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Infinite injective transformations whose centralizers have simple structure

Research Article

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Abstract: For an infinite set X, denote by $\Gamma(X)$ the semigroup of all injective mappings from X to X under function composition. For $\alpha \in \Gamma(X)$, let $C(\alpha) = \{\beta \in \Gamma(X) : \alpha\beta = \beta\alpha\}$ be the centralizer of α in $\Gamma(X)$. The aim of this paper is to determine those elements of $\Gamma(X)$ whose centralizers have simple structure. We find $\alpha \in \Gamma(X)$ such that various Green's relations in $C(\alpha)$ coincide, characterize $\alpha \in \Gamma(X)$ such that the $\mathcal{J}$-classes of $C(\alpha)$ form a chain, and describe Green's relations in $C(\alpha)$ for α with so-called finite ray-cycle decomposition. If α is a permutation, we also find the structure of $C(\alpha)$ in terms of direct and wreath products of familiar semigroups.

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1. Introduction

For a semigroup S and an element $a \in S$, the *centralizer* $C(a)$ of a in S is defined by $C(a) = \{x \in S : ax = xa\}$. It is clear that $C(a)$ is a subsemigroup of S. A significant amount of research has been devoted to studying centralizers in S in the case when S is a semigroup of transformations (full or partial) on a finite set X. For example, the elements of such centralizers have been characterized in [9, 20, 24–26, 31]; Green's relations and regularity have been determined in [16–18]; and some representation theorems have been obtained in [22, 23, 29].

These investigations have been motivated by the fact that, if S is a semigroup of transformations on X that contains the identity id_X, then for any $\alpha \in S$, the centralizer $C(\alpha)$ is a generalization of S in the sense that $S = C(\mathrm{id}_X)$. It is therefore of interest to find out which ideas, approaches, and techniques used to study S can be extended to the centralizers of its elements.

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Centralizers of transformations are also important since they appear in various areas of mathematical research. For example, they play a role in finding the group of automorphisms of a general semigroup [4, Theorem 2.23]. They occur naturally in the theory of unary algebras since the monoid of endomorphisms of any mono-unary algebra $\mathcal{A} = \langle A, f \rangle$ is the centralizer $C(f)$ in the semigroup $T(A)$ of full transformations on A [15], and the group of automorphisms of $\mathcal{A}$ is $C(f) \cap \mathrm{Sym}(A)$, where $\mathrm{Sym}(A)$ is the symmetric group on A [11, 12]. Centralizers also appear in the study of commuting graphs of groups and semigroups (see, for example, [1, 5, 6, 13]). The commuting graph of a finite non-commutative semigroup S is a simple graph whose vertices are all non-central elements of S and two distinct vertices a, b are adjacent if $ab = ba$. Since $ab = ba$ if and only if $b \in C(a)$, the knowledge of the centralizers of transformations is helpful in studying the commuting graph of S whenever S is a semigroup of transformations.

Relatively little has been done regarding centralizers of transformations in the infinite case. For any set X, denote by $T(X)$ the semigroup of full transformations on X (all mappings from X to X) and by $\Gamma(X)$ the semigroup of injective transformations on X (all injective mappings from X to X). In both cases, the operation is the composition of functions. Both semigroups have the symmetric group $\mathrm{Sym}(X)$ of permutations on X as their group of units. (Note that if X is finite, then $\Gamma(X) = \mathrm{Sym}(X)$.) The centralizers of idempotent transformations in an infinite $T(X)$ have been studied in [2, 3, 30]. The author has studied the centralizers of transformations in an infinite $\Gamma(X)$ [19], where he characterized the elements of $C(\alpha)$ and determined Green's relations in $C(\alpha)$, including the partial orders of $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-classes.

The present paper follows up on [19]. The structure of the semigroup $\Gamma(X)$ in terms of Green's relations is simple: $\mathcal{H} = \mathcal{L}, \mathcal{R} = \mathcal{D} = \mathcal{J}$, and the $\mathcal{J}$-classes form a chain [19, Theorem 2.3]. The goal of the present paper is to determine the elements $\alpha \in \Gamma(X)$ whose centralizers have a simple structure, similar to the structure of $\Gamma(X)$. Research along similar lines has been done for other semigroups of transformations, for example for the semigroup $T(X, \rho)$ of transformations on X that preserve an equivalence relation ρ on X [27, 28], and for the semigroup $T(X, \rho, R)$ of transformations on X that preserve both ρ and a cross-section R of X/ρ [3].

In any centralizer $C(\alpha)$, where $\alpha \in \Gamma(X)$, Green's $\mathcal{L}$-relation has the same characterization as Green's $\mathcal{L}$-relation in $\Gamma(X)$: $\beta \,\mathcal{L}\, \gamma$ if and only if β and γ have the same image. However, in contrast with $\Gamma(X)$, Green's relations $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{J}$ in a general $C(\alpha)$ do not have the same characterization. In Section 3, we determine when various Green's relations in $C(\alpha)$ coincide. Also in contrast with $\Gamma(X)$, the $\mathcal{J}$-classes in a general $C(\alpha)$ do not form a chain. In Section 4, we find sufficient and necessary conditions for the $\mathcal{J}$-classes of $C(\alpha)$ to form a chain. Green's relations in $C(\alpha)$ have a particularly simple description when $\alpha \in \Gamma(X)$ has a finite ray-cycle decomposition (see Section 2). We describe Green's relations for such transformations α in Section 5. In Section 6, we assume that α is a permutation of X. Under this assumption, we find the structure of the centralizer $C(\alpha)$ in terms of direct and wreath products of semigroups of injective transformations, the group $\mathbb{Z}$ of integers, and groups $\mathbb{Z}_n$ of integers modulo n.

For the remainder of the paper, we assume that X is an arbitrary infinite set.

2. Centralizers in $\Gamma(X)$

In this section, to make the paper self-contained, we briefly describe some results obtained in [19]. If S is a semigroup and $a, b \in S$, we say that $a \,\mathcal{L}\, b$ if $S^1 a = S^1 b$, $a \,\mathcal{R}\, b$ if $a S^1 = b S^1$, and $a \,\mathcal{J}\, b$ if $S^1 a S^1 = S^1 b S^1$, where S^1 is the semigroup S with an identity adjoined. We define $\mathcal{H}$ as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$, that is, the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$. These five equivalence relations are known as *Green's relations* [10, p. 45]. Green's relations provide one of the most important tools in studying semigroups. For $a \in S$, we denote the equivalence class of a with respect to $\mathcal{J}$ by J_a, and refer to J_a as a $\mathcal{J}$-class. Since $\mathcal{J}$ is defined in terms of principal ideals in S, which are partially ordered by inclusion, we have the induced partial order in the set of the $\mathcal{J}$-classes: $J_a \le J_b$ if $S^1 a S^1 \subseteq S^1 b S^1$.

For $\alpha \in \Gamma(X)$, we denote the image of α by $\mathrm{im}(\alpha)$, the cardinality of $\mathrm{im}(\alpha)$, called the *rank* of α, by $\mathrm{rank}(\alpha)$, and the cardinality of $X \setminus \mathrm{im}(\alpha)$, called the *defect* of α, by $\mathrm{def}(\alpha)$. We will denote by $S(\alpha) = \{x \in X : x\alpha \ne x\}$ the set of elements shifted by α, and by $F(\alpha) = \{x \in X : x\alpha = x\}$ the set of elements fixed by α. (We will write mappings on the right: xf rather than $f(x)$, and compose from left to right: $x(fg)$ rather than $g(f(x))$.)

Green's relations in $\Gamma(X)$ have been determined in [19, Theorem 2.3]: $\alpha \,\mathcal{L}\, \beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$; $\alpha \,\mathcal{R}\, \beta \Leftrightarrow \mathrm{def}(\alpha) = \mathrm{def}(\beta)$; $\mathcal{H} = \mathcal{L}$ and $\mathcal{R} = \mathcal{D} = \mathcal{J}$; $J_\alpha \le J_\beta \Leftrightarrow \mathrm{def}(\alpha) \ge \mathrm{def}(\beta)$; the $\mathcal{J}$-classes in $\Gamma(X)$ form a chain.

Let $\dots, x_{-1}, x_0, x_1, \dots$ be pairwise distinct elements of X. We denote by $(x_0\, x_1\, x_2\, \dots\rangle$ the transformation $\eta \in \Gamma(X)$, called a *ray*, such that $x_i\eta = x_{i+1}$ for all $i \geq 0$ and $y\eta = y$ for all other $y \in X$; by $\langle \dots\, x_{-1}\, x_0\, x_1\, \dots\rangle$ the transformation $\omega \in \Gamma(X)$, called a *double ray*, such that $x_i\omega = x_{i+1}$ for all i and $y\omega = y$ for all other $y \in X$; and by $(x_0\, x_1\, \dots\, x_{n-1})$, where $n \geq 1$, the transformation $\lambda \in \Gamma(X)$, called an *n-cycle*, such that $x_i\lambda = x_{i+1}$ for all i, $0 \leq i < n$, where $x_{n-1}\lambda = x_0$, and $y\lambda = y$ for all other $y \in X$ (see [19, Definition 3.1]).

We say that $\alpha, \beta \in \Gamma(X)$ are *disjoint* if $S(\alpha) \cap S(\beta) = \emptyset$. Let M be a set of pairwise disjoint transformations in $\Gamma(X)$. The *formal product* of elements of M, denoted by $\prod_{\alpha \in M} \alpha$, is a transformation in $\Gamma(X)$ defined by

$$x\left(\prod_{\alpha \in M} \alpha\right) = \begin{cases} x\alpha & \text{if } x \in S(\alpha) \text{ for some } \alpha \in M, \\ x & \text{otherwise.} \end{cases}$$

If $A = \emptyset$, we agree that $\prod_{\alpha \in A} \alpha = \mathrm{id}_X$. (See [19, Definition 3.2].)

If $\alpha \in \Gamma(X)$ with $\alpha \neq \mathrm{id}_X$, then there exist unique sets: A of rays, B of double rays, and C of cycles of length at least 2 such that the transformations in $A \cup B \cup C$ are pairwise disjoint and

$$\alpha = \left(\prod_{\eta \in A} \eta\right)\left(\prod_{\omega \in B} \omega\right)\left(\prod_{\lambda \in C} \lambda\right). \tag{1}$$

(See [21, Proposition 3.3] and [19, Proposition 3.3].) We call the product (1) the *ray-cycle decomposition* of α.

For $\eta = (x_0\, x_1\, x_2\, \dots\rangle$, $\omega = \langle \dots\, x_{-1}\, x_0\, x_1\, \dots\rangle$, $\lambda = (x_0\, x_1\, \dots\, x_{n-1})$, and any β in $\Gamma(X)$, we define:

$$\eta\beta^* = (x_0\beta\, x_1\beta\, x_2\beta\, \dots\rangle, \qquad \omega\beta^* = \langle \dots\, x_{-1}\beta\, x_0\beta\, x_1\beta\, \dots\rangle, \qquad \lambda\beta^* = (x_0\beta\, x_1\beta\, \dots\, x_{n-1}\beta).$$

(See [19, Definition 3.5].) For $\alpha, \beta \in \Gamma(X)$, we will say that α is *contained* in β, and write $\alpha \sqsubset \beta$, if $x\alpha = x\beta$ for every $x \in S(\alpha)$. Note that all rays, double rays, and cycles from the ray-cycle decomposition of α are contained in α.

For $\alpha \in \Gamma(X)$, let A, B, and C be the sets that occur in the ray-cycle decomposition of α (see (1)). By A_α, B_α, and C_α we will mean the following sets:

$$A_\alpha = A, \qquad B_\alpha = B, \qquad C_\alpha = C \cup \big\{\{x\} : x \in F(\alpha)\big\}, \qquad C_\alpha^n = \big\{\lambda \in C_\alpha : \lambda \text{ is a cycle of length } n\big\},$$

where $n \geq 1$ and it is understood that $C_\alpha^1 = \big\{\{x\} : x \in F(\alpha)\big\}$. For $\beta \in \Gamma(X)$, we extend the definition of β^* by $\{x\}\beta^* = \{x\beta\}$ for every $\{x\} \in C_\alpha^1$. For $\lambda \in C_\alpha^n$, we will write $\lambda = (x_0\, x_1\, \dots\, x_{n-1})$, with the understanding that if $n = 1$, then $\lambda = \{x_0\}$ and $S(\{x_0\}) = \{x_0\}$. (See [19, Notation 3.7].)

The elements of the centralizer $C(\alpha)$ have been characterized in [19, Theorem 3.9].

Theorem 2.1.

Let $\alpha, \beta \in \Gamma(X)$. Then $\beta \in C(\alpha)$ if and only if for all $\eta \in A_\alpha$, $\omega \in B_\alpha$, and $\lambda \in C_\alpha$:

(1) *either there is a unique $\eta_1 \in A_\alpha$ such that $\eta\beta^* \sqsubset \eta_1$ or there is a unique $\omega_1 \in B_\alpha$ such that $\eta\beta^* \sqsubset \omega_1$;*

(2) *$\omega\beta^* \in B_\alpha$ and $\lambda\beta^* \in C_\alpha$.*

Let $\alpha \in \Gamma(X)$. For $\beta \in C(\alpha)$, we define a mapping $h_\beta : A_\alpha \cup B_\alpha \cup C_\alpha \to A_\alpha \cup B_\alpha \cup C_\alpha$ by

$$\delta h_\beta = \begin{cases} \eta & \text{if } \delta \in A_\alpha \text{ and } \delta\beta^* \sqsubset \eta \text{ for some } \eta \in A_\alpha, \\ \omega & \text{if } \delta \in A_\alpha \text{ and } \delta\beta^* \sqsubset \omega \text{ for some } \omega \in B_\alpha, \\ \delta\beta^* & \text{if } \delta \in B_\alpha \cup C_\alpha. \end{cases}$$

Note that h_β is well defined by Theorem 2.1. (See [19, Definition 4.1].)

Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. The following conditions occur in the descriptions of Green's relations in $C(\alpha)$.

$$|A_\alpha \setminus A_\alpha h_\beta| + \big|B_\alpha \setminus (A_\alpha h_\beta \cup B_\alpha h_\beta)\big| = |A_\alpha \setminus A_\alpha h_\gamma| + \big|B_\alpha \setminus (A_\alpha h_\gamma \cup B_\alpha h_\gamma)\big|. \tag{2}$$

$$\big|B_\alpha \setminus (A_\alpha h_\beta \cup B_\alpha h_\beta)\big| = \big|B_\alpha \setminus (A_\alpha h_\gamma \cup B_\alpha h_\gamma)\big|. \tag{3}$$

$$|C_\alpha^n \setminus C_\alpha^n h_\beta| = |C_\alpha^n \setminus C_\alpha^n h_\gamma| \qquad \text{for every } n \geq 1. \tag{4}$$

Green's relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{D}$ in $C(\alpha)$ have been characterized in [19, Theorems 4.4, 4.7, 4.9].

Theorem 2.2.

Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. Then

(1) *$\beta \mathcal{L} \gamma$ in $C(\alpha)$ if and only if* $\operatorname{im}(\beta) = \operatorname{im}(\gamma)$.

(2) *$\beta \mathcal{R} \gamma$ in $C(\alpha)$ if and only if*

 (a) *for every $\eta \in A_\alpha$,*

$$\eta h_\beta \in A_\alpha \;\text{ and }\; |S(\eta h_\beta) \setminus \operatorname{im}(\beta)| = k \quad \Leftrightarrow \quad \eta h_\gamma \in A_\alpha \;\text{ and }\; |S(\eta h_\gamma) \setminus \operatorname{im}(\gamma)| = k;$$

 (b) *conditions (2), (3), and (4) hold.*

(3) *$\beta \mathcal{D} \gamma$ in $C(\alpha)$ if and only if*

 (a) *there is a bijection $f : A_\alpha \cap A_\alpha h_\beta \to A_\alpha \cap A_\alpha h_\gamma$ such that for every $\eta \in A_\alpha \cap A_\alpha h_\beta$,*

$$|S(\eta) \setminus \operatorname{im}(\beta)| = |S(\eta f) \setminus \operatorname{im}(\gamma)|;$$

 (b) $|B_\alpha \cap A_\alpha h_\beta| = |B_\alpha \cap A_\alpha h_\gamma|$;

 (c) *conditions (2), (3), and (4) hold.*

The partial order of $\mathcal{J}$-classes in $C(\alpha)$ has been determined in [19, Theorem 4.8].

Theorem 2.3.

Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. Then $J_\gamma \leq J_\beta$ if an only if

(1) *There are injective mappings $f : A_\alpha \cap A_\alpha h_\gamma \to A_\alpha \cap A_\alpha h_\beta$ and $g : B_\alpha \cap A_\alpha h_\gamma \to (A_\alpha \cup B_\alpha) h_\beta$ such that $|S(\eta) \setminus \operatorname{im}(\gamma)| \geq |S(\eta f) \setminus \operatorname{im}(\beta)|$ for all $\eta \in A_\alpha \cap A_\alpha h_\gamma$, $\operatorname{im}(f) \cap \operatorname{im}(g) = \emptyset$, and*

$$|A_\alpha \setminus A_\alpha h_\gamma| + \big|B_\alpha \setminus (A_\alpha h_\gamma \cup B_\alpha h_\gamma)\big| \geq |A_\alpha \setminus A_\alpha h_\beta| + \big|B_\alpha \setminus (A_\alpha h_\beta \cup B_\alpha h_\beta)\big| + \big|A_\alpha h_\beta \setminus (\operatorname{im}(f) \cup \operatorname{im}(g))\big|;$$

(2) $|B_\alpha \setminus (A_\alpha h_\gamma \cup B_\alpha h_\gamma)| \geq |B_\alpha \setminus (A_\alpha h_\beta \cup B_\alpha h_\beta)|$;

(3) $|C_\alpha^n \setminus C_\alpha^n h_\gamma| \geq |C_\alpha^n \setminus C_\alpha^n h_\beta|$ *for every $n \geq 1$.*

3. Centralizers in which Green's relations coincide

In the semigroup $\Gamma(X)$, Green's relations $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{J}$ coincide (see Section 2). In this section, we describe the transformations $\alpha \in \Gamma(X)$ such that $\mathcal{R} = \mathcal{D}$ in $C(\alpha)$, and the transformations $\alpha \in \Gamma(X)$ such that $\mathcal{D} = \mathcal{J}$ in $C(\alpha)$. These descriptions will show that, in general, $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{J}$ are distinct in the centralizer $C(\alpha)$.

Theorem 3.1.
Let $\alpha \in \Gamma(X)$. Then $\mathcal{R} = \mathcal{D}$ in $C(\alpha)$ if and only if $|A_\alpha| \le 1$.

Proof. Let $A = A_\alpha$.

($\Rightarrow$) We will prove the contrapositive. Suppose $|A| > 1$ and let $\eta_1 = \langle x_0\, x_1\, x_2\, \dots\rangle$ and $\eta_2 = \langle y_0\, y_1\, y_2\, \dots\rangle$ be two distinct rays in A. Define $\beta \in \Gamma(X)$ by: $x_i\beta = y_{i+1}$ and $y_i\beta = x_i$ for every $i \ge 0$, and $x\beta = x$ for all other $x \in X$. Define $\gamma \in \Gamma(X)$ by: $x_i\gamma = x_i$ and $y_i\gamma = y_{i+1}$ for every $i \ge 0$, and $x\gamma = x$ for all other $x \in X$. Then $\beta, \gamma \in C(\alpha)$ by Theorem 2.1.

We have $(\beta,\gamma) \notin \mathcal{R}$ since β and γ do not satisfy (2a) of Theorem 2.2 ($|S(\eta_1 h_\beta) \setminus \operatorname{im}(\beta)| = 1$ but $|S(\eta_1 h_\gamma) \setminus \operatorname{im}(\gamma)| = 0$). On the other hand, $(\beta,\gamma) \in \mathcal{D}$ since β and γ satisfy (3a) of Theorem 2.2 (with $f : A \cap Ah_\beta \to A \cap Ah_\gamma$ defined by $\eta f = \eta$) and they also satisfy (3b) and (3c) of Theorem 2.2. Hence $\mathcal{R} \ne \mathcal{D}$.

($\Leftarrow$) Suppose $|A| \le 1$. Then (3a) of Theorem 2.2 implies (2a) of Theorem 2.2, and so $\mathcal{D} \subseteq \mathcal{R}$. Hence $\mathcal{R} = \mathcal{D}$ since $\mathcal{R} \subseteq \mathcal{D}$ in every semigroup. □

To describe the centralizers in which $\mathcal{D} = \mathcal{J}$, we will need the following lemma.

Lemma 3.2.
Let P and Q be finite sets with $|P| = |Q|$ and let $\delta_1 : P \to \{0,1,2,\dots\}$ and $\delta_2 : Q \to \{0,1,2,\dots\}$ be mappings. Suppose there are bijections $f_1 : P \to Q$ and $f_2 : Q \to P$ such that $p\delta_1 \ge (pf_1)\delta_2$ and $q\delta_2 \ge (qf_2)\delta_1$ for all $p \in P$ and $q \in Q$. Then $p\delta_1 = (pf_1)\delta_2$ for all $p \in P$.

Proof. For an integer $n \ge 0$, let $P_n = \{p \in P : p\delta_1 = n\}$ and $Q_n = \{q \in Q : q\delta_2 = n\}$. Let $n \ge 0$. We claim that for all $p \in P_n$ and $q \in Q_n$, we have $pf_1 \in Q_n$ and $qf_2 \in P_n$.

We proceed by induction on n. It is clear that the claim is true for $n = 0$. Let $n \ge 1$ and suppose the claim is true for every k, $0 \le k < n$. Let $p \in P_n$ and let $q = pf_1$ with $q\delta_2 = k$. Then $n = p\delta_1 \ge q\delta_2 = k$. Suppose to the contrary that $n > k$. By the inductive hypothesis, $p_1 f_1 \in Q_k$ and $q_1 f_2 \in P_k$ for all $p_1 \in P_k$ and $q_1 \in Q_k$. Since f_1 and f_2 are injective, it follows that $|P_k| = |Q_k|$ and, since P_k and Q_k are finite, that f_1 restricted to P_k is a bijection from P_k to Q_k. Hence, there is $p_2 \in P_k$ such that $q = p_2 f_1$. But then $p_2 f_1 = q = pf_1$, and so $p_2 = p$, which is a contradiction. Hence $n = k$, that is, $pf_1 \in Q_n$. By a similar argument, $qf_2 \in P_n$ for every $q \in Q_n$, and the claim follows by induction.

The result follows immediately from the claim. □

Theorem 3.3.
Let $\alpha \in \Gamma(X)$. Then $\mathcal{D} = \mathcal{J}$ in $C(\alpha)$ if and only if A_α is finite.

Proof. Let $A = A_\alpha$, $B = B_\alpha$, and $C_n = C_\alpha^n$, $n \ge 1$.

($\Rightarrow$) We will prove the contrapositive. Suppose A is infinite and let

$$\eta_1 = \langle x_0^1\, x_1^1\, x_2^1\, \dots\rangle, \qquad \eta_2 = \langle x_0^2\, x_1^2\, x_2^2\, \dots\rangle, \qquad \eta_3 = \langle x_0^3\, x_1^3\, x_2^3\, \dots\rangle, \qquad \dots,$$

be pairwise distinct rays in A.

Define $\beta : X \to X$ by: $\eta_n\beta^* = \eta_{2n}$ for every $n \ge 1$, and $x\beta = x$ for all other $x \in X$. Define $\gamma : X \to X$ by: $\eta_1\gamma^* = \langle x_1^1\, x_2^1\, x_3^1\, \dots\rangle$, $\eta_n\gamma^* = \eta_{2n}$ for every $n \ge 2$, and $x\gamma = x$ for all other $x \in X$. Then $\beta, \gamma \in C(\alpha)$ by Theorem 2.1.

We have $\eta_n h_\beta = \eta_{2n}$ for every $n \geq 1$, $\eta_1 h_\gamma = \eta_1$, and $\eta_n h_\gamma = \eta_{2n}$ for every $n \geq 2$. Thus $A \cap Ah_\beta = \{\eta_{2n} : n \geq 1\}$ and $A \cap Ah_\gamma = \{\eta_1\} \cup \{\eta_{2n} : n \geq 2\}$.

Define $f_1 : A \cap Ah_\beta \to A \cap Ah_\gamma$ by $\eta_{2n} f_1 = \eta_{2n+2}$, $n \geq 1$, $f_2 : A \cap Ah_\gamma \to A \cap Ah_\beta$ by: $\eta_1 f_2 = \eta_2$ and $\eta_{2n} f_2 = \eta_{2n}$, $n \geq 2$. By the definition of β and γ, the sets $B \cap Ah_\beta$ and $B \cap Ah_\gamma$ are empty. Let $g_1 : B \cap Ah_\gamma \to (A \cup B)h_\beta$ and $g_2 : B \cap Ah_\beta \to (A \cup B)h_\gamma$ be empty mappings. For every $n \geq 1$,

$$|S(\eta_{2n}) \setminus \operatorname{im}(\beta)| = 0 = |S(\eta_{2n+2}) \setminus \operatorname{im}(\gamma)| = |S(\eta_{2n} f_1) \setminus \operatorname{im}(\gamma)|.$$

Further, since $|S(\eta_{2n}) \setminus \operatorname{im}(\beta)| = 0$ for every $n \geq 1$, we have

$$|S(\eta_1) \setminus \operatorname{im}(\gamma)| \geq |S(\eta_2) \setminus \operatorname{im}(\beta)| = |S(\eta_1) f_2 \setminus \operatorname{im}(\beta)| \quad \text{and}$$
$$|S(\eta_{2n}) \setminus \operatorname{im}(\gamma)| \geq |S(\eta_{2n}) \setminus \operatorname{im}(\beta)| = |S(\eta_{2n} f_2) \setminus \operatorname{im}(\beta)| \quad \text{for every } n \geq 2.$$

By the definitions of β and γ, we also have $|A \setminus Ah_\gamma| = |A \setminus Ah_\beta| = \aleph_0$, $B \setminus (Ah_\beta \cup Bh_\beta) = B \setminus (Ah_\gamma \cup Bh_\gamma) = \emptyset$, $\big|Ah_\beta \setminus \big(\operatorname{im}(f_1) \cup \operatorname{im}(g_1)\big)\big| = \big|Ah_\gamma \setminus \big(\operatorname{im}(f_2) \cup \operatorname{im}(g_2)\big)\big| = 1$, and $C_n \setminus C_n h_\beta = C_n \setminus C_n h_\gamma = \emptyset$ for every $n \geq 1$. Hence $J_\gamma \leq J_\beta$ and $J_\beta \leq J_\gamma$ by Theorem 2.3, and so $(\beta, \gamma) \in \mathcal{J}$.

However, there is no bijection $f : A \cap Ah_\beta \to A \cap Ah_\gamma$ such that $|S(\eta_{2n}) \setminus \operatorname{im}(\beta)| = |S(\eta_{2n} f) \setminus \operatorname{im}(\gamma)|$, $n \geq 1$, since $|S(\eta_1) \setminus \operatorname{im}(\gamma)| = |\{x_0\}| = 1$ and $|S(\eta_{2n}) \setminus \operatorname{im}(\beta)| = 0$ for every $n \geq 1$. Thus $(\beta, \gamma) \notin \mathcal{D}$ by (3a) of Theorem 2.2, and so $\mathcal{D} \neq \mathcal{J}$.

($\Leftarrow$) Let A be finite. Suppose $(\beta, \gamma) \in \mathcal{J}$, that is, $J_\gamma \leq J_\beta$ and $J_\beta \leq J_\gamma$. By Theorem 2.3, there are injective mappings $f_1 : A \cap Ah_\beta \to A \cap Ah_\gamma$ and $f_2 : A \cap Ah_\gamma \to A \cap Ah_\beta$ such that

$$|S(\eta_1) \setminus \operatorname{im}(\beta)| \geq |S(\eta_1 f_1) \setminus \operatorname{im}(\gamma)| \quad \text{and} \quad |S(\eta_2) \setminus \operatorname{im}(\gamma)| \geq |S(\eta_2 f_2) \setminus \operatorname{im}(\beta)| \tag{5}$$

for all $\eta_1 \in A \cap Ah_\beta$ and $\eta_2 \in A \cap Ah_\gamma$. Note that f_1 and f_2 must be bijections since $A \cap Ah_\beta$ and $A \cap Ah_\gamma$ are finite sets. Define $\delta_1 : A \cap Ah_\beta \to \{0, 1, 2, \dots\}$ and $\delta_2 : A \cap Ah_\gamma \to \{0, 1, 2, \dots\}$ by: $\eta_1 \delta_1 = |S(\eta_1) \setminus \operatorname{im}(\beta)|$ and $\eta_2 \delta_2 = |S(\eta_2) \setminus \operatorname{im}(\gamma)|$, $\eta_1 \in A \cap Ah_\beta$, $\eta_2 \in A \cap Ah_\gamma$. By (5), $\eta_1 \delta_1 \geq (\eta_1 f_1)\delta_2$ and $\eta_2 \delta_2 \geq (\eta_2 f_2)\delta_1$ for all $\eta_1 \in A \cap Ah_\beta$ and $\eta_2 \in A \cap Ah_\gamma$. Thus, by Lemma 3.2,

$$|S(\eta) \setminus \operatorname{im}(\beta)| = \eta\delta_1 = (\eta f_1)\delta_2 = |S(\eta f_1) \setminus \operatorname{im}(\gamma)|$$

for every $\eta \in A \cap Ah_\beta$. Hence β and γ satisfy condition (3a) of Theorem 2.2. Since A is finite and h_β and h_γ are injective mappings, we have

$$|A \setminus Ah_\beta| = |B \cap Ah_\beta| = |A| - |A \cap Ah_\beta| \quad \text{and} \quad |A \setminus Ah_\gamma| = |B \cap Ah_\gamma| = |A| - |A \cap Ah_\gamma|. \tag{6}$$

We already know that $f_1 : A \cap Ah_\beta \to A \cap Ah_\gamma$ is a bijection. Thus $|A \cap Ah_\beta| = |A \cap Ah_\gamma|$, which, together with (6), gives

$$|B \cap Ah_\beta| = |B \cap Ah_\gamma| \quad \text{and} \quad |A \setminus Ah_\beta| = |A \setminus Ah_\gamma|. \tag{7}$$

Since $J_\gamma \leq J_\beta$ and $J_\beta \leq J_\gamma$, we have by Theorem 2.3 that

$$|B \setminus (Ah_\gamma \cup Bh_\gamma)| = |B \setminus (Ah_\beta \cup Bh_\beta)| \quad \text{and} \quad |C_n \setminus C_n h_\gamma| = |C_n \setminus C_n h_\beta| \quad \text{for every } n \geq 1. \tag{8}$$

Hence β and γ satisfy: condition (3b) of Theorem 2.2 (by (7)), condition (2) (by (7) and (8)), condition (3) (by (8)), and condition (4) (by (8)).

Thus $(\beta, \gamma) \in \mathcal{D}$, and so $\mathcal{J} \subseteq \mathcal{D}$. Hence $\mathcal{D} = \mathcal{J}$ since $\mathcal{D} \subseteq \mathcal{J}$ in every semigroup. □

Corollary 3.4.

Let $\alpha \in \Gamma(X)$. Then in $C(\alpha)$,

(1) *if $\mathcal{R} = \mathcal{D}$ then $\mathcal{R} = \mathcal{D} = \mathcal{J}$;*

(2) *$\mathcal{R} = \mathcal{D} = \mathcal{J}$ if and only if $|A_\alpha| \leq 1$.*

Proof. It follows immediately from Theorems 3.1 and 3.3. □

4. Centralizers in which $\mathcal{J}$-classes form a chain

In the semigroup $\Gamma(X)$, the $\mathcal{J}$-classes form a chain (see Section 2). In the centralizer $C(\alpha)$, this is true only for very special $\alpha \in \Gamma(X)$ described below.

Theorem 4.1.
Let $\alpha \in \Gamma(X)$. The partially ordered set of $\mathcal{J}$-classes in $C(\alpha)$ is a chain if and only if exactly one of the following conditions is satisfied:

(1) *$A_\alpha = \emptyset$, B_α is finite, and C_α^n is infinite for at most one $n \geq 1$;*

(2) *$A_\alpha = \emptyset$, B_α is infinite, and C_α^n is finite for all $n \geq 1$; or*

(3) *$|A_\alpha| = 1$, B_α is finite, and C_α^n is finite for all $n \geq 1$.*

Proof. Let $A = A_\alpha$, $B = B_\alpha$, and $C_n = C_\alpha^n$, $n \geq 1$.

($\Rightarrow$) We will prove the contrapositive. Suppose none of (1)–(3) holds. (Note that (1), (2), and (3) are mutually exclusive, so it is impossible for at least two of them to hold.)

Suppose B is infinite. Let $\omega_1, \omega_2, \omega_3, \dots$ be pairwise distinct double rays in B. Since (2) does not hold, either $A \neq \emptyset$ or C_n is infinite for some $n \geq 1$. Let $A \neq \emptyset$. Define $\beta : X \to X$ by: $\eta\beta^* = \eta$ for every $\eta \in A$, $\omega_i\beta^* = \omega_{i+1}$ for every $i \geq 1$, and $x\beta = x$ for all other $x \in X$. Define $\gamma : X \to X$ by: $\eta\gamma^* = \langle x_1\, x_2\, x_3\, \dots\rangle$ for every $\eta = \langle x_0\, x_1\, x_2\, \dots\rangle \in A$, $\omega_i\gamma^* = \omega_i$ for every $i \geq 1$, and $x\gamma = x$ for all other $x \in X$. Then $\beta, \gamma \in C(\alpha)$ by Theorem 2.1. We have $B \setminus (Ah_\beta \cup Bh_\beta) = \{\omega_1\}$ and $B \setminus (Ah_\gamma \cup Bh_\gamma) = \emptyset$. Thus $J_\gamma \nleq J_\beta$ by (2) of Theorem 2.3. We also have that $|S(\eta) \setminus \operatorname{im}(\beta)| = 0$ for every $\eta \in A \cap Ah_\beta$, and $|S(\eta) \setminus \operatorname{im}(\gamma)| = 1$ for every $\eta \in A \cap A\gamma$. Thus, since $A \neq \emptyset$ and $A \cap Ah_\beta = A \cap Ah_\gamma = A$, there is no injection $f : A \cap Ah_\beta \to A \cap Ah_\gamma$ such that $|S(\eta) \setminus \operatorname{im}(\beta)| \geq |S(\eta f) \setminus \operatorname{im}(\gamma)|$ for every $\eta \in A \cap Ah_\beta$, and so $J_\beta \nleq J_\gamma$ by (1) of Theorem 2.3.

Let C_n be infinite for some $n \geq 1$, say $n = m$. Let $\lambda_1, \lambda_2, \lambda_3, \dots$ be pairwise distinct cycles in C_m. Then we can easily define $\beta, \gamma \in C(\alpha)$ such that $B \setminus (Ah_\beta \cup Bh_\beta) = \{\omega_1\}$, $B \setminus (Ah_\gamma \cup Bh_\gamma) = \emptyset$, $C_m \setminus C_m h_\beta = \emptyset$, and $C_m \setminus C_m h_\gamma = \{\lambda_1\}$. For any such β and γ, the $\mathcal{J}$-classes J_β and J_γ are incompatible by Theorem 2.3.

Suppose B is finite. We consider two cases.

Case 1. C_n is finite for all $n \geq 1$.

Then, since (3) does not hold, $|A| \neq 1$, that is, $A = \emptyset$ or $|A| \geq 2$. But the former is impossible since (1) does not hold. Thus $|A| \geq 2$. Let $\eta_1 = \langle x_0\, x_1\, x_2\, \dots\rangle$ and $\eta_2 = \langle y_0\, y_1\, y_2\, \dots\rangle$ be distinct rays in A. Define $\beta : X \to X$ by: $\eta_1\beta^* = \eta_1$, $\eta_2\beta^* = \langle y_2\, y_3\, y_4\, \dots\rangle$, $\eta\beta^* = \langle z_2\, z_3\, z_4\, \dots\rangle$ for every $\eta = \langle z_0\, z_1\, z_2\, \dots\rangle \in A$ with $\eta \neq \eta_1, \eta_2$, and $x\beta = x$ for all other $x \in X$. Define $\gamma : X \to X$ by: $\eta_1\gamma^* = \langle x_1\, x_2\, x_3\, \dots\rangle$, $\eta_2\gamma^* = \langle y_1\, y_2\, y_3\, \dots\rangle$, $\eta\gamma^* = \langle z_1\, z_2\, z_3\, \dots\rangle$ for every $\eta = \langle z_0\, z_1\, z_2\, \dots\rangle \in A$ with $\eta \neq \eta_1, \eta_2$, and $x\gamma = x$ for all other $x \in X$. Then $\beta, \gamma \in C(\alpha)$ by Theorem 2.1. Note that $A \cap Ah_\beta = A \cap Ah_\gamma = A$. By the definition of β, we have $|S(\eta_1) \setminus \operatorname{im}(\beta)| = 0$, $|S(\eta_2) \setminus \operatorname{im}(\beta)| = 2$, and $|S(\eta) \setminus \operatorname{im}(\beta)| = 2$ for every $\eta \in A$ with $\eta \neq \eta_1, \eta_2$. By the definition of γ, we have $|S(\eta_1) \setminus \operatorname{im}(\gamma)| = 1$, $|S(\eta_2) \setminus \operatorname{im}(\gamma)| = 1$, and $|S(\eta) \setminus \operatorname{im}(\gamma)| = 1$ for every $\eta \in A$ with $\eta \neq \eta_1, \eta_2$. It follows that there is no injective mapping $f : A \to A$ such that $|S(\eta) \setminus \operatorname{im}(\gamma)| \geq |S(\eta f) \setminus \operatorname{im}(\beta)|$ for all $\eta \in A$, and there is no injective mapping $f' : A \to A$ such that $|S(\eta) \setminus \operatorname{im}(\beta)| \geq |S(\eta f') \setminus \operatorname{im}(\gamma)|$ for all $\eta \in A$. Thus $J_\gamma \nleq J_\beta$ and $J_\beta \nleq J_\gamma$ by (1) of Theorem 2.3.

Case 2. C_n is infinite for some $n \geq 1$, say $n = m$.

Let $\lambda_1, \lambda_2, \lambda_3, \dots$ be pairwise distinct cycles in C_m. Let $A \neq \emptyset$. Define $\beta : X \to X$ by: $\eta\beta^* = \eta$ for every $\eta \in A$, $\lambda_i\beta^* = \lambda_{i+1}$ for every $i \geq 1$, and $x\beta = x$ for all other $x \in X$. Define $\gamma : X \to X$ by: $\eta\gamma^* = \langle x_1\, x_2\, x_3\, \dots\rangle$ for every $\eta = \langle x_0\, x_1\, x_2\, \dots\rangle \in A$, $\lambda_i\gamma^* = \lambda_i$ for every $i \geq 1$, and $x\gamma = x$ for all other $x \in X$. Then $\beta, \gamma \in C(\alpha)$ by Theorem 2.1. We have $C_m \setminus C_m h_\beta = \{\lambda_1\}$ and $C_m \setminus C_m h_\gamma = \emptyset$. Thus $J_\gamma \nleq J_\beta$ by (3) of Theorem 2.3. We also have that $|S(\eta) \setminus \operatorname{im}(\beta)| = 0$ for every $\eta \in A \cap Ah_\beta$, and $|S(\eta) \setminus \operatorname{im}(\gamma)| = 1$ for every $\eta \in A \cap A\gamma$. Thus, since $A \neq \emptyset$ and $A \cap Ah_\beta = A \cap Ah_\gamma = A$, there is no injection $f : A \cap Ah_\beta \to A \cap Ah_\gamma$ such that $|S(\eta) \setminus \operatorname{im}(\beta)| \geq |S(\eta f) \setminus \operatorname{im}(\gamma)|$ for every $\eta \in A \cap Ah_\beta$, and so $J_\beta \nleq J_\gamma$ by (1) of Theorem 2.3.

Let $A = \emptyset$. Then, since (1) does not hold, there is $k \geq 1$ with $k \neq m$ such that $C_k \neq C_m$ and C_k is infinite. Let $\mu_1, \mu_2, \mu_3, \ldots$ be pairwise distinct cycles in C_k. Then we can define $\beta, \gamma \in C(\alpha)$ such that $C_m \setminus C_m h_\beta = \emptyset$, $C_m \setminus C_m h_\gamma = \{\lambda_1\}$, $C_k \setminus C_k h_\beta = \{\mu_1\}$, and $C_k \setminus C_k h_\gamma = \emptyset$. For any such β and γ, the $\mathcal{J}$-classes J_β and J_γ are incompatible by Theorem 2.3.

($\Leftarrow$) Suppose that exactly one of the conditions (1)–(3) is satisfied. Let $\beta, \gamma \in C(\alpha)$. We want to prove that $J_\gamma \leq J_\beta$ or $J_\beta \leq J_\gamma$.

Suppose B is infinite. Then (2) holds, and so $A = \emptyset$ and C_n is finite for every $n \geq 1$. Thus $C_n h_\beta = C_n h_\gamma = C_n$, and so $|C_n \setminus C_n h_\gamma| = 0 = |C_n \setminus C_n h_\beta|$ for every $n \geq 1$. Hence, by Theorem 2.3, $J_\gamma \leq J_\beta$ (if $|B \setminus B h_\gamma| \geq |B \setminus B h_\beta|$) or $J_\beta \leq J_\gamma$ (if $|B \setminus B h_\beta| \geq |B \setminus B h_\gamma|$).

Suppose B is finite. Then $B h_\beta = B h_\gamma = B$ and $B \cap A h_\beta = B \cap A h_\gamma = \emptyset$, and so $B \setminus (A h_\beta \cup B h_\beta) = B \setminus (A h_\gamma \cup B h_\gamma) = \emptyset$. Since (2) or (3) holds, we have that either $A = \emptyset$ or C_n is finite for every $n \geq 1$.

Suppose $A = \emptyset$ and C_n is finite for every $n \geq 1$. Then $C_n h_\beta = C_n h_\gamma = C_n$, and so $|C_n \setminus C_n h_\gamma| = 0 = |C_n \setminus C_n \beta|$ for every $n \geq 1$. Thus $J_\gamma \leq J_\beta$ by Theorem 2.3 (with f and g being empty mappings) and, similarly, $J_\beta \leq J_\gamma$. Hence $J_\beta = J_\gamma$.

Suppose $A = \emptyset$ and C_n is infinite for some $n \geq 1$. Then, by (1), C_n is infinite for exactly one $n \geq 1$, say $n = m$. Thus, by Theorem 2.3, $J_\gamma \leq J_\beta$ (if $|C_m \setminus C_m h_\gamma| \geq |C_m \setminus C_m h_\beta|$) or $J_\beta \leq J_\gamma$ (if $|C_m \setminus C_m h_\beta| \geq |C_m \setminus C_m h_\gamma|$).

Suppose $A \neq \emptyset$ and C_n is finite for every $n \geq 1$. Then (3) holds, and so A is a one-element set, say $A = \{\eta\}$. Note that $A \cap A h_\beta = A \cap A h_\gamma = A = \{\eta\}$. Thus, by Theorem 2.3, $J_\gamma \leq J_\beta$ (if $|S(\eta) \setminus \operatorname{im}(\gamma)| \geq |S(\eta) \setminus \operatorname{im}(\beta)|$) or $J_\beta \leq J_\gamma$ (if $|S(\eta) \setminus \operatorname{im}(\beta)| \geq |S(\eta) \setminus \operatorname{im}(\gamma)|$).

Hence, in all possible cases, J_β and J_γ are compatible, which completes the proof. □

5. Transformations with finite ray-cycle decomposition

The descriptions of Green's relations in $C(\alpha)$ simplify considerably when the sets A_α, B_α, and C_α^n (for every $n \geq 1$) are finite. If this happens, we will say that $\alpha \in \Gamma(X)$ has a *finite ray-cycle decomposition*. Let $\alpha \in \Gamma(X)$ have a finite ray-cycle decomposition, and let $\beta \in C(\alpha)$. Recall that h_β is injective and that it maps B_α to B_α. Thus, since B_α is finite, h_β restricted to B_α is a bijection from B_α to B_α. Similarly, for every $n \geq 1$, h_β restricted to C_α^n is a bijection from C_α^n to C_α^n. Further, h_β cannot map elements of A_α to B_α, and so h_β restricted to A_α is a bijection from A_α to A_α. Therefore, we have the following:

(1) $A_\alpha \cap A_\alpha h_\beta = A_\alpha$;

(2) $A_\alpha \setminus A_\alpha h_\beta = B_\alpha \cap A_\alpha h_\beta = B_\alpha \setminus (A_\alpha h_\beta \cup B_\alpha h_\beta) = \emptyset$;

(3) $C_\alpha^n \setminus C_\alpha^n h_\beta = \emptyset$ for every $n \geq 1$.

The following result follows immediately from (1)–(3) and Theorems 2.2 and 3.3.

Theorem 5.1.
Let $\alpha \in \Gamma(X)$ be a transformation with a finite ray-cycle decomposition, and let $\beta, \gamma \in C(\alpha)$. Then in $C(\alpha)$:

(1) *$\beta \mathcal{R} \gamma$ if an only if $|S(\eta h_\beta) \setminus \operatorname{im}(\beta)| = |S(\eta h_\gamma) \setminus \operatorname{im}(\gamma)|$ for every $\eta \in A_\alpha$.*

(2) *$\beta \mathcal{D} \gamma$ if and only if there is a bijection $f : A_\alpha \to A_\alpha$ such that $|S(\eta) \setminus \operatorname{im}(\beta)| = |S(\eta f) \setminus \operatorname{im}(\gamma)|$ for every $\eta \in A_\alpha$.*

(3) *$\mathcal{J} = \mathcal{D}$.*

6. The structure of the centralizer of a permutation

Let $\alpha \in \operatorname{Sym}(X)$. In this section, we prove that $C(\alpha)$ is isomorphic to a semigroup constructed using direct and wreath products of the semigroup $\Gamma(B_\alpha)$ of injective transformations on B_α, the semigroups $\Gamma(C_\alpha^n)$ of injective transformations on C_α^n, $n \geq 1$, the group $\mathbb{Z}$ of integers, and the groups $\mathbb{Z}_n$ of integers modulo n, $n \geq 1$.

First note that for every $\alpha \in \Gamma(X)$,

$$\alpha \in \mathrm{Sym}\,(X) \quad \Longleftrightarrow \quad A_\alpha = \emptyset. \tag{9}$$

The following theorem follows immediately from (9), Theorem 2.2, Corollary 3.4, and Theorem 4.1.

Theorem 6.1.

Let $\alpha \in \mathrm{Sym}\,(X)$. Then in $C(\alpha)$:

(1) *$\beta \mathcal{R} \gamma$ if and only if $|B_\alpha \setminus B_\alpha h_\beta| = |B_\alpha \setminus B_\alpha h_\gamma|$ and $|C_\alpha^n \setminus C_\alpha^n h_\beta| = |C_\alpha^n \setminus C_\alpha^n h_\gamma|$ for every $n \geq 1$.*

(2) *$\mathcal{R} = \mathcal{D} = \mathcal{J}$.*

(3) *The $\mathcal{J}$-classes form a chain if and only if at most one of the sets $B_\alpha, C_\alpha^1, C_\alpha^2, C_\alpha^3, \ldots$ is infinite.*

Let M be a set, S be a semigroup of transformations on M, and T be a semigroup. Denote by T^M the set of all mappings $f : M \to T$ and note that T^M with multiplication defined by

$$i(fg) = (if)(ig), \qquad f, g \in T^M, \quad i \in M,$$

is a semigroup. Define a multiplication on the set $T^M \times S$ by

$$(f, u)(g, v) = (fg^u, uv), \tag{10}$$

where $g^u \in T^M$ is defined by $i(g^u) = (iu)g$, so for every $i \in M$,

$$i(fg^u) = (if)[(iu)g].$$

It is straightforward to verify that $T^M \times S$ with multiplication (10) is a semigroup.

Definition 6.2.

The set $T^M \times S$ with multiplication (10) is called the *wreath product* of T and S (with respect to the set M), and denoted by $T \wr S$.

Note that $T \wr S$ is completely determined by T, S, and the set M. If S is a group of permutations on a set M and T is a group, then $T \wr S$ is the group wreath product (see [7, page 46], [14, page 79], and [29, page 272]).

We will be interested in the wreath products $\mathbb{Z} \wr \Gamma(M)$ and $\mathbb{Z}_n \wr \Gamma(M)$, where $\mathbb{Z}$ is the group of integers and $\mathbb{Z}_n$ is the group of integers modulo n. The wreath product $\mathbb{Z} \wr \Gamma(M)$ is the set $\mathbb{Z}^M \times \Gamma(M)$ with multiplication

$$(f, u)(g, v) = (f + g^u, uv) \qquad \text{with} \qquad i(f + g^u) = (if) + (iu)g, \tag{11}$$

where $f, g \in \mathbb{Z}^M$, $u, v \in \Gamma(M)$, and $i \in M$. The wreath product $\mathbb{Z}_n \wr \Gamma(M)$ is the set $\mathbb{Z}_n^M \times \Gamma(M)$ with multiplication as in (11) except that $f, g \in \mathbb{Z}_n^M$ and '$+$' is the addition modulo n.

For semigroups S and T, we write $S \cong T$ if S is isomorphic to T.

Proposition 6.3.

Let $\alpha \in \mathrm{Sym}\,(X)$ be a product of double rays (that is, $A_\alpha = \emptyset$ and $C_\alpha^n = \emptyset$ for every $n \geq 1$). Then $C(\alpha) \cong \big(\mathbb{Z} \wr \Gamma(B_\alpha)\big)$.

Proof. Let $B = B_\alpha$. For every $\omega \in B$, we fix an element $\omega_0 \in S(\omega)$. For every $i \in \mathbb{Z}$, let $\omega_i = \omega_0\alpha^i$. Then $\omega = \langle \dots \omega_{-1}\, \omega_0\, \omega_1 \dots \rangle$. We will define an isomorphism ϕ from $C(\alpha)$ to $\mathbb{Z} \wr \Gamma(B)$. Let $\beta \in C(\alpha)$. For $\omega \in B$, let $\rho = \omega h_\beta$. Then $\omega_0\beta = \rho_i$ for some $i \in \mathbb{Z}$. Define $f_\beta : B \to \mathbb{Z}$ by $\omega f_\beta = i$. Recall that $h_\beta : B \to B$ is injective, that is, $h_\beta \in \Gamma(B)$. We define $\phi : C(\alpha) \to \mathbb{Z}^B \times \Gamma(B)$ by

$$\beta\phi = (f_\beta, h_\beta), \qquad \beta \in C(\alpha).$$

To prove that ϕ is a homomorphism, let $\beta, \gamma \in C(\alpha)$. Then

$$(\beta\gamma)\phi = (f_{\beta\gamma}, h_{\beta\gamma}), \qquad (\beta\phi)(\gamma\phi) = (f_\beta, h_\beta)(f_\gamma, h_\gamma) = (f_\beta f_\gamma^{h_\beta}, h_\beta h_\gamma).$$

Let $\omega \in B$, $\rho = \omega h_\beta$, $\sigma = \rho h_\gamma$, $i = \omega f_\beta$, and $j = \rho f_\gamma$. Then

$$\omega(f_\beta f_\gamma^{h_\beta}) = \omega f_\beta + (\omega h_\beta) f_\gamma = i + \rho f_\gamma = i + j. \tag{12}$$

On the other hand,

$$\omega_0(\beta\gamma) = \rho_i\gamma = (\rho_0\alpha^i)\gamma = \rho_0(\alpha^i\gamma) = \rho_0(\gamma\alpha^i) = \sigma_j\alpha^i = \sigma_{i+j}, \tag{13}$$

where $\omega_0\beta = \rho_i$ since $\omega f_\beta = i$, $\rho_0\gamma = \sigma_j$ since $\rho f_\gamma = j$, and $\alpha^i\gamma = \gamma\alpha^i$ since $\gamma \in C(\alpha)$. By (13) and the definition of $f_{\beta\gamma}$, we have that $\omega f_{\beta\gamma} = i + j$. Hence $f_{\beta\gamma} = f_\beta f_\gamma^{h_\beta}$ by (12). Thus, since $h_{\beta\gamma} = h_\beta h_\gamma$ (see [19, Lemma 4.2]),

$$(\beta\gamma)\phi = (f_{\beta\gamma}, h_{\beta\gamma}) = (f_\beta f_\gamma^{h_\beta}, h_\beta h_\gamma) = (\beta\phi)(\gamma\phi),$$

that is, ϕ is a homomorphism.

To prove that ϕ is one-to-one, let $\beta, \gamma \in C(\alpha)$ and suppose $\beta\phi = \gamma\phi$, that is, $f_\beta = f_\gamma$ and $h_\beta = h_\gamma$. Let $\omega \in B$, $\rho = \omega h_\beta = \omega h_\gamma$, and $i = \omega f_\beta = \omega f_\gamma$. Then for every $j \in \mathbb{Z}$,

$$\omega_j\beta = (\omega_0\alpha^j)\beta = \omega_0(\alpha^j\beta) = \omega_0(\beta\alpha^j) = (\omega_0\beta)\alpha^j = \rho_i\alpha^j,$$

and, similarly, $\omega_j\gamma = \rho_i\alpha^j$. Thus $\beta = \gamma$.

Finally, to prove that ϕ is onto, let $(f, h) \in \mathbb{Z}^B \times \Gamma(B)$. Let $\omega \in B$, $\rho = \omega h$, and $i = \omega f$. Define $\beta : X \to X$ by

$$\omega_j\beta = \rho_{i+j}, \qquad j \in \mathbb{Z}.$$

It is clear that β is injective. By Theorem 2.1, we have that $\beta \in C(\alpha)$. We also have $h_\beta = h$ (by the definition of β) and $f_\beta = f$ (since $\omega_0\beta = \rho_{i+0} = \rho_i$, and so $\omega f_\beta = i = \omega f$). Hence $(f, h) = (f_\beta, h_\beta) = \beta\phi$. □

Proposition 6.4.
Let $\alpha \in \mathrm{Sym}(X)$ be a product of cycles of fixed length n (that is, $A_\alpha = B_\alpha = \emptyset$ and $C_\alpha^m = \emptyset$ for every $m \neq n$). Then $C(\alpha) \cong \big(\mathbb{Z}_n \wr \Gamma(C_\alpha^n)\big)$.

Proof. Let $C_n = C_\alpha^n$. For every $\lambda \in C_n$, we fix an element $\lambda_0 \in S(\lambda)$. For every $i \in \mathbb{Z}_n = \{0, 1, \dots, n-1\}$, let $\lambda_i = \lambda_0\alpha^i$. Then $\lambda = (\lambda_0\, \lambda_1 \dots \lambda_{n-1})$. We will define an isomorphism ϕ from $C(\alpha)$ to $\mathbb{Z}_n \wr \Gamma(C_n)$. Let $\beta \in C(\alpha)$. For $\lambda \in C_n$, let $\mu = \lambda h_\beta$. Then $\lambda_0\beta = \mu_i$ for some $i \in \mathbb{Z}_n$. Define $f_\beta : C_n \to \mathbb{Z}_n$ by $\lambda f_\beta = i$. We define $\phi : C(\alpha) \to \mathbb{Z}_n^{C_n} \times \Gamma(C_n)$ by

$$\beta\phi = (f_\beta, h_\beta), \qquad \beta \in C(\alpha).$$

Now, the argument from the proof of Proposition 6.3 can be used (almost verbatim) to prove that ϕ is an isomorphism. □

Let I be a nonempty index set and let $\{S_i : i \in I\}$ be a collection of semigroups indexed by I. The direct product of the collection, denoted $\bigotimes_{i\in I} S_i$, is the set of all mappings $p : I \to \bigcup_{i\in I} S_i$ such that $ip \in S_i$ for every $i \in I$, with multiplication defined by $i(pq) = (ip)(iq)$, $p, q \in \bigotimes_{i\in I} S_i$, $i \in I$. The direct product of any collection of semigroups is a semigroup [10, page 5]. If S_i is a group for every $i \in I$, then $\bigotimes_{i\in I} S_i$ is a group [8, Exercise 15, page 157]. If $I = \{1, 2, \ldots, m\}$, we write the direct product $\bigotimes_{i\in I} S_i$ as $S_1 \times S_2 \times \cdots \times S_m$ and elements $p \in \bigotimes_{i\in I} S_i$ as m-tuples $(1p, 2p, \ldots, mp)$.

Theorem 6.5.

Let $\alpha \in \mathrm{Sym}(X)$. Then

$$C(\alpha) \cong \big(\mathbb{Z} \wr \Gamma(B_\alpha)\big) \times \bigotimes_{n\ge 1} \big(\mathbb{Z}_n \wr \Gamma(C^n_\alpha)\big).$$

Proof. Let $B = B_\alpha$ and $C_n = C^n_\alpha$ for every $n \ge 1$. Let $X_0 = \bigcup_{\omega\in B} S(\omega)$ and let $X_n = \bigcup_{\lambda\in C_n} S(\lambda)$, $n \ge 1$. Let $\beta \in \Gamma(X)$. For each integer $m \ge 0$, denote by α_m and β_m the restrictions of α and β to X_m, respectively. Then $\alpha_m \in \mathrm{Sym}(X_m)$ for every $m \ge 0$. By Theorem 2.1,

$$\beta \in C(\alpha) \iff \beta_m \in C(\alpha_m), \qquad m \ge 0. \tag{14}$$

It follows from (14) that the mapping $\phi : C(\alpha) \to \bigotimes_{m\ge 0} C(\alpha_m)$ defined by

$$m(\beta\phi) = \beta_m, \qquad \beta \in C(\alpha), \quad m \ge 0,$$

is an isomorphism. By Proposition 6.3, $C(\alpha_0) \cong \big(\mathbb{Z} \wr \Gamma(B)\big)$. By Proposition 6.4, $C(\alpha_n) \cong \big(\mathbb{Z}_n \wr \Gamma(C_n)\big)$ for every $n \ge 1$. Thus

$$C(\alpha) \cong \bigotimes_{m\ge 0} C(\alpha_m) \cong \big(\mathbb{Z} \wr \Gamma(B)\big) \times \bigotimes_{n\ge 1} \big(\mathbb{Z}_n \wr \Gamma(C_n)\big),$$

which concludes the proof. □

For a monoid S, we denote by $U(S)$ the group of units of S. Let M be a set, S be a monoid of transformations of M with identity id_M ($i\,\mathrm{id}_M = i$ for every $i \in M$), and T be a monoid with identity 1. Then the wreath product $T \wr S$ is a monoid with identity (ι, id_M), where $\iota : M \to T$ is the mapping such that $i\iota = 1$ for every $i \in M$. It is straightforward to show that the group of units of $T \wr S$ is the wreath product $U(T) \wr \big(S \cap \mathrm{Sym}(M)\big)$.

For every $\alpha \in \Gamma(X)$, the centralizer $C(\alpha)$ is a monoid with identity id_X and the group of units $U(C(\alpha)) = C(\alpha) \cap \mathrm{Sym}(X)$. Theorem 6.5 and the foregoing observations give the following result.

Theorem 6.6.

Let $\alpha \in \mathrm{Sym}(X)$. Then

$$U(C(\alpha)) \cong \big(\mathbb{Z} \wr \mathrm{Sym}(B_\alpha)\big) \times \bigotimes_{n\ge 1} \big(\mathbb{Z}_n \wr \mathrm{Sym}(C^n_\alpha)\big).$$

Theorem 6.6 generalizes the result obtained for permutations of a finite set [29]. Suppose X is finite and let $\alpha \in \mathrm{Sym}(X)$. Then $B_\alpha = \emptyset$ and $C^n_\alpha = \emptyset$ for almost all $n \ge 1$. Let $1 \le n_1 < n_2 < \ldots < n_k$ be the integers such that $C^{n_i}_\alpha \ne \emptyset$ for every $i \in \{1, 2, \ldots, k\}$ and $C^n_\alpha = \emptyset$ for all other $n \ge 1$. Denote by $C'(\alpha)$ the centralizer of α relative to $\mathrm{Sym}(X)$ and note that $C'(\alpha) = C(\alpha) \cap \mathrm{Sym}(X)$. Then, by Theorem 6.6,

$$C'(\alpha) \cong \big(\mathbb{Z}_{n_1} \wr \mathrm{Sym}(C^{n_1}_\alpha)\big) \times \big(\mathbb{Z}_{n_2} \wr \mathrm{Sym}(C^{n_2}_\alpha)\big) \times \cdots \times \big(\mathbb{Z}_{n_k} \wr \mathrm{Sym}(C^{n_k}_\alpha)\big),$$

which is the result stated in [29, (2.12) and (2.13)].

We finish this paper with some problems concerning the structure of centralizers in semigroups of transformations other than $\Gamma(X)$.

(1) Determine the structure of centralizers in various subsemigroups of $T(X)$, for example in the semigroup $\Omega(X)$ of all surjective transformations.

(2) Determine the structure of centralizers in various subsemigroups of the semigroup $P(X)$ of partial transformations on X, for example in the symmetric inverse semigroup $I(X)$ of all partial injective transformations.

(3) Determine the structure of centralizers in various subsemigroups of the semigroup $L(V)$ of all linear transformations on a vector space V.

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