2}{8\varepsilon} \tag{40}$$

as $\varepsilon \downarrow 0$.

Proof. We imitate the proof of Deheuvels and Martynov [14, Theorem 1.7], which originates from the idea of Li [21]. By (26) and (5) we have

$$\int_0^T \left(X_t^{(\alpha)}\right)^2 \mathrm{d}t = T^2 \sum_{k=1}^{\infty} \frac{\xi_k^2}{\left(z_k^{(v)}\right)^2} \tag{41}$$

with independent standard normally distributed random variables ξ_k, $k \in \mathbb{N}$. We are going to replace the zeros $z_k^{(v)}$ on the right-hand side of (41) by the first terms ('leading terms') on the right-hand side of (45). We use Theorem 2 in Li [21] to prove that this replacement can be done. Namely, given any two convergent series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with positive terms, the convergence condition $\sum_{k=1}^{\infty} |1 - a_k/b_k| < \infty$ implies that

$$P\left(\sum_{k=1}^{\infty} a_k \xi_k^2 \le \varepsilon\right) = (1 + o(1))\sqrt{\prod_{k=1}^{\infty} \frac{b_k}{a_k}}\; P\left(\sum_{k=1}^{\infty} b_k \xi_k^2 \le \varepsilon\right)$$

as $\varepsilon \downarrow 0$. Observe that $\sum_{k=1}^{\infty} |1 - a_k/b_k| < \infty$ yields the convergence of $\prod_{k=1}^{\infty}(b_k/a_k)$. Indeed, by mean value theorem,

$$\ln\frac{b_k}{a_k} = \ln\frac{b_k}{a_k} - \ln 1 = \frac{1}{\theta_k}\left(\frac{a_k}{b_k} - 1\right), \qquad k \in \mathbb{N},$$

with some θ_k from the interval with endpoints a_k/b_k and 1. Since $\lim_{k\to\infty} a_k/b_k = 1$, we have that the sequence θ_k, $k \in \mathbb{N}$, is bounded and hence the series $\sum_{k=1}^{\infty} \ln(b_k/a_k)$ is absolutely convergent. In what follows let $a_k = \left(z_k^{(v)}\right)^{-2}$, $b_k = \left(k + (v - 1/2)/2\right)^{-2}\pi^{-2}$, $k \in \mathbb{N}$. Then with the aid of (45) and the fact that $z_1^{(v)} \le z_k^{(v)}$, $k \in \mathbb{N}$, we obtain

$$\sum_{k=1}^{\infty} \left|1 - \frac{a_k}{b_k}\right| = \sum_{k=1}^{\infty} \left|1 - \left(\pi\frac{k + (v - 1/2)/2}{z_k^{(v)}}\right)^2\right| = \sum_{k=1}^{\infty} \frac{c_1}{k^2} < \infty$$

for some constant $c_1 > 0$, which yields that

$$\mathsf{P}\left(\sum_{k=1}^{\infty} \frac{\xi_k^2}{\left(z_k^{(\nu)}\right)^2} \le \frac{\varepsilon}{T^2}\right) = (1 + o(1))\sqrt{\prod_{k=1}^{\infty}\left(\pi \frac{k + (\nu - 1/2)/2}{z_k^{(\nu)}}\right)^{-2}}\,\mathsf{P}\left(\sum_{k=1}^{\infty}\frac{\xi_k^2}{\left(k + (\nu - 1/2)/2\right)^2} \le \frac{\varepsilon\pi^2}{T^2}\right)$$

$$= (c_2 + o(1))\,\mathsf{P}\left(\sum_{k=1}^{\infty}\frac{\xi_k^2}{\left(k + (\nu - 1/2)/2\right)^2} \le \frac{\varepsilon\pi^2}{T^2}\right) \tag{42}$$

as $\varepsilon \downarrow 0$, with a suitable constant $c_2 > 0$. By Li [21, formula (3.4), p. 14], for an arbitrary $d > -1$,

$$\mathsf{P}\left(\sum_{k=1}^{\infty}\frac{\xi_k^2}{(k + d)^2} \le \varepsilon\right) = (c_3 + o(1))\,\varepsilon^{-d}\exp\frac{-\pi^2}{8\varepsilon} \tag{43}$$

holds as $\varepsilon \downarrow 0$, with some constant $c_3 > 0$. Combining (41), (42) and (43) we obtain (40). $\qquad\square$

Remark 3.9.
In case of $\alpha \ge 1/2$, Corollary 3.8 can be improved in the sense that the constant c can be explicitly given. Namely, by Nazarov [28, Lemma 3.2], if ξ_k, $k \in \mathbb{N}$, are independent standard normally distributed random variables, then for all $\nu \ge 0$,

$$\mathsf{P}\left(\sum_{n=1}^{\infty}\frac{\xi_n^2}{\left(z_n^{(\nu)}\right)^2} \le \varepsilon^2\right) \sim \sqrt{\frac{\sqrt{\pi}}{2^{\nu - 1/2}\,\Gamma(1 + \nu)}}\,\frac{\sqrt{2}\,\varepsilon_1^{1/2 - \nu}}{\sqrt{\pi \mathcal{D}_1}}\exp\frac{-\mathcal{D}_1}{2\varepsilon_1^2},$$

as $\varepsilon \to 0$, where

$$\varepsilon_1 = \varepsilon\sqrt{2\sin(\pi/2)} = \sqrt{2}\,\varepsilon \qquad \text{and} \qquad \mathcal{D}_1 = \frac{1}{2\sin(\pi/2)} = \frac{1}{2}.$$

Hence

$$\mathsf{P}\left(\sum_{n=1}^{\infty}\frac{\xi_n^2}{\left(z_n^{(\nu)}\right)^2} \le \varepsilon^2\right) \sim \frac{2^{3/2 - \nu}\,\pi^{-1/4}}{\sqrt{\Gamma(1 + \nu)}}\,\varepsilon^{1/2 - \nu}\exp\frac{-1}{8\varepsilon^2} \qquad \text{as } \varepsilon \downarrow 0.$$

Then for all $T > 0$ and $\varepsilon > 0$ we have

$$\mathsf{P}\left(T^2\sum_{n=1}^{\infty}\frac{\xi_n^2}{\left(z_n^{(\nu)}\right)^2} \le \varepsilon\right) \sim \frac{2^{3/2 - \nu}\,\pi^{-1/4}}{\sqrt{\Gamma(1 + \nu)}}\,\frac{\varepsilon^{1/4 - \nu/2}}{T^{1/2 - \nu}}\exp\frac{-T^2}{8\varepsilon} \qquad \text{as } \varepsilon \downarrow 0.$$

By (41), this yields that in case of $\alpha \ge 1/2$ the constant c in Corollary 3.8 takes the following form

$$c = \frac{2^{3/2 - \nu}\,\pi^{-1/4}}{\sqrt{\Gamma(1 + \nu)}\,T^{1/2 - \nu}}.$$

The reason for restricting ourselves to the case $\alpha \ge 1/2$ is that Lemma 3.2 in Nazarov [28] is valid for $\nu \ge 0$, while in Corollary 3.8 we have $\nu = \alpha - 1/2$, $\alpha > 0$.

4. Appendix

In the next proposition we list some properties of the Bessel functions J_ν and Y_ν of the first and second kind, respectively (introduced in Section 2).

Proposition 4.1.

(i) *For all $\nu \in \mathbb{R}$, J_ν is continuous on $(0, \infty)$, and in case of $\nu \geq 0$ it can be continuously extended to $[0, \infty)$ by putting $J_\nu(0) = 0$ if $\nu > 0$ and $J_\nu(0) = 1$ if $\nu = 0$. However, in case of $-1/2 < \nu < 0$ we have $\lim_{x \downarrow 0} J_\nu(x) = \infty$, and $\lim_{x \downarrow 0} \left(\sqrt{x} J_\nu(x) \right) = 0$ holds for all $\nu > -1/2$. Further, for any $\nu > 0$,*

$$Y_\nu(x) = (1 + o(1)) \frac{-\Gamma(\nu)}{\pi} \left(\frac{x}{2} \right)^{-\nu} \qquad \text{as } x \downarrow 0. \tag{44}$$

(ii) *By Watson [36, p. 44], $(0, \infty) \times (-1, \infty) \ni (x, \nu) \mapsto J_\nu(x)$ is an analytic function of both variables.*

(iii) *By the Bessel–Lommel theorem, see, e.g., Watson [36, pp. 478, 482], if $\nu > -1$, then J_ν has infinitely many positive real zeros with multiplicities one. We denote them by $z_k^{(\nu)}$, $k \in \mathbb{N}$, where we assume*

$$0 < z_1^{(\nu)} < z_2^{(\nu)} < \ldots < z_k^{(\nu)} < z_{k+1}^{(\nu)} < \ldots$$

By Watson [36, p. 508] or Korenev [19, p. 96], for fixed $k \in \mathbb{N}$, $z_k^{(\nu)}$ is a strictly increasing and continuous function of $\nu \in (-1, \infty)$.

(iv) *By Korenev [19, p. 96], if $\nu > -1$, then*

$$z_k^{(\nu)} = \left(k + \frac{1}{2} \left(\nu - \frac{1}{2} \right) \right) \pi + O\left(\frac{1}{k} \right) \qquad \text{as } k \to \infty. \tag{45}$$

(v) *If $\nu > -1$, then J_ν can be written with the help of its zeros $z_k^{(\nu)}$, $k \in \mathbb{N}$, as*

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1)} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(z_k^{(\nu)})^2} \right), \qquad x \in (0, \infty). \tag{46}$$

This is the so-called Euler formula, see Watson [36, p. 498].

(vi) *By Bowman [8, p. 107], for $\nu > -1$ and for all zeros $z_k^{(\nu)}$, $k \in \mathbb{N}$, of J_ν, it holds that*

$$\int_0^{z_k^{(\nu)}} x J_\nu^2(x) \, dx = \frac{\left(z_k^{(\nu)} \right)^2}{2} J_{\nu+1}^2\left(z_k^{(\nu)} \right) = \frac{\left(z_k^{(\nu)} \right)^2}{2} J_{\nu-1}^2\left(z_k^{(\nu)} \right). \tag{47}$$

(vii) *The Bessel functions $J_{1/2}$ and $J_{-1/2}$ (of the first kind) can be expressed in closed forms by elementary functions:*

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad \text{and} \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \qquad x \in (0, \infty),$$

see, e.g., Watson [36, pp. 54, 55].

Proof. We only have to check (44), since all other statements are immediate consequences of the definitions or references given for them. If $\nu \in \mathbb{N}$, then (44) follows by Abramowitz and Stegun [1, p. 360, 9.1.9]. If $\nu > 0$ and $\nu \notin \mathbb{N}$, then

$$\lim_{x \downarrow 0} \left(x^\nu Y_\nu(x) \right) = -\frac{1}{\sin(\pi\nu)} \lim_{x \downarrow 0} \left(x^\nu J_{-\nu}(x) \right) = -\frac{2^\nu}{-\nu \, \Gamma(-\nu) \sin(\pi\nu)} = -\frac{2^\nu \Gamma(\nu)}{\pi},$$

where the last equality follows from the identity $\Gamma(z)\Gamma(-z) = -\pi/(z \sin(\pi z))$, $z > 0$. $\qquad\square$

Acknowledgements

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