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Metric subregularity of order q and the solving of inclusions

Research Article

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Abstract: We consider some metric regularity properties of order q for set-valued mappings and we establish several characterizations of these concepts in terms of Hölder-like properties of the inverses of the mappings considered. In addition, we show that even if these properties are weaker than the classical notions of regularity for set-valued maps, they allow us to solve variational inclusions under mild assumptions.

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1. Introduction

The origins of the concept of metric regularity go back to the Banach open mapping theorem established in the early 1930s (see, e.g., [5]). The latter can be formulated in many ways; here, we state it as follows:

Banach open mapping theorem.
For any linear and bounded mapping $A \in \mathcal{L}(X, Y)$ acting between two Banach spaces the following are equivalent:

(i) *A is surjective;*

(ii) *A is open (at every point);*

(iii) *there is a constant $\kappa > 0$ such that for all $y \in Y$ there exists $x \in X$ with $Ax = y$ and $\|x\| \leq \kappa\|y\|$.*

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The item (ii) above means that for any $x \in X$ and any neighborhood U of x the set $A(U)$ is a neighborhood of Ax in Y while assertion (iii) is clearly equivalent to the existence of a constant $\kappa > 0$ such that

$$d\big(0, A^{-1}(y)\big) \le \kappa \|y\| \qquad \text{for all} \quad y \in Y. \tag{1}$$

Keeping in mind that A is a linear mapping, one can show that relation (1) is equivalent to the existence of a positive constant κ such that

$$d\big(x, A^{-1}(y)\big) \le \kappa d(y, Ax) \qquad \text{for all} \quad x \in X, \quad y \in Y.$$

The above property is known as metric regularity (see e.g., [4, 8] for a comprehensive study on this topic) and can be extended in the set-valued framework as in Definition 1.1 below where a set-valued mapping F from a space X to the subsets of a space Y is indicated by $F : X \rightrightarrows Y$.

Definition 1.1.

A mapping $F : X \rightrightarrows Y$ is said to be *metrically regular at* $\bar{x}$ *for* $\bar{y}$ if $\bar{y} \in F(\bar{x})$ and there exist some positive constants κ, a and b such that

$$d\big(x, F^{-1}(y)\big) \le \kappa d(y, F(x)) \qquad \text{for all} \quad x \in B_a(\bar{x}), \quad y \in B_b(\bar{y}).$$

While the metric regularity of a bounded and linear operator is equivalent to the openness of the operator at every point, it has been shown (see, e.g., [6, 23]) that the metric regularity of a set-valued mapping F is equivalent to a stronger concept of openness known as *linear openness* (also known as *covering property*). We recall that a mapping $F : X \rightrightarrows Y$ is said to be linearly open at $\bar{x}$ for $\bar{y}$, where $\bar{y} \in F(\bar{x})$, if there is a positive constant κ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V + \kappa r B \subset F(x + rB) \qquad \text{whenever} \quad x + rB \subset U \text{ as } r > 0.$$

For single-valued mappings the linear openness relates to the conventional openness property, actually, it is stronger than the latter since it ensures the uniformity of covering around the point $\bar{x}$ with linear rate κ. For more details on this topic the reader could refer, for instance, to [21–23] and the references therein. In [6], Borwein and Zhuang considered the so-called *linear openness of (at) order* $p \ge 1$, the definition of which is given below:

A set-valued mapping $F : X \rightrightarrows Y$ is linearly open of order $p \ge 1$ at $\bar{x}$ for $\bar{y}$, where $\bar{y} \in F(\bar{x})$, if there is a positive constant κ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V + \kappa r^p B \subset F(x + rB) \qquad \text{whenever} \quad x + rB \subset U \text{ as } r > 0.$$

One can immediately note that if a mapping F is linearly open of order p at $\bar{x}$ for $\bar{y}$ then it enjoys the same property of order p' for any $p' > p$ (on a potentially smaller neighborhood U of $\bar{x}$ so that $r \in (0, 1)$). In particular, linear openness implies linear openness of order p for each $p > 1$. In their paper, Borwein and Zhuang showed that the linear openness of order p of a set-mapping F acting between metric spaces is actually equivalent to the *metric regularity of order* $q = 1/p$ of F and is also equivalent to a Hölder-type regularity property of its inverse F^{-1}. We state here the definition of metric regularity of order q ($[q]$-metric regularity for short).

Definition 1.2 ($[q]$-metric regularity).

Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. We say that F is $[q]$-*metrically regular,* $q \in (0, 1)$*, at* $\bar{x}$ *for* $\bar{y}$ if there are positive constants a, b and a constant $\kappa \ge 0$ such that

$$d\big(x, F^{-1}(y)\big) \le \kappa [d(y, F(x))]^q \qquad \text{for all} \quad x \in B_a(\bar{x}), \quad y \in B_b(\bar{y}). \tag{2}$$

Lately, Yen et al. [28] proposed alternative proofs, in normed spaces, of the equivalences first established by Borwein and Zhuang through a new systematic study of the linear openness at a positive order rate. It is worth noting that the concept of metric regularity of order q naturally occurs in many problems of variational analysis; for instance, consider the following differential inclusion:

$$x'(s) \in F(x(s)), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{DI}$$

where the mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally Lipschitz and also convex and compact valued. Then, one can prove (see, e.g., [12]) that the reachable set

$$R(t) = \{x(t) \,|\, x(\cdot) \text{ is a trajectory of (DI)}\}$$

happens to be metrically regular of order $q < 1$ whenever some mild assumptions on its variations are satisfied. Additional examples of metrically regular mappings of order q along with new developments in this theory can be found in [13]. It is also important to mention that interesting results regarding metric regularity of order q were obtained by Frankowska in two seminal works (see [11, 12]). The reader could also refer to [17] where a concept of *ρ-metric regularity* with respect to a function ρ is discussed.

In 2009, Kummer [19] investigated the existence and stability of solutions to inclusions of the form $F(x) \ni p$ where F was a set-valued mapping acting from a complete metric space X to a linear normed space P. To this end, he explored regularity properties for set-valued mappings involving an exponent q. In particular he considered the calmness with an exponent q and connected this property to certain iteration schemes of descent type. We will see in the next section that the calmness with an exponent q of a set-valued mapping is closely tied to some kind of metric regularity property of its inverse.

In the last two decades, a variety of authors investigated iterative frameworks for solving variational inclusions or generalized equations in the case when the set-valued mapping involved enjoyed some metric regularity properties (or when its inverse satisfied some Lipschitz-like properties), see for example [2, 7, 10, 14, 20]. To our knowledge there are no similar studies when the mappings we are dealing with enjoy a metric regularity property of order q. Carrying out such a study is the main purpose of this work. Our particular interest is in showing the feasibility of establishing converging methods for solving inclusions involving mappings that enjoy a so-called *metric subregularity property of order q*. Nevertheless, let us point out that our purpose is not, at this time, to provide an implementable algorithm in the proper sense.

The remainder of this paper is organized as follows. In Section 2, we found it useful to provide a few characterizations of the metric regularity of order q and we define the notion of metric subregularity of order q for which we provide also some characterizations. By doing so, we emphasize the similarities between standard metric (sub)regularity and metric (sub)regularity of order q. The result established in this section (especially Proposition 2.8) will play a central role in Section 3 where we show how the metric subregularity of order q can be used to prove the superlinear convergence of a method that we propose for solving variational inclusions in the finite dimensional setting.

Notation

Throughout, X and Y stand for real Banach spaces. The closed unit ball is denoted by B while $B_r(a)$ stands for the closed ball of radius r centered at a. We denote by $d(x, C)$ the distance from a point x to a set C, that is, $d(x, C) = \inf_{y \in C} \|x - y\|$.

Let F be a set-valued mapping from X into the subsets of Y, indicated by $F : X \rightrightarrows Y$. Here $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F and the range of F is the set $\operatorname{rge} F = \{y \in Y \mid F(x) \ni y \text{ for some } x\}$. The inverse of F, denoted by F^{-1}, is defined by: $x \in F^{-1}(y) \;\Leftrightarrow\; y \in F(x)$.

2. Characterizations of $[q]$-metric regularity

In this section we present several characterizations, in terms of Hölder-type properties, of two important concepts of metric regularity of order q. We start with the *standard* metric regularity of order q as presented in Definition 1.2.

Before going further, according to Kummer's definition (see [19]), we consider the Aubin property with exponent $q > 0$, called $[q]$-Aubin property, for set-valued mappings. Its definition reads as follows:

Definition 2.1 (Kummer [19]).

Let $S : Y \rightrightarrows X$ be a set-valued mapping and let $(\bar{y}, \bar{x}) \in \operatorname{gph} S$. Let $q \in (0, 1]$, we say that S *obeys the* $[q]$*-Aubin property at* $\bar{y}$ *for* $\bar{x}$ if

$$\begin{aligned} &\text{there exist} \quad a, b, \kappa > 0 \quad \text{such that for all} \quad y, y' \in B_b(\bar{y}) \\ &\qquad x \in S(y) \cap B_a(\bar{x}) \quad \Longrightarrow \quad B_{\kappa\|y-y'\|^q}(x) \cap S(y') \neq \emptyset. \end{aligned} \tag{3}$$

Note that Kummer called this property Aubin property $[q]$ at $(\bar{y}, \bar{x})$, we prefer here the terminology $[q]$-Aubin property at $\bar{y}$ for $\bar{x}$ since it is consistent with the one we use for describing the metric regularity.

We will now state some characterizations of the metric regularity of order q of a set-valued mapping in terms of Hölder-like properties of its inverse. First, we recall that given two subsets A and B of X, the excess of A beyond B, denoted by $e(A, B)$, is given by $e(A, B) = \sup_{x \in A} d(x, B)$. Along with this definition we use the following convention:

$$e(\emptyset, B) = \begin{cases} 0 & \text{when } B \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 2.2 (Characterizations of the $[q]$-metric regularity).

Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. We denote by S the inverse of F; i.e., $S = F^{-1}$. The following assertions are equivalent.

(i) *The mapping F is $[q]$-metrically regular at $\bar{x}$ for $\bar{y}$.*

(ii) *The mapping S has the $[q]$-Aubin property at $\bar{y}$ for $\bar{x}$.*

(iii) *There exist positive constants a, b and a constant $\kappa \geq 0$ such that*

$$e\big(S(y) \cap B_a(\bar{x}), S(y')\big) \leq \kappa \|y - y'\|^q \qquad \textit{for all} \quad y, y' \in B_b(\bar{y}). \tag{4}$$

(iv) *There exist positive constants a, b and a constant $\kappa \geq 0$ such that*

$$S(y) \cap B_a(\bar{x}) \subset S(y') + \kappa \|y - y'\|^q B \qquad \textit{for all} \quad y, y' \in B_b(\bar{y}). \tag{5}$$

(v) *There exist positive constants a, b and a constant $\kappa \geq 0$ such that*

$$e\big(S(y) \cap B_a(\bar{x}), S(y')\big) \leq \kappa \|y - y'\|^q \qquad \textit{for all} \quad y \in Y, \; y' \in B_b(\bar{y}). \tag{6}$$

Proof. We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i).

(i) ⇒ (ii). Assume that the mapping F is $[q]$-metrically regular at $\bar{x}$ for $\bar{y}$, then there are positive constants a and b, and a constant $\kappa \geq 0$ such that

$$d\big(x, F^{-1}(y)\big) \leq \kappa [d(y, F(x))]^q \qquad \text{for all} \quad x \in B_a(\bar{x}), \; y \in B_b(\bar{y}). \tag{7}$$

Take two arbitrary points y and y' in $B_b(\bar{y})$ and let $x \in S(y) \cap B_a(\bar{x})$. Then from (7) we get

$$d(x, S(y')) \leq \kappa \big[d\big(y', S^{-1}(x)\big)\big]^q \leq \kappa \|y - y'\|^q,$$

hence for any $\tilde{\kappa} > \kappa$ one has
$$d(x, S(y')) < \tilde{\kappa}\|y - y'\|^q.$$
Consequently, there exists a point x' in $S(y')$ such that $\|x - x'\| < \tilde{\kappa}\|y - y'\|^q$, i.e.,
$$B_{\tilde{\kappa}\|y-y'\|^q}(x) \cap S(y') \neq \emptyset,$$
and the mapping S obeys the $[q]$-Aubin property at $\bar{y}$ for $\bar{x}$.

(ii) ⇒ (iii). Let a, b, and κ be as in Definition 2.1, i.e., satisfying (3). Let $y, y' \in B_b(\bar{y})$ and take an arbitrary $x \in S(y) \cap B_a(\bar{x})$. Then there exists $x' \in S(y')$ such that
$$\|x - x'\| \leq \kappa\|y - y'\|^q$$
which implies that
$$d(x, S(y')) \leq \kappa\|y - y'\|^q.$$
Since x has been chosen arbitrarily in $S(y) \cap B_a(\bar{x})$ we infer
$$e\big(S(y) \cap B_a(\bar{x}), S(y')\big) \leq \kappa\|y - y'\|^q,$$
thus, assertion (iii) holds.

(iii) ⇒ (iv). The proof follows from the very definition of the excess.

(iv) ⇒ (v). Assume that (5) holds with constants a, b and κ. Let $0 < a' < a$ and $0 < b' < b$ be such that
$$\kappa b'^q + a' \leq \kappa(b - b')^q. \tag{8}$$
It is not difficult to see that such constants a' and b' exist. Indeed, if we take $b' < b/2$, i.e., $b' < b - b'$, a simple adjustment of a' suffices to obtain the desired inequality.

Moreover, for any $y' \in B_{b'}(\bar{y})$, we have from (5) that
$$\bar{x} \in S(\bar{y}) \cap B_a(\bar{x}) \subset S(y') + \kappa\|\bar{y} - y'\|^q B,$$
thus
$$d(\bar{x}, S(y')) \leq \kappa\|\bar{y} - y'\|^q \leq \kappa b'^q,$$
and hence
$$e\big(B_{a'}(\bar{x}), S(y')\big) \leq \kappa b'^q + a'. \tag{9}$$
Take any $y \in Y$. If $y \in B_b(\bar{y})$, then (5) yields $S(y) \cap B_a(\bar{x}) \subset S(y') + \kappa\|y - y'\|^q B$. So for any $x \in S(y) \cap B_a(\bar{x})$ we have $d(x, S(y')) \leq \kappa\|y - y'\|^q$. And it follows that
$$e\big(S(y) \cap B_a(\bar{x}), S(y')\big) \leq \kappa\|y - y'\|^q \qquad \text{for all} \quad y, y' \in B_b(\bar{y}). \tag{10}$$
Otherwise, assume that $y \notin B_b(\bar{y})$, i.e., $\|y - \bar{y}\| > b$. Then we have
$$\|y' - y\| \geq \|y - \bar{y}\| - \|y' - \bar{y}\| > b - b'.$$
So $\|y' - y\|^q > (b - b')^q$ and, thanks to (8), we get $\kappa b'^q + a' \leq \kappa(b - b')^q \leq \kappa\|y - y'\|^q$. Then, (9) yields
$$e\big(B_{a'}(\bar{x}), S(y')\big) \leq \kappa\|y - y'\|^q.$$

Since $S(y) \cap B_{a'}(\bar{x})$ is obviously a subset of $B_{a'}(\bar{x})$, inequality (6) holds for any $y \notin B_b(\bar{y})$. Keeping in mind relation (10) we complete the proof.

(v) ⇒ (i). Assume that relation (6) holds for some constants a, b and κ. We intend to show that inequality (2) is valid. Take $x \in B_a(\bar{x})$ and $y \in B_b(\bar{y})$. Let $z \in F(x)$ (if $F(x) = \emptyset$ then there is nothing to prove) we have $x \in F^{-1}(z) \cap B_a(\bar{x})$. Then, from (6), we get

$$d\big(x, F^{-1}(y)\big) \leq \kappa \|z - y\|^q.$$

Since z is an arbitrary point in $F(x)$ we obtain

$$d\big(x, F^{-1}(y)\big) \leq \kappa [d(y, F(x))]^q,$$

and the proof is complete. □

Remark 2.3.

Note that the equivalence between assertions (i) and (ii) does not mean that relations (2) and (3) are themselves equivalent. Indeed the growth constant κ in (3) may be slightly greater than the one in (2); nevertheless they happen to be equal in finite dimension whenever the mapping S is closed-valued (i.e, $S(y)$ is closed for any $y \in Y$). One can formulate the same remark regarding the equivalence between assertions (iii) and (iv). Moreover, the infimum of κ such that the five assertions in Proposition 2.2 hold is the same whichever formulation is adopted.

Assertion (v) provides an alternative description of the $[q]$-metric regularity in terms of excess which appears to be very useful. It asserts that one can simplify the characterization given in (iii) by letting the point y in (4) lie in the whole space Y.

The second concept of $[q]$-metric regularity we wish to consider here is the $[q]$-*metric subregularity* of a set-valued mapping. It is closely tied, and actually equivalent, to a calmness-type property of its inverse. In [19], Kummer introduced the following notion of $[q]$-calmness for a set-valued mapping:

Definition 2.4 (Kummer [19]).

Consider a set-valued mapping $S : Y \rightrightarrows X$ and take $(\bar{y}, \bar{x}) \in \operatorname{gph} S$. Let $q \in (0, 1]$, we say that S is $[q]$-*calm at* $\bar{y}$ *for* $\bar{x}$ if

$$\begin{array}{c} \text{there exist} \quad a, b, \kappa > 0 \quad \text{such that for all} \quad y \in B_b(\bar{y}) \\ x \in S(y) \cap B_a(\bar{x}) \quad \Longrightarrow \quad B_{\kappa\|y - \bar{y}\|^q}(x) \cap S(\bar{y}) \neq \emptyset. \end{array} \tag{11}$$

Kummer also established sufficient conditions to ensure the $[q]$-calmness of the set of feasible constraints of an optimization problem. More precisely, he considered the mapping $S_h : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$S_h \;:\; p \mapsto \big\{x \in \mathbb{R}^n \mid h_i(x) \leq p_i,\ i = 1, \ldots, m\big\},$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^2$ function satisfying $h(\bar{x}) = 0$ for some $\bar{x}$ in $\mathbb{R}^n$. Then, denoting by H the mapping $\max_i h_i$ and by ∂^c the Clarke subdifferential, he showed the following result:

Theorem 2.5 ([19, Theorem 4.11]).

The mapping S_h is 1/2-calm at 0 for $\bar{x}$ if $0 \notin \partial^c H(\bar{x})$ or if (otherwise) the contingent derivative $C(\partial^c H)$ is injective at $(\bar{x}, 0)$.

Additional (sufficient) conditions for $[q]$-calmness (when $q = 1/2$) can be found in [1, 18]. It turns out that the $[q]$-calmness of a mapping S is equivalent to the so-called $[q]$-metric subregularity of its inverse S^{-1} the definition of which is given below.

Definition 2.6 ([q]-metric subregularity).
Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. We say that the mapping F is *[q]-metrically subregular at $\bar{x}$ for $\bar{y}$*, $q \in (0, 1]$, if there are constants $\kappa \geq 0$ and $a > 0$ such that

$$d\big(x, F^{-1}(\bar{y})\big) \leq \kappa[d(\bar{y}, F(x))]^q \qquad \text{for all} \quad x \in B_a(\bar{x}). \tag{12}$$

The only difference between [q]-metric regularity and [q]-metric subregularity is that the data $\bar{y}$ is now fixed and is no more allowed to vary. Hence, obviously, [q]-metric regularity at some reference point implies [q]-metric subregularity at the same point. Besides, the [q]-metric subregularity is also tied to the standard concept of metric subregularity (see for instance [8] for an extensive discussion of this topic); more precisely, the metric subregularity of a mapping $F : X \rightrightarrows Y$ at $\bar{x}$ for $\bar{y}$ can also be described by inequality (12) with $q = 1$. It is clear that the [q]-metric subregularity is a weaker condition than the (standard) metric subregularity.

We exhibit now two examples of [q]-metrically subregular mappings. The first one is given by the simple mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $F(x) = \{x^2\}$. A straightforward computation, left to the reader, shows that F is metrically subregular at 0 for 0 of order $q = 1/2$ with a constant $\kappa = 1$.

Now, let H be a real Hilbert space and consider the space of all proper lower semicontinuous convex functions from H into $\mathbb{R} \cup \{\infty\}$, denoted by $\Gamma(H)$. For any $f \in \Gamma(H)$, we denote by ∂f the subdifferential of convex analysis of the function f. Then the following statement provides us with the second example we would like to present:

Proposition 2.7.
Let $f \in \Gamma(H)$ and $\bar{y} \in \partial f(\bar{x})$. If there is a neighborhood U of $\bar{x}$ along with a positive constant c such that the function f satisfies

$$f(x) \geq f(\bar{x}) - \langle \bar{y}, \bar{x} - x \rangle + c\big[d\big(x, (\partial f)^{-1}(\bar{y})\big)\big]^2 \qquad \textit{whenever} \quad x \in U, \tag{13}$$

then ∂f is [q]-metrically subregular at $\bar{y}$ for $\bar{x}$ for any $q \in (0, 1)$.

Note that, in the special case when $\bar{y} = 0$ (i.e., when dealing with critical points of the function f) condition (13) becomes the following quadratic growth condition:

$$f(x) \geq \inf f + c\big[d\big(x, (\partial f)^{-1}(\bar{y})\big)\big]^2 \qquad \text{for } x \text{ close to } \bar{x}.$$

Proposition 2.7 is a direct consequence of [3, Theorem 3.3] where it has been shown that relation (13) characterizes the (standard) metric subregularity of the mapping ∂f at $\bar{y}$ for $\bar{x}$.

As in the case of the [q]-metric regularity, we are able to provide several charaterizations of the [q]-metric subregularity.

Proposition 2.8 (Characterizations of the [q]-metric subregularity).
Consider a set-valued mapping $F : X \rightrightarrows Y$ and take $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. We denote by S the inverse of F; i.e., $S = F^{-1}$. Then the following assertions are equivalent.

(i) *The mapping F is [q]-metrically subregular at $\bar{x}$ for $\bar{y}$.*

(ii) *The mapping S is [q]-calm at $\bar{y}$ for $\bar{x}$.*

(iii) *There exist a positive constant a and a constant $\kappa \geq 0$ such that*

$$e\big(S(y) \cap B_a(\bar{x}), S(\bar{y})\big) \leq \kappa \|y - \bar{y}\|^q \qquad \textit{for all} \quad y \in Y. \tag{14}$$

(iv) *There exist a positive constant a and a constant $\kappa \geq 0$ such that*

$$S(y) \cap B_a(\bar{x}) \subset S(\bar{y}) + \kappa \|y - \bar{y}\|^q B \qquad \textit{for all} \quad y \in Y. \tag{15}$$

Proof. We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

(i) ⇒ (ii). Since the mapping F is $[q]$-metrically subregular at $\bar{x}$ for $\bar{y}$, there is a positive constant a along with a constant $\kappa \geq 0$ such that

$$d\big(x, F^{-1}(\bar{y})\big) \leq \kappa[d(\bar{y}, F(x))]^q \qquad \text{for all} \quad x \in B_a(\bar{x}). \tag{16}$$

Take an arbitrary point y in Y and let $x \in S(y) \cap B_a(\bar{x})$ (if $S(y) \cap B_a(\bar{x}) = \emptyset$ there is nothing to prove and we are done). Then from (16) we get

$$d(x, S(\bar{y})) \leq \kappa\big[d\big(\bar{y}, S^{-1}(x)\big)\big]^q \leq \kappa\|y - \bar{y}\|^q.$$

Hence for any $\tilde{\kappa} > \kappa$ we have $d(x, S(\bar{y})) < \tilde{\kappa}\|y - \bar{y}\|^q$ and it follows that there exists a point x' in $S(\bar{y})$ such that $\|x - x'\| < \tilde{\kappa}\|y - \bar{y}\|^q$, i.e.,

$$B_{\tilde{\kappa}\|y-\bar{y}\|^q}(x) \cap S(\bar{y}) \neq \emptyset,$$

so the mapping S is $[q]$-calm at $\bar{y}$ for $\bar{x}$.

(ii) ⇒ (iii). Assume that the mapping S is $[q]$-calm at $\bar{y}$ for $\bar{x}$. There are positive constants a, b, κ such that relation (11) holds. First we show that

$$d\big(x, F^{-1}(\bar{y})\big) \leq \kappa\big[d\big(\bar{y}, F(x) \cap B_b(\bar{y})\big)\big]^q \qquad \text{for all} \quad x \in B_a(\bar{x}). \tag{17}$$

Let $x \in B_a(\bar{x})$. If $F(x) \cap B_b(\bar{y}) = \emptyset$ then relation (17) is obviously valid. Otherwise, take $y \in F(x) \cap B_b(\bar{y})$. Then $x \in F^{-1}(y) \cap B_a(\bar{x})$ and thanks to (11) we obtain the existence of an element $x' \in F^{-1}(\bar{y})$ such that $\|x - x'\| \leq \kappa\|y - \bar{y}\|^q$. It follows that $d\big(x, F^{-1}(\bar{y})\big) \leq \kappa\|y - \bar{y}\|^q$. Passing to the infimum over y in the latter inequality a straightforward computation yields (17).

Now we show that there is a positive constant a' such that

$$d\big(x, F^{-1}(\bar{y})\big) \leq \kappa[d(\bar{y}, F(x))]^q \qquad \text{for all} \quad x \in B_{a'}(\bar{x}). \tag{18}$$

Let $b' \in (0, b)$, then $B_{b'}(\bar{y}) \subset B_b(\bar{y})$. Set $a' = \min\{a, \kappa b'^q\}$, then assertion (17) is still valid when we replace a with a' and b with b'. Hence we have

$$d\big(x, F^{-1}(\bar{y})\big) \leq \kappa\big[d\big(\bar{y}, F(x) \cap B_{b'}(\bar{y})\big)\big]^q \qquad \text{for all} \quad x \in B_{a'}(\bar{x}).$$

Take an arbitrary x in $B_{a'}(\bar{x})$. If $F(x) \cap B_{b'}(\bar{y}) \neq \emptyset$ then $d\big(\bar{y}, F(x) \cap B_{b'}(\bar{y})\big) = d(\bar{y}, F(x))$ and thus relation (18) holds. Otherwise, $F(x) \cap B_{b'}(\bar{y}) = \emptyset$ and consequently

$$[d(\bar{y}, F(x))]^q \geq b'^q \geq \frac{1}{\kappa}\|x - \bar{x}\| \geq \frac{1}{\kappa} d\big(x, F^{-1}(\bar{y})\big),$$

so (18) holds for any $x \in B_{a'}(\bar{x})$.

Now we show that inequality (14) holds for some constants that we will make precise. To this end, we consider any $y \in Y$. If $S(y) \cap B_{a'}(\bar{x}) = \emptyset$ then we are done; otherwise, take an arbitrary point x in $S(y) \cap B_{a'}(\bar{x})$. Thanks to (18) we get $d(x, S(\bar{y})) \leq \kappa\|y - \bar{y}\|^q$, and by passing to the supremum over x we get

$$e\big(S(y) \cap B_{a'}(\bar{x}), S(\bar{y})\big) \leq \kappa\|y - \bar{y}\|^q.$$

We obtain the desired conclusion.

(iii) ⇒ (iv). The proof is similar to the proof of (iii) ⇒ (iv) in Proposition 2.2.

(iv) ⇒ (i). Consider two constants a and κ such that relation (15) holds and let $x \in B_a(\bar{x})$. We show that

$$d\big(x, F^{-1}(\bar{y})\big) \leq \kappa[d(\bar{y}, F(x))]^q. \tag{19}$$

If $F(x) = \emptyset$ then inequality (19) is obviously valid. Otherwise, take $y \in F(x)$; then $x \in F^{-1}(y) \cap B_a(\bar{x})$ and thanks to (15) we get $x \in F^{-1}(\bar{y}) + \kappa\|y - \bar{y}\|^q B$. It follows that $d\big(x, F^{-1}(\bar{y})\big) \leq \kappa\|y - \bar{y}\|^q$. The latter being valid for all $y \in F(x)$ we obtain relation (19) and the proof is complete. □

Remark 2.9.
We learn from the proof of Proposition 2.8 (see (ii) ⇒ (iii)) that there is no need to consider a neighborhood of $\bar{y}$ in the definition of the $[q]$-calmness. Indeed, since assertions (ii) and (iii) are equivalent, an equivalent formulation of relation (11) in Definition 2.4 is given by: there exist $a, b, \kappa > 0$ such that for all $y \in Y$

$$x \in S(y) \cap B_a(\bar{x}) \quad \Longrightarrow \quad B_{\kappa \|y-\bar{y}\|^q}(x) \cap S(\bar{y}) \neq \emptyset,$$

where y may lie in the whole space Y.

Before ending this section, it is worth mentioning another characterization of the calmness with exponent q, due to Kummer [19]. It reads as follows.

Proposition 2.10.
Consider a set-valued mapping $S : Y \rightrightarrows X$ and $(\bar{y}, \bar{x}) \in \operatorname{gph} S$. Then S is $[q]$-calm, $q \in (0, 1)$, at $\bar{y}$ for $\bar{x}$ if and only if there exist $\varepsilon > 0$, $\alpha > 0$ such that

$$\alpha d(x, S(\bar{y})) \leq \big(\operatorname{dist} \big((\bar{y}, x), \operatorname{gph} S \big) \big)^q$$

for all $x \in B_\varepsilon(\bar{x})$, where $\operatorname{dist}((y, x), (y', x')) = \max \{\|y - y'\|, \|x - x'\|\}$.

Our focus for the remainder of this paper will involve metric subregularity of order q.

3. Successive approximations of $[q]$-metrically subregular inclusions

Problem statement

In this section we consider the inclusion

$$f(x) + G(x) \ni 0, \tag{20}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous single-valued map while $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a closed set-valued mapping; i.e., with closed graph. Such inclusions are known as *generalized equations* and where introduced by Robinson in the 1970s as an abstract model of variational problems (see, e.g., the survey [24]). Actually, they may serve as a general model for a wide variety of variational problems including linear and non-linear complementarity problems, systems of non-linear equations, variational inequalities (e.g., first-order necessary conditions for non-linear programming), etc. In particular, they may characterize optimality or equilibrium and then have several applications. For instance, the Walrasian law of competitive equilibria of exchange economies [26] can be formulated as a variational inclusion and so can the Wardrop principle of user equilibrium in traffic theory [27]. For more examples, the reader could refer to [9], where an extensive documentation of applications of finite-dimensional non-linear complementarity problems in engineering and equilibrium modelling is available.

Our purpose is to provide an iterative procedure for solving (20) and to show how the metric subregularity of order q can be an efficient tool for establishing the superlinear convergence of such a method.

We denote by $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ the solution map to (20) defined by

$$S(y) = (f + G)^{-1}(y) = \{x \in \mathbb{R}^n \mid y \in f(x) + G(x)\}.$$

It is worth pointing out that the solution set to (20), denoted by S (note that $S = S(0)$), is a closed nonempty subset of $\mathbb{R}^n$, consequently every point of $\mathbb{R}^n$ has a closest point in S (with respect to $\|\cdot\|$).

Algorithms for solving inclusions often generate a sequence of iterates by subsequently solving subproblems that can be simpler or more robust than the original problem. In our case, for a given $z \in \mathbb{R}^n$, we associate to (20) the approximate generalized equation

$$0 \in A(z, x) + G(x), \tag{21}$$

where $A : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping constructed on the basis of a single-valued approximation a, in a sense which we will make precise shortly, to the function f:

$$A(z, x) = a(z, x) + M\|x - z\|^{\mu} B, \tag{22}$$

for some real constants $M \geq 0$ and $\mu > 1$. The solution set of (20) being nonempty, we fix $\bar{x} \in S$. In addition we consider a real constant $\varepsilon \in (0, 1)$.

We assume that the mapping $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is an approximation to f in a neighborhood of $\bar{x}$, namely,

$$\|f(x) - a(z, x)\| \leq M\|x - z\|^{\mu} \qquad \text{for all} \quad x, z \in B_{\varepsilon}(\bar{x}), \tag{23}$$

where the constants M and μ are the ones introduced in (22). Relation (23) measures the quality of the approximation $a(z, x)$ to f at a point x. When f is a smooth function it is not difficult to define an approximation in the sense of (23). Indeed, it turns out that a special case of such approximations is given by the so-called (n, α)-*point based approximations*, an extension of the concept of point-based approximations introduced by Robinson in [25], which are easy to define in the differentiable setting (for instance, any Fréchet differentiable function such that its derivative is Lipschitz continuous or Hölder continuous admits a (n, α)-point-based approximation). For more examples and details on this topic the reader could refer to [15, 25] and references therein.

Iterative procedure for solving (20)

We are now going to propose an iterative scheme for the problem (20) by solving subsequently approximate inclusions closely related to (21). For any $z \in \mathbb{R}^n$ we denote by $\Sigma(z)$ the solution set of (21), i.e.,

$$\Sigma(z) = \{x \in \mathbb{R}^n \mid 0 \in A(z, x) + G(x)\}.$$

Moreover, since S is a closed and nonempty subset of $\mathbb{R}^n$, any element $z \in \mathbb{R}^n$ admits (at least) one closest point in S; such a point will be denoted by π_z. We then consider the following subset $Z(z)$ of $\Sigma(z)$, for any $z \in \mathbb{R}^n$, defined by

$$Z(z) = \left\{x \in \Sigma(z) \;\middle|\; \|x - \pi_z\| \leq \|z - \pi_z\|\right\}.$$

Let us now present the algorithm we will study in the sequel. Given any starting point x_0 in some neighborhood of $\bar{x}$, for $k = 0, 1, 2, \ldots$ compute x_{k+1} such that

$$\|x_{k+1} - \pi_{x_k}\| \leq \|x_k - \pi_{x_k}\| \tag{24}$$

and satisfying

$$0 \in A(x_k, x_{k+1}) + G(x_{k+1}).$$

The combination of these two conditions is equivalent to the simpler inclusion

$$x_{k+1} \in Z(x_k). \tag{25}$$

The essence of the method is to replace the set-valued mapping $f + G$ with the approximate mapping $A(\cdot, \cdot) + G(\cdot)$. A zero of this approximation then provides additional information that can be used to refine the approximation and thereby restart the process. By computing, for each k, an element x_{k+1} satisfying (25) we build a sequence such that each new iterate gets closer to S than the previous one. Indeed, assertion (24) implies that $d(x_{k+1}, S) \leq d(x_k, S)$. Moreover, it is worth noting that, combining relations (22) and (23) and keeping in mind that G is a closed set-valued mapping, if the sequence x_k converges to some element $x^* \in \mathbb{R}^n$ then necessarily $x^* \in S$.

Local behavior of the method

The following lemma asserts that the subproblem (21) admits at least one solution lying in $Z(z)$ whenever the point z is close enough to the solution $\bar{x}$ to (20).

Lemma 3.1.

Let $z \in B_{\varepsilon/3}(\bar{x})$, then $Z(z) \neq \emptyset$.

Proof. Since $\pi_z \in S$, we have

$$0 \in a(z, \pi_z) + G(\pi_z) + (f(\pi_z) - a(z, \pi_z)).$$

Moreover, keeping in mind that $\bar{x} \in S$, from the very definition of π_z we have $\|z - \pi_z\| \leq \|z - \bar{x}\|$. Then

$$\|\pi_z - \bar{x}\| \leq \|\pi_z - z\| + \|z - \bar{x}\| \leq 2\,\|z - \bar{x}\| \leq \frac{2}{3}\,\varepsilon,$$

hence both z and π_z are in $B_\varepsilon(\bar{x})$. Thus relation (23) yields

$$0 \in a(z, \pi_z) + G(\pi_z) + M\,\|\pi_z - z\|^\mu B = A(z, \pi_z) + G(\pi_z).$$

It follows that $\pi_z \in \Sigma(z)$ and, obviously, $\pi_z \in Z(z)$. Therefore $Z(z)$ is not empty. □

Lemma 3.2.

Let $z \in B_{\varepsilon/3}(\bar{x})$. Assume that the mapping $f + G$ is $[q]$-metrically subregular at $\bar{x}$ for 0, $q < 1$, with a constant κ such that $2\kappa\, 2^{q(\mu+1)} M^q < 1$. Then

$$\|x - \pi_x\| \leq \frac{1}{2}\,\|z - \pi_z\|^{\mu q} \qquad \text{whenever} \quad x \in Z(z).$$

Proof. Let x be an arbitrary point in $Z(z)$. From the definition of the set $Z(z)$ we have

$$\|x - \bar{x}\| \leq \|x - \pi_z\| + \|\pi_z - z\| + \|z - \bar{x}\| \leq 2\,\|z - \pi_z\| + \|z - \bar{x}\| \leq 3\,\|z - \bar{x}\| \leq \varepsilon.$$

Hence, $x \in B_\varepsilon(\bar{x})$. Since x is a solution of the generalized equation

$$0 \in f(x) + G(x) + A(z, x) - f(x),$$

there exists $y \in f(x) - A(z, x)$ such that

$$x \in S(y) \cap B_\varepsilon(\bar{x}).$$

Since the mapping $f + G$ is $[q]$-metrically subregular at $\bar{x}$ for 0, its inverse, which turns out to be equal to S, is $[q]$-calm at 0 for $\bar{x}$ (see Proposition 2.8). Hence, from Proposition 2.8 again, there is a positive constant a and a constant $\tilde{\kappa} > \kappa$ satisfying $2\tilde{\kappa}\, 2^{q(\mu+1)} M^q < 1$ such that

$$S(u) \cap B_a(\bar{x}) \subset S(0) + \tilde{\kappa}\,\|u\|^q B \qquad \text{for all} \quad u \in Y.$$

Considering a smaller ε if necessary, we obtain $x \in S + \tilde{\kappa}\,\|y\|^q B$. Then

$$\|x - \pi_x\| \leq \tilde{\kappa}\,\|y\|^q. \tag{26}$$

Since $y \in f(x) - A(z, x)$, from relation (22) together with (23) we get $\|y\| \leq 2M\|x - z\|^\mu$. Moreover,

$$\|x - z\| \leq \|x - \pi_z\| + \|\pi_z - z\| \leq 2\,\|z - \pi_z\|, \qquad \text{because} \quad x \in Z(z).$$

Hence,

$$\|y\| \leq 2^{\mu+1} M \|z - \pi_z\|^{\mu}.$$

Then, from (26), we have

$$\|x - \pi_x\| \leq 2^{q(\mu+1)} M^q \tilde{\kappa} \, \|z - \pi_z\|^{\mu q}.$$

Since $2\tilde{\kappa}\, 2^{q(\mu+1)} M^q < 1$, one has

$$\|x - \pi_x\| \leq \frac{1}{2} \|z - \pi_z\|^{\mu q},$$

and we are done. □

We are now ready to state our convergence theorem. The result we obtain is local since we need our initial guess to be sufficiently close to a solution to the problem.

Theorem 3.3.

Assume that the mapping $f+G$ is $[q]$-metrically subregular at $\bar{x}$ for $0, q < 1$, with a constant κ such that $2\kappa\, 2^{q(\mu+1)} M^q < 1$. In addition, let the constants μ and q be such that $\mu q > 1$. Then, there is a neigborhood Ω of $\bar{x}$ such that for any initial guess $x_0 \in \Omega$ there exists a sequence x_k whose elements are in $B_{\varepsilon/3}(\bar{x})$, satisfying (25) and converging superlinearly with order μq to some solution to the generalized equation (20).

Proof. Take an arbitrary point x_0 in $B_{\varepsilon/15}(\bar{x})$. First we show the existence of a sequence x_k starting from x_0 such that $x_{k+1} \in Z(x_k) \cap B_{\varepsilon/3}(\bar{x})$ for $k = 0, 1, 2, \ldots$ Thanks to Lemma 3.1 the set $Z(x_0)$ is nonempty, therefore we can take x_1 in $Z(x_0)$. Moreover,

$$\|x_1 - x_0\| \leq \|x_1 - \pi_{x_0}\| + \|\pi_{x_0} - x_0\| \leq 2\,\|x_0 - \pi_{x_0}\| \leq 2\,\|x_0 - \bar{x}\|.$$

It follows

$$\|x_1 - \bar{x}\| \leq \|x_1 - x_0\| + \|x_0 - \bar{x}\| \leq 3\,\|x_0 - \bar{x}\| \leq \frac{\varepsilon}{5},$$

that is $x_1 \in B_{\varepsilon/5}(\bar{x}) \subset B_{\varepsilon/3}(\bar{x})$. Hence $Z(x_1) \neq \emptyset$ and there is $x_2 \in Z(x_1)$.

Now, proceeding by induction, suppose that there are elements $x_0, x_1, \ldots, x_n$ in $B_{\varepsilon/3}(\bar{x})$ such that $x_{k+1} \in Z(x_k)$ for $k = 0, 1, \ldots, n-1$. From Lemma 3.1 there is an element $x_{n+1} \in Z(x_n)$. Let us show that $x_{n+1} \in B_{\varepsilon/3}(\bar{x})$. First, since $x_{k+1} \in Z(x_k)$ for $k = 0, 1, \ldots, n$, we have

$$\|x_{k+1} - x_k\| \leq \|x_{k+1} - \pi_{x_k}\| + \|\pi_{x_k} - x_k\| \leq 2\,\|x_k - \pi_{x_k}\|, \qquad k = 0, 1, \ldots, n.$$

Hence we get

$$\|x_{n+1} - x_0\| = \left\| \sum_{k=0}^{n} (x_{k+1} - x_k) \right\| \leq \sum_{k=0}^{n} \|x_{k+1} - x_k\| \leq 2 \sum_{k=0}^{n} \|x_k - \pi_{x_k}\|. \tag{27}$$

Then, using Lemma 3.2, we have for $k \geq 1$

$$\begin{aligned} \|x_k - \pi_{x_k}\| &\leq \frac{1}{2}\,\|x_{k-1} - \pi_{x_{k-1}}\|^{\mu q} \leq \frac{1}{2^{1+\mu q}}\,\|x_{k-2} - \pi_{x_{k-2}}\|^{(\mu q)^2} \\ &\leq \frac{1}{2^{1+\mu q+(\mu q)^2}}\,\|x_{k-3} - \pi_{x_{k-3}}\|^{(\mu q)^3} \leq \ldots \leq \frac{1}{2^{1+\mu q+(\mu q)^2+\ldots+(\mu q)^{(k-1)}}}\,\|x_0 - \pi_{x_0}\|^{(\mu q)^k}. \end{aligned}$$

Since $\mu q > 1$, a straightforward computation shows that

$$2^{1+\mu q+(\mu q)^2+\ldots+(\mu q)^{(k-1)}} \geq 2^k.$$

Then

$$\|x_k - \pi_{x_k}\| \leq \frac{1}{2^k}\,\|x_0 - \pi_{x_0}\|^{(\mu q)^k} \qquad \text{for all} \quad k \geq 1. \tag{28}$$

Combining relations (27) and (28) we obtain

$$\|x_{n+1}-x_0\| \leq 2\sum_{k=1}^{n}\frac{1}{2^k}\|x_0-\pi_{x_0}\|^{(\mu q)^k}+2\|x_0-\pi_{x_0}\|.$$

From the definition of π_0, together with the fact that $\bar{x}\in S$, we get

$$\|x_0-\pi_{x_0}\| \leq \|x_0-\bar{x}\| \leq \frac{\varepsilon}{15} < 1.$$

So we infer

$$\|x_{n+1}-x_0\| \leq 2\|x_0-\pi_{x_0}\|\sum_{k=1}^{n}\frac{1}{2^k}+2\|x_0-\pi_{x_0}\| \leq 4\|x_0-\pi_{x_0}\|. \tag{29}$$

Hence,

$$\|x_{n+1}-\bar{x}\| \leq \|x_{n+1}-x_0\|+\|x_0-\bar{x}\| \leq 4\|x_0-\pi_{x_0}\|+\|x_0-\bar{x}\| \leq 5\|x_0-\bar{x}\| \leq \frac{\varepsilon}{3}.$$

Consequently, x_{n+1} is in $B_{\varepsilon/3}(\bar{x})$ and there is a sequence x_k satisfying $x_{k+1}\in Z(x_k)$ for $k=0,1,2,\dots$ and the elements of which are in the ball $B_{\varepsilon/3}(\bar{x})$. In particular our algorithm (25) is well defined. Moreover, since the sequence x_k is bounded in $\mathbb{R}^n$ it admits a cluster point. We show that it is unique. To this end, assume that there are two different cluster points $\hat{x}$ and $\tilde{x}$. As we mentioned it previously, necessarily, both $\hat{x}$ and $\tilde{x}$ are in S. Since $\|\hat{x}-\tilde{x}\|>0$, there are positive integers i and j, $j>i$, such that

$$\max\{\|x_i-\hat{x}\|,\|x_j-\tilde{x}\|\} \leq \frac{1}{3}\|\hat{x}-\tilde{x}\|, \qquad d(x_i,S) \leq \frac{1}{16}\|\hat{x}-\tilde{x}\|.$$

Hence,

$$\|x_j-x_i\| \geq \|\hat{x}-\tilde{x}\|-\|x_i-\hat{x}\|-\|x_j-\tilde{x}\| \geq \frac{1}{3}\|\hat{x}-\tilde{x}\|. \tag{30}$$

Now, invoking the same arguments we used to establish relation (29), we have

$$\|x_j-x_i\| \leq 4\|x_i-\pi_{x_i}\|.$$

Then, from the choice of x_i, one has

$$\|x_j-x_i\| \leq 4\,d(x_i,S) \leq \frac{1}{4}\|\hat{x}-\tilde{x}\|. \tag{31}$$

Since (31) contradicts (30), we deduce that the sequence x_k has exactly one cluster point and therefore is convergent to some point $x^*\in S$. To complete the proof we show that the sequence x_k converges superlinearly to x^*. Once again, by the same arguments that the ones used to establish (29), we get for all integers l greater than 1,

$$\|x_{k+l}-x_{k+1}\| \leq 4\|x_{k+1}-\pi_{x_{k+1}}\|.$$

Moreover, thanks to Lemma 3.2 we have

$$\|x_{k+l}-x_{k+1}\| \leq 2\|x_k-\pi_{x_k}\|^{\mu q} \leq 2\|x_k-x^*\|^{\mu q}.$$

Letting l go to infinity we obtain

$$\|x_{k+1}-x^*\| \leq 2\|x_k-x^*\|^{\mu q},$$

that is the sequence x_k converges superlinearly to x^*. □

Remark 3.4.

The results stated in this section strongly rely on the existence of a projection over the closed set of solutions S. Hence, one way to extend this theory to the infinite dimensional setting is to assume that the solution set to (20) is a nonempty *proximinal* subset of a Banach space Y (in that case, f and G would be mappings acting between two Banach spaces X and Y). We recall that a subset A of a normed space $(E, \|\cdot\|)$ is said to be proximinal if every point of E has a closest point in A (with respect to $\|\cdot\|$). Any boundedly weakly compact set is proximinal; in particular so is any nonempty closed convex set in a reflexive Banach space. In finite dimensions, any nonempty closed set is proximinal. We did not carry out such a study because the finite dimensional setting is the one that covers most of the applications of our results. Indeed, for instance, both the differential inclusion (DI) in Section 1 and the optimization problem in Section 2 occur in $\mathbb{R}^n$. Of course, more practical applications lie also in the finite dimensional framework, for example, when investigating strong stationary solutions to an electricity spot market model, see e.g., [16], where metric regularity plays a central role in solving of such problems.

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