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The group $\mathrm{Sp}_{10}(\mathbb{Z})$ is (2,3)-generated

Research Article

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Abstract: It is proved that the group $\mathrm{Sp}_{10}(\mathbb{Z})$ is generated by an involution and an element of order 3.

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1. Introduction

A group is called $(2,3)$-generated if it can be generated by an involution and an element of order 3. For instance, $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the projective images of the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

of (projective) order 2 and 3, respectively. Moreover, it is well known that $\mathrm{PSL}_2(\mathbb{Z})$ is isomorphic to the free product of the cyclic group of order 2 and the cyclic group of order 3. Thus, the problem of $(2,3)$-generation is closely related to the problem of description of the normal subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

L. Di Martino and N. Vavilov conjectured in [1, 2] that, for any finitely generated commutative ring R, elementary Chevalley groups over R are $(2,3)$-generated provided their rank is large enough. For classical matrix groups over finite fields this conjecture was settled affirmatively in [4]. For matrix groups over other finitely generated rings see e.g. [8, 9]. The latter results are only asymptotic, i.e., they do not give the answer for low-dimensional groups. However, for certain

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series of groups and certain rings the problem can be solved completely. For instance, joint efforts of several authors [6, 7, 11–14] led to the discovery that the groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$ are $(2,3)$-generated precisely when $n \geq 5$.

It turns out that the problem for the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{Z})$ is more delicate than for $\mathrm{SL}_n(\mathbb{Z})$, because all general results in the symplectic case either required invertibility of 2 in the ring under consideration [6] or dealt only with groups over finite fields [5]. Moreover, these methods cannot be directly transferred to $\mathrm{Sp}_{2n}(\mathbb{Z})$. The evidence arising from the solution of a similar problem for $\mathrm{SL}_n(\mathbb{Z})$ shows that low-dimensional cases require a separate treatment.

We began a systematic study of the $(2,3)$-generation problem for $\mathrm{Sp}_{2n}(\mathbb{Z})$ in [10], where the cases $n \leq 4$ were considered. We also conjectured that $\mathrm{Sp}_{2n}(\mathbb{Z})$ is $(2,3)$-generated precisely when $n \geq 4$. In the present paper we make the next step and give the affirmative answer for $\mathrm{Sp}_{10}(\mathbb{Z})$. Our methods are similar to those developed in [10], but even a subtle increase of the dimension led to a significant growth of computational efforts. It might be rather difficult to proceed in the same manner for larger values of n.

2. Main result and notation

Let I_n be the $n \times n$ identity matrix. Recall that up to conjugation

$$\mathrm{Sp}_{2n}(\mathbb{Z}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{Z}) : g^T J g = J \right\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Our main result is the following theorem.

Theorem 2.1.

The group $\mathrm{Sp}_{10}(\mathbb{Z})$ is $(2,3)$-generated. More precisely, define

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & -2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 3 & -3 & 1 \\ 0 & 1 & 1 & 0 & -2 & -1 & -3 & 0 & 1 & 3 \\ 0 & -2 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & -2 \\ 0 & 0 & 0 & -1 & -1 & -3 & -1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 2 & -2 & 1 & -1 & -3 & -3 & 2 & -5 & 6 & 3 \\ -2 & 5 & -3 & -4 & 3 & 2 & -7 & 0 & -1 & 1 \\ 3 & 3 & -2 & -1 & 2 & 5 & 0 & 5 & -1 & 3 \\ -1 & 0 & 0 & 2 & 1 & 1 & -1 & 1 & -4 & -3 \\ 1 & 1 & -1 & 1 & 2 & 3 & 1 & 3 & -3 & -2 \\ 1 & -1 & 1 & 3 & 0 & 2 & 2 & 3 & -3 & -1 \\ -3 & 5 & -3 & -4 & 4 & 2 & -6 & 1 & -2 & -1 \\ 2 & -3 & 2 & 3 & -3 & -1 & 3 & -1 & 1 & 1 \\ 3 & -4 & 3 & 7 & -3 & 1 & 4 & 1 & -3 & -1 \\ -2 & 4 & -3 & -3 & 5 & 3 & -3 & 2 & -3 & -3 \end{pmatrix}.$$

Then $x^2 = y^3 = I_{10}$ and $\mathrm{Sp}_{10}(\mathbb{Z}) = \langle x, y \rangle$.

Remark 2.2.

The method of finding x and y is similar to that used in [10]. The starting point is a pair of parametric matrices

$$x_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & r_1 & 0 & 0 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 1 & r_2 & 0 & 0 & 0 & 0 & r_4 \\ 1 & 0 & 0 & 0 & -r_1 & 0 & 0 & 0 & 0 & -r_3 \\ 0 & 1 & 0 & 0 & -r_2 & 0 & 0 & 0 & 0 & -r_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & r_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -r_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad y_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & r_7 & r_9 & 0 & r_{11} & 0 & r_{13} \\ 0 & 1 & 0 & 0 & r_8 & r_{10} & 0 & r_{12} & 0 & r_{14} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

of order 2 and 3, respectively. Next we try to find $r_1, \dots, r_{14}$ such that x_0, y_0 fix *some* skew-symmetric form J_0, which remains non-degenerate modulo any prime p. This assumption implies certain relations between $r_1, \dots, r_{14}$, which allow to reduce the search area. For instance, the following parameters will suit us:

$$r_1 = r_2 = r_{14} = -1,\ r_3 = 1,\ r_4 = -4,\ r_5 = 3,\ r_6 = -2,\ r_7 = r_{10} = r_{11} = 2,\ r_8 = r_9 = r_{12} = 0,\ r_{13} = 4.$$

Finally, we find an invertible integral matrix Z such that $J_0 = Z^T J Z$ and set $x = Z x_0 Z^{-1}$, $y = Z y_0 Z^{-1}$. In our case

$$Z = \begin{pmatrix} 0 & 5 & 0 & 4 & 0 & 3 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 & -4 & 2 & -1 & 1 \\ 2 & 8 & 1 & 7 & 0 & 5 & 5 & 1 & 2 & 1 \\ -1 & -5 & -1 & -4 & 0 & -3 & -3 & -1 & -1 & -2 \\ 0 & 0 & -2 & 1 & 0 & -2 & 2 & -3 & 1 & -2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -3 & 1 & -1 & -3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & -2 & -1 & -1 & 0 & -3 & 0 & -2 & 1 & 0 \end{pmatrix}.$$

We omit further computational details.

The claim about the orders of x and y is trivial. It is also straightforward to verify that $\langle x, y \rangle \subseteq \mathrm{Sp}_{10}(\mathbb{Z})$. To prove the converse inclusion we use the well-known fact (e.g. see Theorem 5.3.4 in [3]) that $\mathrm{Sp}_{2n}(\mathbb{Z})$ coincides with the elementary symplectic group $\mathrm{ESp}_{2n}(\mathbb{Z})$. Recall the definition of $\mathrm{ESp}_{2n}(\mathbb{Z})$. For $1 \le i, j \le 2n$, let $e_{i,j}$ be the $2n \times 2n$ matrix with 1 in the ith row and jth column and zeros elsewhere. Define

$$P_{i,j}(k) = \begin{cases} I_{2n} + k \cdot (e_{i,j+n} + e_{j,i+n}) & 1 \le i < j \le n, \\ I_{2n} + k \cdot e_{i,i+n} & 1 \le i = j \le n, \end{cases}$$

$$Q_{i,j}(k) = \begin{cases} I_{2n} + k \cdot (e_{i+n,j} + e_{j+n,i}) & 1 \le i < j \le n, \\ I_{2n} + k \cdot e_{i+n,i} & 1 \le i = j \le n, \end{cases}$$

$$R_{i,j}(k) = I_{2n} + k \cdot (e_{i,j} - e_{j+n,i+n}) \qquad 1 \le i \ne j \le n.$$

Notice that, for $k \in \mathbb{Z}$,

$$P_{i,j}(k) = (P_{i,j}(1))^k, \qquad Q_{i,j}(k) = (Q_{i,j}(1))^k, \qquad R_{i,j}(k) = (R_{i,j}(1))^k.$$

Following [3, Chapter 5] we define $\mathrm{ESp}_{2n}(\mathbb{Z})$ as the group generated by the matrices $P_{i,j}(1)$, $Q_{i,j}(1)$, $1 \le i \le j \le n$, and $R_{i,j}(1)$, $1 \le i \ne j \le n$.

Thus, to prove the inclusion $\mathrm{ESp}_{10}(\mathbb{Z}) \subseteq \langle x, y \rangle$ it is enough to show that $P_{i,j}(1), Q_{i,j}(1), R_{i,j}(1) \in \langle x, y \rangle$. We split the proof into several steps, which are presented in the next section. In the proof we construct a sequence of matrices in $\langle x, y \rangle$. In order to assist the reader and make the construction more transparent we use the following notation:

- A_i are upper block-triangular matrices in $\langle x, y \rangle$ of the shape

$$\begin{pmatrix} I_5 & L \\ 0 & I_5 \end{pmatrix}; \tag{1}$$

- B_i are lower block-triangular matrices in $\langle x, y \rangle$ of the shape

$$\begin{pmatrix} I_5 & 0 \\ L & I_5 \end{pmatrix}; \tag{2}$$

- C_i are upper block-triangular matrices in $\langle x, y\rangle$ of the shape

$$\begin{pmatrix} K & L \\ 0 & M \end{pmatrix}, \tag{3}$$

where K, L and M are 5×5 matrices;

- D_i are block-diagonal matrices in $\langle x, y\rangle$ of the shape

$$\begin{pmatrix} K & 0 \\ 0 & \left(K^T\right)^{-1} \end{pmatrix}, \tag{4}$$

where K is a 5×5 matrix and T denotes the transpose of a matrix;

- g_i are auxiliary matrices from $\langle x, y\rangle$ with no prescribed shape.

3. Detailed proofs

To assist the reader in verifying the proof, the corresponding Magma file is also available as a supplementary information to the article.

Lemma 3.1.
We have $P_{1,1}(4) \in \langle x, y\rangle$ and $P_{1,i}(2), R_{1,i}(2) \in \langle x, y\rangle$ for $2 \le i \le 5$.

Proof. First of all, let us define

$$\begin{aligned}
g_1 &= y\,(xy)^3\left(xy^2\right)^4,\\
g_2 &= (xy)^2\left(xy^2\right)^2(xy)^3,\\
g_3 &= y\,\left(xy^2\right)^2xy\,\left(xyxy^2\right)^2,\\
C_1 &= \left(\left(xyxy^2\right)^3g_1\right)^4.
\end{aligned}$$

Now we can construct first matrices of shape (1):

$$\begin{aligned}
A_1 &= \left(y^{-1}g_2yx\right)^{-1}C_1y^{-1}g_2yx\cdot g_3^{-1}C_1^{-1}g_3 = P_{1,1}(4)\,P_{1,3}(2)\,P_{1,5}(2),\\
A_2 &= \left(g_2^{-1}C_1g_2x\right)^2 = P_{1,1}(-4)\,P_{1,3}(-4),\\
A_3 &= xA_1A_2xA_1 = P_{1,3}(-4),\\
A_4 &= A_3A_2^{-1} = P_{1,1}(4).
\end{aligned}$$

This gives the first inclusion stated in the lemma. Let us set

$$\begin{aligned}
g_4 &= \left(xy^2\right)^3,\\
g_5 &= xyxy^2x,\\
g_6 &= y\,\left(xy^2\right)^2(xy)^2\left(xy^2\right)^2xy\,\left(xy^2\right)^3x,\\
g_7 &= \left(xyxy^2\right)^2\left(xy^2\right)^3x,\\
g_8 &= \left(xy^2\right)^3\left((xy)^2xy^2xy\,\left(xy^2\right)^2\right)^2.
\end{aligned}$$

Using these matrices and $A_1, \dots, A_4$ we can find more matrices of the desired shape (1):

$$\begin{aligned}
A_5 &= g_5^{-1}C_1g_5 \cdot g_2xC_1xg_2^{-1} \cdot A_3^2A_4^2 = P_{1,2}(4),\\
A_6 &= \left(xg_5^{-1}C_1g_5\right)^2 \cdot \left(g_5xg_1^{-1}\right)^{-1}C_1g_5xg_1^{-1} \cdot \left(g_2^2xg_4\right)^{-1}C_1^{-1}g_2^2xg_4 \cdot A_1 = P_{1,4}(4),\\
A_7 &= \left(g_2^{-1}g_4x\right)^{-1}A_1g_2^{-1}g_4x \cdot g_6^{-1}A_1g_6 \cdot A_1^{-4}A_3^{-1}A_4^{-4}A_5A_6 = P_{1,5}(2),\\
A_8 &= A_4^{-1}A_1A_7^{-1} = P_{1,3}(2),\\
A_9 &= (g_2xg_4)^{-1}A_8g_2xg_4 \cdot A_4^{-1}A_8^{-2} = P_{1,2}(2).
\end{aligned}$$

We have already proved that $P_{1,2}(2), P_{1,3}(2), P_{1,5}(2) \in \langle x, y\rangle$. Before proving that $P_{1,4}(2)$ belongs to $\langle x, y\rangle$ we have to construct a few block-diagonal matrices of shape (4). Let us consider

$$\begin{aligned}
D_1 &= (xA_6)^2A_4^{-4}A_6^{-1}A_7^2A_9^{-2} = R_{1,2}(-4),\\
D_2 &= xD_1x \cdot A_4^{-8}A_6^{-3}A_7^2A_8^6 = R_{1,4}(-4),\\
D_3 &= g_7^{-1}D_1^{-1}g_7 \cdot D_2A_4^{-8}A_6^3A_7^6 = R_{1,5}(8),\\
D_4 &= g_8^{-1}C_1g_8 \cdot D_2^3D_3^{-1}A_4^{74}A_6^7A_7^4A_8^{-20}A_9^{-5} = R_{1,4}(2),\\
D_5 &= g_4^{-1}C_1^{-1}g_4 \cdot D_4A_4^{-2}A_6^{-1}A_7^{-2}A_8^2A_9 = R_{1,5}(-4).
\end{aligned}$$

In particular, we have just shown the inclusion $R_{1,4}(2) \in \langle x, y\rangle$. Finally, let us set

$$\begin{aligned}
g_9 &= \left(xy^2\right)^6(xy)^2xy^2xy\left(xy^2\right)^2,\\
C_2 &= \left(g_2^{-1}g_4xg_2xg_4\right)^{-1}A_8g_2^{-1}g_4xg_2xg_4 \cdot D_4D_5^{-2}A_4^{25}A_6^{-1}A_7^{-4}A_9^2 = P_{1,4}(2)\,R_{1,5}(2).
\end{aligned}$$

Now we are able to complete the proof by constructing the following matrices:

$$\begin{aligned}
D_6 &= (g_2xg_4)^{-1}C_2g_2xg_4 \cdot A_4^{-1}A_6^2A_8^{-16}A_9^{-6} = R_{1,5}(2),\\
A_{10} &= C_2D_6^{-1} = P_{1,4}(2),\\
D_7 &= g_9^{-1}C_1^{-1}g_9 \cdot D_4^3A_4^{28}A_7^4A_8^{24}A_9^6A_{10}^{-9} = R_{1,2}(2),\\
D_8 &= \left(g_2^{-1}g_4xg_2xg_4\right)^{-1}A_9g_2^{-1}g_4xg_2xg_4 \cdot D_4^{-5}D_6^{-6}A_4^{104}A_7^{10}A_8^{-4}A_9^{-5}A_{10}^9 = R_{1,3}(2).
\end{aligned}$$ □

Let us define two subsets of $\mathbb{Z}^{10}$:

$$\begin{aligned}
U_1 &= \left\{u = (u_1, \dots, u_{10})^T : u_6 = u_7 = \dots = u_{10} = 0\right\},\\
U_2 &= \left\{u = (u_1, \dots, u_{10})^T : u_1 = u_2 = \dots = u_5 = 0\right\}.
\end{aligned} \tag{5}$$

Remark 3.2.

Clearly, $I_{10} + uv^TJ + vu^TJ$ has shape (1) if $u, v \in U_1$ and has shape (2) if $u, v \in U_2$.

Lemma 3.3.

We have $P_{1,1}(2), P_{i,j}(2) \in \langle x, y\rangle$, where $2 \le i \le j \le 5$.

Proof. First, we explain further constructions that are used in the proof of the lemma. Let u, v be two integral column-vectors orthogonal with respect to J, i.e., $v^TJu = 0$. A direct computation shows that

$$S = I_{10} + uv^TJ + vu^TJ \in \mathrm{Sp}_{10}(\mathbb{Z}). \tag{6}$$

If we take

$$u = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{7}$$

and

$$v = \big((a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5)/2, b_2, b_3, b_4, b_5, 0, -a_2, -a_3, -a_4, -a_5\big)^T, \tag{8}$$

where all coefficients a_i and b_i are even, then we can write S as

$$S = \prod_{i=2}^{5} R_{1,i}(a_i) \cdot \prod_{i=2}^{5} P_{1,i}(b_i). \tag{9}$$

In other words, since a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5 are assumed to be even, S can be written as a product of suitable powers of D_7, D_8, D_4, D_6, A_9, A_8, A_{10}, and A_7. Hence, such S belongs to $\langle x, y\rangle$ by Lemma 3.1. Assume further that $g \in \langle x, y\rangle$ and $g^{-1}u$, $g^{-1}v$ belong to U_1. Then $g^{-1}Sg \in \langle x, y\rangle$ and

$$g^{-1}Sg = I_{10} + \big(g^{-1}u\big)\big(g^{-1}v\big)^T g^T Jg + \big(g^{-1}v\big)\big(g^{-1}u\big)^T g^T Jg = I_{10} + \big(g^{-1}u\big)\big(g^{-1}v\big)^T J + \big(g^{-1}v\big)\big(g^{-1}u\big)^T J$$

has shape (1) by the remark preceding the statement of the lemma. Moreover, the above conditions on v guarantee that $g^{-1}Sg$ belongs to $\big\langle P_{i,i}(4), P_{i,j}(2) : 1 \le i < j \le 5\big\rangle$.

Let us describe the strategy which is used in further computations.

1. We search for $g \in \langle x, y\rangle$ such that the last five entries in the first column of g^{-1} vanish (this is equivalent to the condition $g^{-1}u \in U_1$, where u is given by (7)).

2. We find v of the form (8) such that $g^{-1}v \in U_1$, and find the corresponding decomposition (9) for S. After that we evaluate $g^{-1}Sg$.

3. Finally, to simplify the subsequent calculations we multiply $g^{-1}Sg$ by suitable powers of

$$P_{1,1}(4), P_{1,2}(2), \dots, P_{1,5}(2) \in \langle x, y\rangle$$

(i.e., respectively by powers of $A_4, A_9, A_8, A_{10}, A_7$ defined in the proof of Lemma 3.1) and obtain matrices from $\big\langle P_{i,i}(4), P_{i,j}(2) : 2 \le i < j \le 5\big\rangle$.

Now we present the results of computation based on the above strategy. Let

$$\begin{aligned}
g_{10} &= y^2\big((xy)^7xy^2\big)^2xy\,\big(xy^2\big)^2xyxy^2,\\
g_{11} &= y^2\Big(\big(xy^2\big)^3xy\,\big(xy^2\big)^4\Big)^2xyxy^2x,\\
g_{12} &= y^2\big(xy^2\big)^4xy\,\big(xy^2\big)^9xy\,\big(xy^2\big)^4xyxy^2x,\\
g_{13} &= yxy^2\big((xy)^6xy^2(xy)^3\big)^2,\\
g_{14} &= yxy^2xyxy^2\Big(xy\,\big(xy^2\big)^5\Big)^2\Big(\big(xy^2\big)^2xy\Big)^2xyxy^2,\\
g_{15} &= y\,\big(xy^2\big)^3(xy)^2\big(xy^2\big)^4xyxy^2(xy)^4\big(xy^2\big)^4(xy)^4xy^2xyxy^2x,\\
g_{16} &= \big(xy^2\big)^2xyxy^2xy\,\big(xy^2\big)^9xy\,\big(xy^2\big)^7x,\\
g_{17} &= y^2xy^2(xy)^2\big(xy^2\big)^8xy\,\big(xy^2\big)^2xyxy^2xy\,\big(xy^2\big)^2xy\,\big(xy^2\big)^3,\\
g_{18} &= \big(xy^2\big)^2xy\big(xy^2\big)^4\big(xyxy^2\big)^2xy^2(xy)^2\big(xy^2\big)^2xyxy^2(xy)^2\big(xy^2\big)^2(xy)^2\big(xy^2xy\big)^2\big(xy^2\big)^2xyx,\\
g_{19} &= y^2\Big(\big(xy^2\big)^3xy\Big)^2xy^2xy\,\big(xy^2\big)^2\big(xy^2xy\big)^3xy\,\big(xy^2\big)^2xy\,\big(xyxy^2\big)^2xyx,\\
g_{20} &= y^2xyxy^2(xy)^{10}xy^2(xy)^5xy^2xy\,\big(xy^2\big)^3xyx,\\
g_{21} &= \big(xy^2\big)^4xy\,\big(xy^2\big)^9\Big(xy\big(xy^2\big)^2\Big)^2xyxy^2(xy)^3xy^2xy\,\big(xy^2\big)^3x,\\
g_{22} &= xy^2(xy)^2\big(xy^2\big)^2(xy)^7xy^2\big(xy^2xy\big)^2.
\end{aligned}$$

All these matrices as well as $g_{16}g_2xg_4$ and $g_{11}g_2xg_4$ satisfy the condition $g^{-1}u \in U_1$. Using the idea described at the beginning of the proof we can find 15 upper-block triangular matrices $A_{11}, \dots, A_{25} \in \langle x, y\rangle \cap \langle P_{i,i}(4), P_{i,j}(2) : 2 \le i < j \le 5\rangle$:

$$
\begin{aligned}
A_{11} &= g_{10}^{-1} D_7^{13} A_7^3 A_8^{-1} A_9^{-6} A_{10}^3 g_{10} \cdot A_4^{310} A_7^2 A_8^{708} A_9^{325} A_{10}^{-335},\\
A_{12} &= g_{11}^{-1} D_7 A_7^2 A_8^2 A_9^{-4} A_{10}^{-2} g_{11} \cdot A_4^{11} A_7^2 A_8^{36} A_9^{10} A_{10}^{-17},\\
A_{13} &= g_{12}^{-1} D_7^{-5} A_7^7 A_8^{-22} A_9^{-12} A_{10}^{-15} g_{12} \cdot A_4^{-54} A_7^5 A_8^{-272} A_9^{-92} A_{10}^{107},\\
A_{14} &= g_{13}^{-1} D_7^{45} A_7^{24} A_8^{24} A_9^{-62} A_{10}^{-15} g_{13} \cdot A_4^{2787} A_7^{15} A_8^{16759} A_9^{5571} A_{10}^{-5577},\\
A_{15} &= g_{14}^{-1} D_7 A_7 A_8^2 A_9^{-4} A_{10}^{-2} g_{14} \cdot A_4^{13} A_7^{16} A_8^{41} A_9^{21} A_{10}^{-14},\\
A_{16} &= g_{15}^{-1} D_7^2 A_7 A_8^3 A_9^{-4} A_{10}^{-2} g_{15} \cdot A_4^{12} A_7^{19} A_8^{94} A_9^{59} A_{10}^{-18},\\
A_{17} &= g_{16}^{-1} D_7^{-12} A_7 A_8^{-4} A_9^{11} g_{16} \cdot A_4^{158} A_7^8 A_8^{-867} A_9^{-288} A_{10}^{284},\\
A_{18} &= (g_{16}g_2xg_4)^{-1} D_7^{-12} A_7 A_8^{-4} A_9^{11} g_{16}g_2xg_4 \cdot A_4^{147} A_7^8 A_8^{841} A_9^{274} A_{10}^{-289},\\
A_{19} &= (g_{11}g_2xg_4)^{-1} D_7 A_7^2 A_8^2 A_9^{-4} A_{10}^{-2} g_{11}g_2xg_4 \cdot A_4^{15} A_7^2 A_8^{-62} A_9^{-20} A_{10}^{17},\\
A_{20} &= g_{17}^{-1} D_7^{-34} A_7^5 A_8^{14} A_9^{33} A_{10}^4 g_{17} \cdot A_4^{1186} A_7^{-18} A_8^{2317} A_9^{28} A_{10}^{-2343},\\
A_{21} &= g_{18}^{-1} D_7^2 A_7^7 A_8^{25} A_9^6 A_{10}^{-9} g_{18} \cdot A_4^{-8} A_7^{-4} A_8^{-18} A_9^{-14} A_{10}^4,\\
A_{22} &= g_{19}^{-1} A_8^{-3} A_9^{-1} A_{10} g_{19} \cdot A_4^{-72} A_7^{-59} A_8^{-150} A_9^{-50} A_{10}^{93},\\
A_{23} &= g_{20}^{-1} D_7^{-1} A_7 A_8^5 A_9^2 A_{10}^{-1} g_{20} \cdot A_4^4 A_8 A_9^4 A_{10}^{-1},\\
A_{24} &= g_{21}^{-1} D_7^{-1} A_7 A_8^{-13} A_9^{-6} A_{10}^6 g_{21} \cdot A_4^{-34} A_7^{-43} A_8^{135} A_9^{34} A_{10}^{-26},\\
A_{25} &= g_{22}^{-1} D_7^{11} A_7^{11} A_8^{61} A_9^5 A_{10}^{-18} g_{22} \cdot A_4^{-176} A_7^{-358} A_8^{-984} A_9^{-134} A_{10}^{530}.
\end{aligned}
$$

The matrices $A_{11}, \dots, A_{25}$ can be written in the following way:

$$
\begin{aligned}
A_i = I_{10} &+ k_1^{(i)} e_{2,7} + k_2^{(i)}(e_{2,8} + e_{3,7}) + k_3^{(i)}(e_{2,9} + e_{4,7}) + k_4^{(i)}(e_{2,10} + e_{5,7}) + k_5^{(i)} e_{3,8}\\
&+ k_6^{(i)}(e_{3,9} + e_{4,8}) + k_7^{(i)}(e_{3,10} + e_{5,8}) + k_8^{(i)} e_{4,9} + k_9^{(i)}(e_{4,10} + e_{5,9}) + k_{10}^{(i)} e_{5,10}.
\end{aligned}
$$

For reader's convenience, we present the values of the coefficients $k_j^{(i)}$ in Table 1.

Table 1. The coefficients $k_j^{(i)}$

i	$k_1^{(i)}$	$k_2^{(i)}$	$k_3^{(i)}$	$k_4^{(i)}$	$k_5^{(i)}$	$k_6^{(i)}$	$k_7^{(i)}$	$k_8^{(i)}$	$k_9^{(i)}$	$k_{10}^{(i)}$
11	−340	−738	350	−2	−1592	758	−4	−360	2	0
12	4	0	10	−4	−36	42	−12	−24	4	0
13	152	448	−182	−10	1320	−538	−30	212	10	0
14	−11136	−33500	11148	−30	−100776	33536	−90	−11160	30	0
15	−32	−52	18	−22	−24	10	−22	−4	8	−12
16	−276	−434	82	−88	−680	128	−138	−24	26	−28
17	−520	−1566	512	16	−4716	1542	48	−504	−16	0
18	−508	−1562	538	−16	−4800	1652	−48	−568	16	0
19	−20	−64	14	4	−204	46	12	−8	−4	0
20	0	−56	56	0	−4524	4576	36	−4628	−36	0
21	20	30	−10	10	40	−10	10	0	0	0
22	32	96	−64	40	288	−192	120	120	−76	48
23	0	8	0	0	20	−2	0	0	0	0
24	0	34	0	−34	236	−26	−144	0	26	52
25	92	732	−404	274	5472	−2964	2004	1596	−1078	728

Clearly, the matrices $A_{11}, \dots, A_{25}$ commute pairwise. It turns out that we can express $P_{i,i}(4)$, $P_{i,j}(2)$, $2 \le i < j \le 5$, as certain products of their powers:

$$\begin{aligned}
A_{26} &= A_{11}^{4} A_{12}^{663} A_{13}^{-539} A_{14}^{41} A_{15}^{-2990} A_{16}^{284} A_{17}^{-1062} A_{18}^{-64} A_{19}^{-130} A_{20}^{-7} A_{21}^{-2758} A_{22}^{-441} A_{23}^{-7712} A_{24}^{-410} A_{25}^{20} = P_{2,2}(4),\\
A_{27} &= A_{11}^{-223} A_{12}^{-1296} A_{13}^{-439} A_{14}^{273} A_{15}^{1344} A_{16}^{-754} A_{17}^{-2481} A_{18}^{-3281} A_{19}^{-115} A_{20}^{16} A_{21}^{-4816} A_{22}^{388} A_{23}^{4726} A_{24}^{176} A_{25}^{-45} = P_{2,3}(2),\\
A_{28} &= A_{11}^{-193} A_{12}^{-544} A_{13}^{-422} A_{14}^{430} A_{15}^{-526} A_{16}^{86} A_{17}^{-6663} A_{18}^{-2402} A_{19}^{-2032} A_{20}^{5} A_{21}^{2806} A_{22}^{365} A_{23}^{4126} A_{24}^{-818} A_{25}^{29} = P_{2,4}(2),\\
A_{29} &= A_{11}^{9} A_{12}^{726} A_{13}^{84} A_{14}^{107} A_{15}^{-1073} A_{16}^{478} A_{17}^{-2505} A_{18}^{295} A_{19}^{-1555} A_{20}^{-6} A_{21}^{4432} A_{22}^{3} A_{23}^{-153} A_{24}^{-595} A_{25}^{43} = P_{2,5}(2),\\
A_{30} &= A_{11}^{-165} A_{12}^{-1434} A_{13}^{-605} A_{14}^{287} A_{15}^{-564} A_{16}^{-147} A_{17}^{-4236} A_{18}^{-1920} A_{19}^{-842} A_{20}^{6} A_{21}^{-1107} A_{22}^{186} A_{23}^{1502} A_{24}^{-423} A_{25}^{3} = P_{3,3}(4),\\
A_{31} &= A_{11}^{-288} A_{12}^{-1498} A_{13}^{-1044} A_{14}^{424} A_{15}^{-1500} A_{16}^{-523} A_{17}^{-5165} A_{18}^{-3976} A_{19}^{-1498} A_{20}^{12} A_{21}^{-7035} A_{22}^{101} A_{23}^{-1052} A_{24}^{-413} A_{25}^{-22} = P_{3,4}(2),\\
A_{32} &= A_{11}^{-156} A_{12}^{-858} A_{13}^{189} A_{14}^{144} A_{15}^{2868} A_{16}^{-755} A_{17}^{-402} A_{18}^{-2403} A_{19}^{-604} A_{20}^{16} A_{21}^{-2281} A_{22}^{529} A_{23}^{7875} A_{24}^{453} A_{25}^{-49} = P_{3,5}(2),\\
A_{33} &= A_{11}^{-130} A_{12}^{-333} A_{13}^{-138} A_{14}^{66} A_{15}^{1249} A_{16}^{-787} A_{17}^{1004} A_{18}^{-2361} A_{19}^{196} A_{20}^{13} A_{21}^{-6846} A_{22}^{150} A_{23}^{1666} A_{24}^{552} A_{25}^{-59} = P_{4,4}(4),\\
A_{34} &= A_{11}^{-193} A_{12}^{-1167} A_{13}^{-919} A_{14}^{282} A_{15}^{-2016} A_{16}^{-287} A_{17}^{-3592} A_{18}^{-2583} A_{19}^{-1075} A_{20}^{6} A_{21}^{-6106} A_{22}^{-94} A_{23}^{-3519} A_{24}^{-379} A_{25}^{-11} = P_{4,5}(2),\\
A_{35} &= A_{11}^{-45} A_{12}^{321} A_{13}^{570} A_{14}^{28} A_{15}^{3182} A_{16}^{-439} A_{17}^{713} A_{18}^{-1022} A_{19}^{116} A_{20}^{10} A_{21}^{1505} A_{22}^{494} A_{23}^{8278} A_{24}^{462} A_{25}^{-30} = P_{5,5}(4).
\end{aligned}$$

To complete the proof of Lemma 3.3 it remains to show that $P_{i,i}(2) \in \langle x, y \rangle$ for $1 \le i \le 5$. For this purpose let us define the following matrices:

$$\begin{aligned}
g_{23} &= yxy\left((xy)^4\left(xy^2\right)^2\right)^2,\\
g_{24} &= \left(xyxy^2\right)^3(xy)^2\left(xyxy^2\right)^2(xy)^3xy^2(xy)^2x,\\
g_{25} &= \left(xyxy^2\right)^3,\\
C_3 &= (xy)^3\left(xyxy^2\right)^2,
\end{aligned}$$

and set

$$\begin{aligned}
A_{36} &= (C_3 g_{24})^4 A_8^{-1} A_9^{-1} A_{27}^{2} A_{28}^{-1} A_{30}^{4} A_{31}^{-3} A_{33} = P_{1,1}(2)\, P_{4,4}(2),\\
A_{37} &= A_{36}^{-1} (C_3 g_{23})^{20} A_4^{8} A_7^{15} A_8^{60} A_9^{15} A_{10}^{-15} A_{26}^{8} A_{27}^{60} A_{28}^{-15} A_{29}^{15} A_{30}^{120} A_{31}^{-60} A_{32}^{60} A_{33}^{8} A_{34}^{-15} A_{35}^{8} = P_{2,2}(2)\, P_{5,5}(2),\\
A_{38} &= \left(A_{37} g_{25}^{-1}\right)^2 A_8^{-2} A_9^{-1} A_{26}^{-1} A_{27}^{-2} A_{30}^{-2} A_{35}^{-1} = P_{1,1}(2),\\
A_{39} &= \left(A_{36} g_{25}^{-1}\right)^2 A_7^{-1} A_8^{-4} A_{10} A_{30}^{-8} A_{31}^{4} A_{32}^{-4} A_{33}^{-1} A_{34} A_{38}^{-3} = P_{5,5}(2),\\
A_{40} &= A_{38}^{-1} A_{36} = P_{4,4}(2),\\
A_{41} &= A_{39}^{-1} A_{37} = P_{2,2}(2),\\
A_{42} &= \left(g_2 x g_4 A_{41}^{-1}\right)^{-2} A_8^{-9} A_9^{-6} A_{27}^{-6} A_{30}^{-4} A_{38}^{-9} A_{41}^{-5} = P_{3,3}(2).
\end{aligned}$$

The last five entries complete the proof. □

Lemma 3.4.

We have $R_{1,i}(1), P_{3,5}(1), P_{1,j}(1) \in \langle x, y \rangle$ for $2 \le i \le 5$, $1 \le j \le 5$.

Proof. Let us define

$$\begin{aligned}
g_{26} &= y\,(xy)^2\left(xyxy^2\right)^5(xy)^3\left(xyxy^2\right)^2,\\
g_{27} &= y\,\left(xy^2xyxy^2\right)^2xy\,\left(xyxy^2\right)^2,\\
g_{28} &= (xy)^2\left(xy^2\right)^2\left(xyxy^2\right)^2\left(xy^2\right)^3x,\\
g_{29} &= \left(xyxy^2\right)^3(xy)^2\left(xy^2\right)^2\left(xyxy^2\right)^2xy^2xy\,\left(xyxy^2\right)^2,
\end{aligned}$$

and consider

$$\begin{aligned}
D_9 &= \left(A_{27} g_{26}^{-1} y x y^2 g_5\right)^{-2} A_8 A_{27} A_{31}^{-1} A_{32}^2 = R_{3,5}(2),\\
D_{10} &= \left(g_{26} A_{27}^{-1}\right)^2 A_8^{-2} A_{27} A_{31}^2 A_{42}^{-10} = R_{3,4}(2),\\
D_{11} &= \left(A_{31} x\right)^{-2} A_{27} A_{31} A_{32}^{-1} A_{42}^{-2} = R_{3,2}(2),\\
D_{12} &= \left(A_{31} g_5^{-1} g_{25}^{-1}\right)^2 D_9 A_{27} A_{31}^{-2} A_{32}^{-2} A_{42}^2 = R_{3,1}(2),\\
D_{13} &= \left(g_2 x g_4 C_1\right)^2 D_4^3 D_{10}^2 C_1^{-1} A_8^{72} A_9^{31} A_{10}^{-6} A_{27}^{26} A_{28}^{-4} A_{31}^{-4} A_{38}^{72} A_{41}^{-12} A_{42}^{64} = R_{2,4}(-4),\\
D_{14} &= \left(A_{27} g_{27}^{-1}\right)^2 A_{27}^{-2} A_{28}^2 A_{29}^2 A_{41}^{-6} = R_{2,4}(2)\, R_{2,5}(2),\\
D_{15} &= g_{28}^{-1} D_{14} g_{28} \cdot D_6^{-6} D_9^{-4} D_{13}^{-1} D_{14}^{-2} A_7^{-12} A_8^{-80} A_9^{-33} A_{10}^{12} A_{27}^{6} A_{28}^{6} A_{29}^{-6} A_{31}^{8} A_{32}^{-8} A_{38}^{-204} A_{41}^{18} A_{42}^{-16} = R_{2,5}(2).
\end{aligned}$$

Finally, we can obtain the following matrices:

$$\begin{aligned}
C_4 &= \left(y x y^2 x y \left(x y^2\right)^2 (x y)^2 \left(x y^2\right)^2 (x y)^5 \left(x y x y^2\right)^2 x\right)^{15},\\
C_5 &= C_4 D_6^2 D_9^4 D_{13} D_{14}^2 A_7^{-4} A_8^{39} A_9^{22} A_{10}^{-6} A_{27}^{39} A_{28}^{-6} A_{29}^{-4} A_{31}^{-16} A_{32}^{-5} A_{34} A_{38}^{22} A_{40}^{6} A_{41}^{22} A_{42}^{70}.
\end{aligned}$$

The advantage of C_4 and C_5 is that they have shape (3) and some of their non-diagonal entries are odd. Moreover, C_5 will help us to construct the first matrix of shape (1) with the block L containing only ones and zeros, namely

$$A_{43} = \left(g_2 x g_4 C_5^{-1}\right)^{-2} D_6^{-1} D_9^{-1} D_{15}^{-1} A_7^{-2} A_8^{-7} A_9^{-6} A_{10}^{4} A_{27}^{-12} A_{28}^{7} A_{29}^{-4} A_{31}^{13} A_{32}^{-8} A_{34}^{2} A_{38}^{-6} A_{40}^{-5} A_{41}^{-8} A_{42}^{-21}.$$

Now let us set

$$\begin{aligned}
C_6 &= \left(g_3^{-1} g_6\right)^{-1} A_{43} g_3^{-1} g_6 \cdot D_{10}^2 D_{12} D_{14} D_{15}^{-1} A_8^6 A_9^3 A_{27}^7 A_{28}^{-1} A_{31}^{-3} A_{41}^7 A_{42}^2 = R_{2,1}(-1)\, P_{2,2}(1)\, P_{2,4}(1),\\
C_7 &= \left(g_{25} g_4^{-1} x g_2^{-1}\right)^{-1} C_6 g_{25} g_4^{-1} x g_2^{-1} \cdot D_9^2 D_{10}^2 D_{11} A_8^6 A_{27}^3 A_{31}^{-5} A_{32}^{-1} A_{42}^{34} = R_{3,1}(1)\, R_{3,2}(1)\, R_{3,4}(1)\, P_{3,3}(1)\, P_{3,5}(1),\\
C_8 &= \left(g_{26} g_4^{-1}\right)^{-1} C_6 g_{26} g_4^{-1} \cdot D_9^{-3} D_{10} D_{11} D_{12}^2 A_8^{-1} A_{27}^{-3} A_{31}^2 A_{32}^4 A_{42}^{36} = R_{3,1}(1)\, R_{3,2}(1)\, P_{1,3}(1)\, P_{2,3}(1)\, P_{3,3}(-1),\\
C_9 &= \left(g_{29} g_3^{-1}\right)^{-1} C_6 g_{29} g_3^{-1} \cdot D_9^{-3} D_{11}^3 D_{12}^{-1} A_{27}^{-3} A_{31}^{-1} A_{32}^6 A_{42}^{75} = R_{3,2}(1)\, R_{3,5}(1)\, P_{2,3}(1)\, P_{3,4}(1),\\
C_{10} &= \left(g_7 g_2\right)^{-1} C_6 g_7 g_2 \cdot D_9^2 D_{11}^{-2} D_{12}^2 A_{27}^5 A_{31} A_{32}^{-3} A_{42}^{28} = R_{3,1}(1)\, R_{3,2}(1)\, R_{3,5}(1)\, P_{3,3}(1),\\
C_{11} &= \left(g_3^{-1} g_{26}\right)^{-1} C_6 g_3^{-1} g_{26} \cdot D_9^5 D_{10}^3 D_{11} D_{12}^{-1} A_8^6 A_{27}^4 A_{31}^{-8} A_{32}^{-5} A_{42}^{138} = R_{3,2}(1)\, R_{3,4}(1)\, R_{3,5}(1)\, P_{3,4}(1),\\
C_{12} &= \left(g_{27} g_3^{-1}\right)^{-1} C_6 g_{27} g_3^{-1} \cdot D_9^{-1} D_{10}^2 D_{11}^2 A_8^4 A_{27}^{-1} A_{31}^{-4} A_{32}^3 A_{42}^{42} = R_{3,1}(1)\, R_{3,2}(1)\, R_{3,4}(1)\, R_{3,5}(1)\, P_{2,3}(1)\, P_{3,3}(-1)\, P_{3,5}(1).
\end{aligned}$$

Using them we are now able to prove some of the statements of the lemma. Namely, take

$$\begin{aligned}
A_{44} &= C_7 C_8^{-1} C_{10} C_{12}^{-1} A_8 A_{27} A_{42}^2 = P_{1,3}(1),\\
A_{45} &= C_7 C_8^{-1} C_9 C_{11}^{-1} A_{44} = P_{3,5}(1),\\
A_{46} &= (g_2 x g_4)^{-1} A_{44} g_2 x g_4 \cdot A_8^{-1} A_{38}^{-1} = P_{1,2}(1),\\
A_{47} &= (g_2 x g_5 x)^{-1} A_{44} g_2 x g_5 x \cdot D_4^{-1} D_6^{-1} A_7^2 A_{10}^2 A_{38}^3 A_{44}^{-3} A_{46}^{-1} = P_{1,5}(1).
\end{aligned}$$

Futhermore, let us consider

$$\begin{aligned}
g_{30} &= \left(x y^2\right)^3 (x y)^2 \left(x y^2\right)^2 x y \left(x y^2\right)^7 (x y)^3 x,\\
g_{31} &= \left(\left(x y^2\right)^2 x y\right)^2 (x y)^2,\\
g_{32} &= y \left(x y^2\right)^2 (x y)^2 \left(x y^2\right)^4 x y x y^2.
\end{aligned}$$

Now we finish the proof by constructing the following matrices:

$$
\begin{aligned}
D_{16} &= (g_5g_4)^{-1}A_{44}g_5g_4 \cdot A_{10}A_{38}^{-3}A_{44}^{-3} = R_{1,4}(1),\\
A_{48} &= (g_2xg_{31})^{-1}A_{44}g_2xg_{31} \cdot g_{30}^{-1}A_{47}^{-1}g_{30} \cdot D_6^7D_7^{-1}D_{16}^5A_{10}^{-6}A_{38}^{180}A_{44}^{-9}A_{46}^6A_{47}^{-20} = P_{1,4}(1),\\
D_{17} &= g_{32}^{-1}D_{16}g_{32} \cdot D_6^{-1}D_7^2D_8^{-2}D_{16}^{-11}A_8^{-4}A_9^{-4}A_{38}^{63}A_{47}^{11}A_{48}^{11} = R_{1,5}(1),\\
D_{18} &= \left(g_{31}^{-1}g_4xg_2xg_4\right)^{-1}A_{47}g_{31}^{-1}g_4xg_2xg_4 \cdot D_6D_7^2D_{16}^5D_{17}^{-1}A_{38}^{31}A_{44}^{14}A_{47}^{-3}A_{48}^{-9} = R_{1,2}(1),\\
A_{49} &= \left(g_{31}^{-1}g_4x\right)^{-1}A_{44}g_{31}^{-1}g_4x \cdot D_6^2D_{16}D_{17}^{-1}A_7^{-2}A_9A_{10}^{-1}A_{38}^{10}A_{48}^{-1} = P_{1,1}(1),\\
D_{19} &= g_{30}^{-1}D_{18}g_{30} \cdot D_8^5D_{16}^{14}D_{17}^{-2}D_{18}^{-11}A_{44}^{12}A_{46}^{19}A_{47}^{-14}A_{48}^{-9}A_{49}^{169} = R_{1,3}(1).
\end{aligned}
$$

□

Lemma 3.5.

We have $P_{i,j}(1) \in \langle x, y\rangle$ for $2 \le i \le j \le 5$.

Proof. We argue as at the beginning of the proof of Lemma 3.3, but now we consider $v \in \left(\frac{1}{2}\mathbb{Z}\right)^{10}$ of shape (8) with $a_i, b_j \in \mathbb{Z}$, i.e. without any assumptions on their parity (taking u as in (7) we always have that $uv^TJ + vu^TJ$ is an integral matrix). Again, the matrix S defined by (6) can be represented in the form (9), where now the product is in $\langle x, y\rangle$ by Lemma 3.4. In other words, S is a product of suitable powers of D_{18}, D_{19}, D_{16}, D_{17}, A_{46}, A_{44}, A_{48}, and A_{47}. Set

$$
\begin{aligned}
g_{33} &= y^2xy^2(xy)^2\left(xy^2\right)^5xyxy^2(xy)^4\left(xy^2\right)^4\left(xyxy^2\right)^3\left(xy^2\right)^2(xy)^4\left(xy^2\right)^2x,\\
g_{34} &= xy\left((xy)^7xy^2\right)^2(xy)^5\left(xy^2\right)^2x.
\end{aligned}
$$

It is easy to check that the last five entries in the first column of g_{33}^{-1}, g_{34}^{-1} as well as $(g_{16}g_2xg_4)^{-1}$ vanish. Recall that the same property holds for $g_{10}^{-1}, \ldots, g_{22}^{-1}$ constructed in the proof of Lemma 3.3. Hence, for these matrices g we have $g^{-1}u \in U_1$. Finding suitable vectors v and reasoning in the same way as in the proof of Lemma 3.3, we define the following matrices:

$$
\begin{aligned}
A_{50} &= g_{10}^{-1}D_{18}^{13}A_{44}^{-1}A_{46}^{-6}A_{47}^3A_{48}^3g_{10} \cdot A_{44}^{396}A_{46}^{169}A_{47}^2A_{48}^{-179}A_{49}^{308},\\
A_{51} &= g_{12}^{-1}D_{18}^{-5}A_{44}^{-22}A_{46}^{-12}A_{47}^7A_{48}^{-15}g_{12} \cdot A_{44}^{-92}A_{46}^{-32}A_{47}^5A_{48}^{47}A_{49}^{-48},\\
A_{52} &= g_{14}^{-1}D_{18}A_{44}^2A_{46}^{-4}A_{47}A_{48}^{-2}g_{14} \cdot A_{44}^{29}A_{46}^{17}A_{47}^{12}A_{48}^{-10}A_{49}^{22},\\
A_{53} &= g_{16}^{-1}D_{18}^{-12}A_{44}^{-4}A_{46}^{11}A_{47}g_{16} \cdot A_{44}^{-471}A_{46}^{-156}A_{47}^8A_{48}^{152}A_{49}^{184},\\
A_{54} &= (g_{16}g_2xg_4)^{-1}D_{18}^{-12}A_{44}^{-4}A_{46}^{11}A_{47}g_{16}g_2xg_4 \cdot A_{44}^{445}A_{46}^{142}A_{47}^8A_{48}^{-157}A_{49}^{162},\\
A_{55} &= g_{22}^{-1}D_{18}^{11}A_{44}^{61}A_{46}^5A_{47}^{11}A_{48}^{-18}g_{22} \cdot A_{44}^{-324}A_{46}^{-24}A_{47}^{-138}A_{48}^{200}A_{49}^{-132},\\
A_{56} &= g_{33}^{-1}D_{18}^{-19}A_{44}^{195}A_{46}^{73}A_{47}^{19}A_{48}^{-43}g_{33} \cdot A_{44}^{39674}A_{46}^{15818}A_{47}^{3983}A_{48}^{-7963}A_{49}^{11901},\\
A_{57} &= g_{34}^{-1}D_{18}^{-1}A_{44}^{12}A_{46}^5A_{47}^3A_{48}^{-2}g_{34} \cdot A_{44}^{-85}A_{46}^{-28}A_{47}^2A_{48}^{25}A_{49}^{35}.
\end{aligned}
$$

Clearly, the matrices $A_{51}, \ldots, A_{57}$ can be written in the form (10). The coefficients $k_j^{(i)}$ are presented in Table 2. Moreover, we can simplify the result by making the reduction modulo 2 (the possibility of such simplification follows from Lemmas 3.1 and 3.3; moreover, using $A_{45} = P_{3,5}(1)$ it is possible to make $k_7^{(i)}$ equal 0):

$$
\begin{aligned}
A_{58} &= A_{50}A_{27}^{107}A_{28}^{-48}A_{29}A_{31}^{-111}A_{40}^{51}A_{41}^{46}A_{42}^{242}A_{45}^2,\\
A_{59} &= A_{51}A_{27}^{-22}A_{28}^{16}A_{29}^3A_{31}^{45}A_{34}^{-2}A_{40}^{-23}A_{41}^{-8}A_{42}^{-60}A_{45}^{15},\\
A_{60} &= A_{52}A_{27}^7A_{28}^{-2}A_{29}^4A_{31}^4A_{39}A_{40}^{-1}A_{41}^6A_{42}^{-12}A_{45}^{-1},\\
A_{61} &= A_{53}A_{27}^{194}A_{28}^{-62}A_{29}^{-4}A_{31}^{-187}A_{34}^4A_{40}^{60}A_{41}^{64}A_{42}^{585}A_{45}^{-24},\\
A_{62} &= A_{54}A_{27}^{193}A_{28}^{-68}A_{29}^4A_{31}^{-215}A_{34}^{-4}A_{40}^{76}A_{41}^{61}A_{42}^{606}A_{45}^{24},\\
A_{63} &= A_{55}A_{27}^{-18}A_{28}^{19}A_{29}^{-13}A_{31}^{246}A_{34}^{105}A_{39}^{-72}A_{40}^{-151}A_{41}^5A_{42}^{-378}A_{45}^{-342},\\
A_{64} &= A_{56}A_{27}^{26366}A_{28}^{-5292}A_{29}^{2647}A_{31}^{-13273}A_{34}^{-1332}A_{39}^{667}A_{40}^{2664}A_{41}^{10512}A_{42}^{66130}A_{45}^{13278},\\
A_{65} &= A_{57}A_{27}^{32}A_{28}^{-9}A_{29}^{-1}A_{31}^{-27}A_{34}A_{40}^8A_{41}^{11}A_{42}^{98}A_{45}^{-6}.
\end{aligned}
$$

Table 2. The coefficients $k_j^{(i)}$

i	$k_1^{(i)}$	$k_2^{(i)}$	$k_3^{(i)}$	$k_4^{(i)}$	$k_5^{(i)}$	$k_6^{(i)}$	$k_7^{(i)}$	$k_8^{(i)}$	$k_9^{(i)}$	$k_{10}^{(i)}$
50	−92	−213	97	−1	−484	223	−2	−102	1	0
51	16	44	−31	−5	120	−89	−15	46	5	0
52	−12	−14	5	−7	24	−7	1	2	0	−2
53	−128	−387	124	8	−1170	375	24	−120	−8	0
54	−122	−385	137	−8	−1212	430	−24	−152	8	0
55	−9	36	−37	27	756	−492	342	303	−209	144
56	−21024	−52732	10584	−5294	−132260	26546	−13278	−5328	2665	−1333
57	−21	−64	18	2	−195	55	6	−15	−2	0

The matrices $A_{58}, \dots, A_{65}$ satisfy (10) with $k_j^{(i)} \in \{0,1\}$. The coefficients $k_j^{(i)}$ are listed in Table 3.

Table 3. The coefficients $k_j^{(i)}$

i	$k_1^{(i)}$	$k_2^{(i)}$	$k_3^{(i)}$	$k_4^{(i)}$	$k_5^{(i)}$	$k_6^{(i)}$	$k_7^{(i)}$	$k_8^{(i)}$	$k_9^{(i)}$	$k_{10}^{(i)}$
58	0	1	1	1	0	1	0	0	1	0
59	0	0	1	1	0	1	0	0	1	0
60	0	0	1	1	0	1	0	0	0	0
61	0	1	0	0	0	1	0	0	0	0
62	0	1	1	0	0	0	0	0	0	0
63	1	0	1	1	0	0	0	1	1	0
64	0	0	0	0	0	0	0	0	1	1
65	1	0	0	0	1	1	0	1	0	0

Recall that we already have $P_{3,5}(1) \in \langle x, y\rangle$ by Lemma 3.4. To complete the proof it is enough to construct the following matrices:

$$\begin{aligned}
A_{66} &= A_{58}A_{59}^{-1} = P_{2,3}(1),\\
A_{67} &= A_{59}A_{60}^{-1} = P_{4,5}(1),\\
A_{68} &= A_{61}A_{66}^{-1} = P_{3,4}(1),\\
A_{69} &= A_{62}A_{66}^{-1} = P_{2,4}(1),\\
A_{70} &= A_{59}A_{67}^{-1}A_{68}^{-1}A_{69}^{-1} = P_{2,5}(1),\\
A_{71} &= A_{64}A_{67}^{-1} = P_{5,5}(1),\\
A_{72} &= A_{63}A_{67}^{-1}A_{69}^{-1}A_{70}^{-1} = P_{2,2}(1)\,P_{4,4}(1),\\
A_{73} &= A_{65}A_{68}^{-1}A_{72}^{-1} = P_{3,3}(1),\\
A_{74} &= (g_2xg_4)^{-1}A_{72}g_2xg_4 \cdot A_{41}^{-4}A_{44}^{-9}A_{46}^{-6}A_{49}^{-9}A_{66}^{-18}A_{68}^{6}A_{69}^{2}A_{73}^{-45} = P_{4,4}(1),\\
A_{75} &= A_{72}A_{74}^{-1} = P_{2,2}(1).
\end{aligned}$$

□

The statements of Lemma 3.4 and Lemma 3.5 show us that $P_{i,j}(1) \in \langle x, y\rangle$, $1 \le i \le j \le 5$. Using this fact we can prove that $Q_{i,j}(1)$, $1 \le i \le j \le 5$, are also in $\langle x, y\rangle$.

Lemma 3.6.

For $1 \le i \le j \le 5$ we have $Q_{i,j}(1) \in \langle x, y\rangle$.

Proof. Observe that

$$S = I_{10} \pm uu^T J \in \begin{cases} \langle \mathrm{P}_{i,j}(1) : 1 \le i \le j \le 5 \rangle \subset \langle x, y \rangle & \text{if } u \in U_1, \\ \langle \mathrm{Q}_{i,j}(1) : 1 \le i \le j \le 5 \rangle & \text{if } u \in U_2. \end{cases}$$

Here U_1 and U_2 are the subsets defined in (5) and the matrix J is defined at the beginning of Section 2. Also it is clear that

$$g^{-1}Sg = g^{-1}\left(I_{10} \pm uu^T J\right)g = I_{10} \pm \left(g^{-1}u\right)\left(g^{-1}u\right)^T g^T J g = I_{10} \pm \left(g^{-1}u\right)\left(g^{-1}u\right)^T J$$

provided $g \in \mathrm{Sp}_{10}(\mathbb{Z})$. Thus, if we take $g \in \langle x, y \rangle \subseteq \mathrm{Sp}_{10}(\mathbb{Z})$ and $u \in U_1$ such that $g^{-1}u \in U_2$, then the above matrix $g^{-1}Sg$ belongs to $\langle Q_{i,j}(1) : 1 \le i \le j \le 5 \rangle \cap \langle x, y \rangle$. Set

$$\begin{aligned}
g_{35} &= y\,(xy)^2\left(xy^2(xy)^3\right)^2(xy)^4\left(xy^2\right)^2 xyx, \\
g_{36} &= xy\,\left(xy^2\right)^2(xy)^3\left(xy^2\right)^2, \\
g_{37} &= y\,(xy)^2 xy^2(xy)^3\left(xy^2\right)^2 xyx, \\
g_{38} &= (xy)^5\left(xy^2\right)^2(xy)^4\left(xy^2\right)^5(xy)^3, \\
g_{39} &= y^2xy^2(xy)^2x, \\
g_{40} &= y\,\left(xyxy^2\right)^2\left(xy^2\right)^2 xy, \\
g_{41} &= y^2xyxy^2xy\,\left(xy^2\right)^4 xyx, \\
g_{42} &= yxy^2xy\,\left(xy^2\right)^2 xy\,\left(xy^2\right)^4(xy)^3\left(xy^2\right)^2(xy)^2\left(xy^2\right)^3 xyx, \\
g_{43} &= y\,\left(\left(xy^2\right)^3 xy\right)^2 xy^2, \\
g_{44} &= \left(xy^2\right)^5 xyxy^2x, \\
g_{45} &= \left(xyxy^2\right)^2 xy\,\left(xyxy^2\right)^3 xy\,\left((xy)^2xy^2\right)^2\left(xyxy^2\right)^3 y, \\
g_{46} &= y\,\left(xy^2\right)^2\left(xyxy^2\right)^2(xy)^4x, \\
g_{47} &= \left(xy^2\right)^3\left(xyxy^2\right)^3 x, \\
g_{48} &= y\,(xy)^4\left(xyxy^2\right)^3\left(xy^2\right)^2, \\
g_{49} &= xyxy^2(xy)^7xy^2(xy)^2x, \\
g_{50} &= y^2\left(xy^2\right)^3, \\
g_{51} &= \left(xy^2xy\right)^3 xyxy^2,
\end{aligned}$$

and also

$$\begin{aligned}
u_1 &= (1, 1, 2, -1, 0, 0, 0, 0, 0, 0)^T, \\
u_2 &= (-1, 1, 3, -1, 0, 0, 0, 0, 0, 0)^T, \\
u_3 &= (-2, -7, -20, 1, -5, 0, 0, 0, 0, 0)^T, \\
u_4 &= (-10, -29, -64, 3, -4, 0, 0, 0, 0, 0)^T, \\
u_5 &= (-4, 0, -2, 3, -1, 0, 0, 0, 0, 0)^T, \\
u_6 &= (-7, -19, -31, 5, -2, 0, 0, 0, 0, 0)^T, \\
u_7 &= (-3, -3, -11, 4, -3, 0, 0, 0, 0, 0)^T, \\
u_8 &= (-6, -3, 13, -4, -1, 0, 0, 0, 0, 0)^T, \\
u_9 &= (-6, -6, -19, 7, -2, 0, 0, 0, 0, 0)^T, \\
u_{10} &= (1, -3, -8, 2, -1, 0, 0, 0, 0, 0)^T, \\
u_{11} &= (0, -2, -6, 1, -1, 0, 0, 0, 0, 0)^T, \\
u_{12} &= (-3, -2, -4, 2, 0, 0, 0, 0, 0, 0)^T, \\
u_{13} &= (-5, 0, -2, 3, -2, 0, 0, 0, 0, 0)^T,
\end{aligned}$$

$$
\begin{aligned}
u_{14} &= (2,-2,-5,0,-1,0,0,0,0,0)^T,\\
u_{15} &= (-8,-11,-29,7,-2,0,0,0,0,0)^T,\\
u_{16} &= (-1,-3,-11,3,-2,0,0,0,0,0)^T,\\
u_{17} &= (-1,0,-1,1,-1,0,0,0,0,0)^T.
\end{aligned}
$$

Finally, let us consider

$$
B_i = \begin{cases} g_{34+i}^{-1}\big(I_{10} + u_i u_i^T J\big) g_{34+i} & \text{if } 1 \le i \le 15,\\ g_{34+i}^{-1}\big(I_{10} - u_i u_i^T J\big) g_{34+i} & \text{if } i = 16, 17, \end{cases}
$$

which belong to $\big\langle Q_{i,j}(1) : 1 \le i \le j \le 5 \big\rangle \cap \langle x, y\rangle$. Clearly, $B_1, \dots, B_{17}$ commute pairwise. It turns out that they generate the same subgroup of $\mathrm{Sp}_{10}(\mathbb{Z})$ as $Q_{i,j}(1)$ do. Namely, we can express the matrices $Q_{i,j}(1)$ as the following products of $B_1, \dots, B_{17}$:

$$
\begin{aligned}
Q_{1,1}(1) &= B_1^{-97}B_2^{-105}B_3^{12}B_4^{2}B_5^{14}B_6^{-21}B_7^{95}B_8^{17}B_9^{2}B_{10}^{137}B_{11}^{10}B_{12}^{-29}B_{13}^{-10}B_{14}^{-277}B_{15}^{-19}B_{16}^{-40}B_{17}^{65},\\
Q_{1,2}(1) &= B_1^{34}B_2^{42}B_3^{6}B_4B_5^{-9}B_6^{-10}B_7^{30}B_8^{-2}B_9^{-5}B_{10}^{56}B_{11}^{-15}B_{12}^{36}B_{13}^{-14}B_{14}^{-122}B_{15}^{-6}B_{16}^{-48}B_{17}^{16},\\
Q_{1,3}(1) &= B_1^{-160}B_2^{-221}B_3^{-20}B_4^{-3}B_5^{26}B_6^{34}B_7^{-93}B_8^{8}B_9^{22}B_{10}^{-180}B_{11}^{63}B_{12}^{-122}B_{13}^{54}B_{14}^{396}B_{15}^{19}B_{16}^{182}B_{17}^{-45},\\
Q_{1,4}(1) &= B_1^{-111}B_2^{-140}B_3^{-2}B_5^{16}B_6^{3}B_7^{11}B_8^{11}B_9^{10}B_{10}^{-5}B_{11}^{30}B_{12}^{-59}B_{13}^{16}B_{14}^{23}B_{15}^{-2}B_{16}^{52}B_{17}^{15},\\
Q_{1,5}(1) &= B_1^{14}B_2^{18}B_3^{-6}B_4^{-1}B_5^{2}B_6^{10}B_7^{-39}B_8^{-3}B_9^{2}B_{10}^{-65}B_{11}^{3}B_{12}^{-14}B_{13}^{8}B_{14}^{134}B_{15}^{8}B_{16}^{30}B_{17}^{-25},\\
Q_{2,2}(1) &= B_{17},\\
Q_{2,3}(1) &= B_1^{85}B_2^{121}B_3^{13}B_4^{2}B_5^{-17}B_6^{-22}B_7^{63}B_8^{-4}B_9^{-13}B_{10}^{120}B_{11}^{-36}B_{12}^{75}B_{13}^{-32}B_{14}^{-262}B_{15}^{-13}B_{16}^{-111}B_{17}^{31},\\
Q_{2,4}(1) &= B_1^{107}B_2^{163}B_3^{15}B_4^{2}B_5^{-11}B_6^{-26}B_7^{74}B_8^{-2}B_9^{-16}B_{10}^{136}B_{11}^{-45}B_{12}^{69}B_{13}^{-40}B_{14}^{-298}B_{15}^{-15}B_{16}^{-135}B_{17}^{37},\\
Q_{2,5}(1) &= B_1^{49}B_2^{45}B_3^{-12}B_4^{-2}B_5^{-7}B_6^{21}B_7^{-86}B_8^{-12}B_9B_{10}^{-128}B_{11}^{2}B_{12}^{6}B_{13}^{16}B_{14}^{265}B_{15}^{17}B_{16}^{57}B_{17}^{-57},\\
Q_{3,3}(1) &= B_1^{36}B_2^{21}B_3^{-18}B_4^{-3}B_5^{-2}B_6^{31}B_7^{-119}B_8^{-12}B_9^{6}B_{10}^{-189}B_{11}^{12}B_{12}^{-21}B_{13}^{27}B_{14}^{393}B_{15}^{24}B_{16}^{99}B_{17}^{-75},\\
Q_{3,4}(1) &= B_1^{-102}B_2^{-125}B_3^{4}B_4B_5^{12}B_6^{-7}B_7^{45}B_8^{12}B_9^{6}B_{10}^{57}B_{11}^{21}B_{12}^{-37}B_{13}^{5}B_{14}^{-107}B_{15}^{-9}B_{16}^{13}B_{17}^{35},\\
Q_{3,5}(1) &= B_1^{-76}B_2^{-119}B_3^{-20}B_4^{-3}B_5^{16}B_6^{34}B_7^{-108}B_9^{16}B_{10}^{-193}B_{11}^{42}B_{12}^{-86}B_{13}^{43}B_{14}^{414}B_{15}^{22}B_{16}^{152}B_{17}^{-60},\\
Q_{4,4}(1) &= B_1^{208}B_2^{244}B_3^{-10}B_4^{-2}B_5^{-30}B_6^{18}B_7^{-106}B_8^{-28}B_9^{-12}B_{10}^{-132}B_{11}^{-40}B_{12}^{88}B_{13}^{-6}B_{14}^{254}B_{15}^{21}B_{16}^{-12}B_{17}^{-80},\\
Q_{4,5}(1) &= B_1^{-86}B_2^{-89}B_3^{18}B_4^{3}B_5^{8}B_6^{-31}B_7^{129}B_8^{17}B_9^{-2}B_{10}^{197}B_{11}B_{12}^{-1}B_{13}^{-21}B_{14}^{-404}B_{15}^{-26}B_{16}^{-79}B_{17}^{85},\\
Q_{5,5}(1) &= B_{16}.
\end{aligned}
$$

□

Proof of Theorem 2.1. By Lemmas 3.4, 3.5 and 3.6 we already have that $P_{i,j}(1), Q_{i,j}(1) \in \langle x, y\rangle$, $1 \le i \le j \le 5$. Finally, we set $P_{i,j}(1) = P_{j,i}(1)$, $Q_{i,j}(1) = Q_{j,i}(1)$ for $i > j$ and use the commutator identity

$$
R_{i,k}(1) = P_{i,j}(1)\,Q_{j,k}(1)\,P_{i,j}(-1)\,Q_{j,k}(-1)
$$

which is true for any triple of pairwise distinct indices $1 \le i, j, k \le 5$. We conclude that $R_{i,j}(1) \in \langle x, y\rangle$ for $1 \le i \ne j \le 5$ and hence $\mathrm{ESp}_{10}(\mathbb{Z}) \subseteq \langle x, y\rangle$. Since $\mathrm{Sp}_{10}(\mathbb{Z}) = \mathrm{ESp}_{10}(\mathbb{Z})$ by [3], this finishes the proof of Theorem 2.1. □

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