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Ascents of size less than d in compositions

Research Article

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Abstract: A composition of a positive integer n is a finite sequence $\pi_1\pi_2\ldots\pi_m$ of positive integers such that $\pi_1+\cdots+\pi_m=n$. Let d be a fixed number. We say that we have an ascent of size d or more (respectively, less than d) if $\pi_{i+1}\geq\pi_i+d$ (respectively, $\pi_i<\pi_{i+1}<\pi_i+d$). Recently, Brennan and Knopfmacher determined the mean, variance and limiting distribution of the number of ascents of size d or more in the set of compositions of n. In this paper, we find an explicit formula for the multi-variable generating function for the number of compositions of n according to the number of parts, ascents of size d or more, ascents of size less than d, descents and levels. Also, we extend the results of Brennan and Knopfmacher to the case of ascents of size less than d. More precisely, we determine the mean, variance and limiting distribution of the number of ascents of size less than d in the set of compositions of n.

MSC: 05A05, 05A15

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1. Introduction

A *composition* of $n\in\mathbb{N}$ is an ordered m-tuple $\pi_1\pi_2\ldots\pi_m$ of positive integers whose sum is n. The number of summands or letters, namely m, is called the number of *parts* of the composition. For example, the compositions of 4 are 1111, 112, 121, 211, 22, 13, 31 and 4. It is well known that the number of compositions of n is given by 2^{n-1}. We denote the set of compositions of n with m parts by $C_{n,m}$ and the set of compositions of n by C_n.

Let $d\geq 1$ be any integer and let $\pi=\pi_1\pi_2\ldots\pi_m$ be any composition of n. We say that we have a *descent*, *ascent*, or *level* at i in π if $\pi_i>\pi_{i+1}$, $\pi_i<\pi_{i+1}$, or $\pi_i=\pi_{i+1}$, respectively. We say that we have an *ascent of size d or more* (respectively, *ascent of size less than d*) at i in π if $\pi_{i+1}\geq\pi_i+d$ (respectively, $\pi_i<\pi_{i+1}<\pi_i+d$). For instance,

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there are 2 ascents of size 2 or more (respectively, two ascents of size less than 3) that occur in the composition $\sigma = 141224$, namely 14 and 24 (respectively, 12 and 24). Also σ contains one descent (41), three ascents (14, 12 and 24) and one level (22). We denote the number of descents, ascents of size d or more, ascents of size less than d, levels in π by $\operatorname{des}(\pi)$, $\operatorname{asc}_d^+(\pi)$, $\operatorname{asc}_d^-(\pi)$, $\operatorname{lev}(\pi)$, respectively. Clearly, the number of ascents in a composition π is given by $\operatorname{asc}_1^+(\pi) = \operatorname{asc}_d^+(\pi) + \operatorname{asc}_d^-(\pi)$, for all $d \geq 1$.

In recent years, questions relating to statistics of compositions of n have attracted quite a lot of attention: see [7] and references therein. For instance, Brennan and Knopfmacher [1] studied the mean, variance and asymptotic distribution of the number of ascents of size d or more in compositions of n. In this paper we generalize these results by studying the generating function for the number of compositions of n according to the number of parts, number of ascents of size d or more, number of ascents of size less than d, number of descents and number of levels, as discussed in Section 2. In particular, in Section 3 we determine the mean, variance and asymptotic distribution of the number of ascents of size less than d in compositions of n.

2. Enumeration of compositions according to ascents of size less than d

Let $F_d(z, u, w; p, q, r, s)$ be the multi-variable generating function for the number of compositions of n with exactly m parts, the number of ascents of size d or more, the number of ascents of size less than d, number of descents and number of levels, that is,

$$F_d(z, u, w) = F_d(z, u, w; p, q, r, s) = \sum_{n,m\geq 0} z^n u^m \sum_{\pi\in C_{n,m}} w^{\pi_m} p^{\operatorname{asc}_d^+(\pi)} q^{\operatorname{asc}_d^-(\pi)} r^{\operatorname{des}(\pi)} s^{\operatorname{lev}(\pi)},$$

where we use z to mark (the keyword "mark" is a part of the symbolic combinatorics vocabulary, see [4]) the composition size (that is, the number n), u to mark the number of parts (that is, the number m), w to mark the last part, p to mark the ascents of size d or more, q to mark the ascents of size less than d, r to mark the descents, and s to mark the levels. The main goal of this section is to find an explicit formula for the generating function $F_d(z, u, 1; p, q, r, s)$, that is, the generating function for the number of compositions of n with exactly m parts according to the number of ascents of size d or more, the number of ascents of size less than d, number of descents and number of levels. To do this, we study a general generating function $F_d(z, u, w; p, q, r, s)$ by using the "adding-the-slice" technique [1, 3, 8]. As we show in this section, the "slice" technique is based on adding a new part on the rightmost of a composition with exactly k parts to introduce a composition with exactly $k + 1$ parts. Actually, this is the main reason for creating the variable w which marks the last part of a composition.

Let $\pi = \pi_1\pi_2\dots\pi_m$ be any composition of n with exactly m parts and let $a = \pi_m$ be the value of the last part of the composition π. We proceed from a composition with m parts to a composition with $m+1$ parts. We denote by $F_{d,k}(z, w)$ the generating function for the number of compositions of n with a fixed number m of parts according to the last part, the number of ascents of size d or more, the number of ascents of size less than d, and the number of descents and number of levels. That is,

$$F_{d,m}(z, w) = F_{d,m}(z, w; p, q, r, s) = \sum_{n\geq 0} z^n \sum_{\pi\in C_{n,m}} w^{\pi_m} p^{\operatorname{asc}_d^+(\pi)} q^{\operatorname{asc}_d^-(\pi)} r^{\operatorname{des}(\pi)} s^{\operatorname{lev}(\pi)}.$$

Consider the set of compositions with k parts with last part a. In generating a composition with exactly $k + 1$ parts from a composition with exactly k parts, we have an ascent of size d or more whenever the last part has value at least $a + d$, we have an ascent of size less than d whenever the new last part has any value from $a + 1$ to $a + d - 1$, we have a descent whenever the new last part has any value from 1 to $a - 1$, and for the value a of the new last part we have a level. This gives the following rule for adding a new part or "slice" to the end of the composition:

$$\begin{aligned} w^a &\to r\sum_{j=1}^{a-1}(zw)^j + s(zw)^a + q\sum_{j=a+1}^{a+d-1}(zw)^j + p\sum_{j\geq a+d}(zw)^j \\ &= r\,\frac{zw-(zw)^a}{1-zw} + s(zw)^a + q\,\frac{(zw)^{a+1}-(zw)^{a+d}}{1-zw} + p\,\frac{(zw)^{a+d}}{1-zw}, \end{aligned}$$

which implies the following result.

Lemma 2.1.
For all $k \geq 1$,

$$F_{d,k+1}(z,w) = \frac{rzw}{1-zw} F_{d,k}(z,1) + \frac{s-r+(q-s)\,zw+(p-q)(zw)^d}{1-zw} F_{d,k}(z,zw). \tag{1}$$

Let $G_d(z,u,w) = F_d(z,u,w) - 1 = \sum_{k\geq 1} F_{d,k}(z,w)\,u^k$. By multiplying (1) by u^{k+1} and summing over $k \geq 1$, we obtain that

$$G_d(z,u,w) - F_{d,1}(z,w)\,u = \frac{rzuw}{1-zw} G_d(z,u,1) + \frac{\left(s-r+(q-s)\,zw+(p-q)(zw)^d\right)u}{1-zw} G_d(z,u,zw).$$

Using the initial condition $F_{d,1}(z,w) = \frac{zw}{1-zw}$ which holds immediately from the definitions, we obtain

$$G_d(z,u,w) = \frac{zuw}{1-zw} + \frac{rzuw}{1-zw} G_d(z,u,1) + \frac{\left(s-r+(q-s)\,zw+(p-q)(zw)^d\right)u}{1-zw} G_d(z,u,zw). \tag{2}$$

In order to give an explicit formula for the generating function $G_d(z,u,w)$, we need the following lemma.

Lemma 2.2.
Let $A(z,w)$ be any bivariate analytic generating function in the poly-disk $|z| \leq \rho < 1$ and $|w| \leq 1$. Assume that $A(z,w)$ satisfies $A(z,0) = 0$ and

$$A(z,w) = f(z,w) + g(z,w)\,A(z,zw).$$

Then

$$A(z,w) = \sum_{j\geq 0} \left(f(z,z^j w) \prod_{i=0}^{j-1} g(z,z^i w) \right).$$

Proof. Iterating the recursion for $A(z,w)$, we obtain

$$\begin{aligned} A(z,w) &= f(z,w) + g(z,w)\,A(z,zw) = f(z,w) + g(z,w)\,f(z,zw) + g(z,w)\,g(z,zw)\,A(z,z^2w) \\ &= f(z,w) + g(z,w)\,f(z,zw) + g(z,w)\,g(z,zw)\,f(z,z^2w) + g(z,w)\,g(z,zw)\,g(z,z^2w)\,A(z,z^3w). \end{aligned}$$

Continuing this iteration and using the fact that $A(z,z^jw) \to 0$ as $j \to \infty$ for $|z| \leq \rho < 1$ and $|w| \leq 1$, we obtain the result. $\square$

Note that the above lemma can be seen as a two dimensional analogue of [3, equation (9)] (for another similar result see [7, Lemma 5.45]).

Applying Lemma 2.2 on the equation (2), we obtain

$$G_d(z,u,w) = \sum_{j\geq 1} \left[\left(\frac{z^j u^j w}{1-z^j w} + \frac{rz^j u^j w}{1-z^j w} G_d(z,u,1) \right) \prod_{i=1}^{j-1} \frac{s-r+(q-s)\,z^i w+(p-q)(z^i w)^d}{1-z^i w} \right], \tag{3}$$

which implies the following result.

Theorem 2.3.
The generating function $F_d(z,u,1;p,q,r,s)$ is given by

$$\frac{1+\sum_{j\geq 1}\left(\frac{(1-r)z^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r+(q-s)z^i+(p-q)z^{id}}{1-z^i}\right)}{1-\sum_{j\geq 1}\left(\frac{rz^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r+(q-s)z^i+(p-q)z^{id}}{1-z^i}\right)}.$$

Proof. Substituting $w = 1$ in (3) we obtain that

$$G_d(z,u,1) = \sum_{j\geq 1}\left[\left(\frac{z^j u^j}{1-z^j} + \frac{rz^j u^j}{1-z^j}G_d(z,u,1)\right)\prod_{i=1}^{j-1}\frac{s-r+(q-s)z^i+(p-q)z^{id}}{1-z^i}\right].$$

If we solve this equation for $G_d(z,u,1)$, then

$$G_d(z,u,1) = \frac{\sum_{j\geq 1}\left(\frac{z^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r+(q-s)z^i+(p-q)z^{id}}{1-z^i}\right)}{1-\sum_{j\geq 1}\left(\frac{rz^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r+(q-s)z^i+(p-q)z^{id}}{1-z^i}\right)}.$$

Thus, by $F_d(z,u,1) = 1 + G_d(z,u,1)$ we get the desired result. □

We end this section by presenting several applications for the above theorem.

Example 2.4.

Theorem 2.3 gives

$$F_d(z,u,1;1,1,1,1) = \frac{1}{1-\frac{zu}{1-z}} = \frac{1-z}{1-z(1+u)} = 1 + \sum_{n\geq 1} u(1+u)^{n-1}z^n = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}\binom{n-1}{k-1}u^k z^n,$$

which implies that the number of compositions of n with k parts is given by $\binom{n-1}{k-1}$, which is a well known result.

Example 2.5.

Another application of Theorem 2.3 gives

$$F_d(z,u,1;p,p,r,s) = \frac{1+(1-r)\sum_{j\geq 1}\left(\frac{z^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r+(p-s)z^i}{1-z^i}\right)}{1-r\sum_{j\geq 1}\left(\frac{z^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r+(p-s)z^i}{1-z^i}\right)},$$

which is the generating function for the number of compositions of n according to the number of parts, ascents, descents and levels. On the other hand, Heubach and Mansour [6] found another expression for the generating function $F_d(z,u,1;p,p,r,s)$. They showed that

$$F_d(z,u,1;p,p,r,s) = \frac{1+(1-r)\sum_{j\geq 1}\left(\frac{z^j u}{1-z^j u(s-r)}\prod_{i=1}^{j-1}\frac{1-z^i u(s-p)}{1-z^i u(s-r)}\right)}{1-r\sum_{j\geq 1}\left(\frac{z^j u}{1-z^j u(s-r)}\prod_{i=1}^{j-1}\frac{1-z^i u(s-p)}{1-z^i u(s-r)}\right)}.$$

Comparing the two results gives the following identity

$$\sum_{j\geq 1}\left(\frac{z^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{s-r-(s-p)z^i}{1-z^i}\right) = \sum_{j\geq 1}\left(\frac{z^j u}{1-z^j u(s-r)}\prod_{i=1}^{j-1}\frac{1-z^i u(s-p)}{1-z^i u(s-r)}\right).$$

Example 2.6.

Theorem 2.3 for $q = r = s = 1$ gives

$$\frac{1}{F_d(z,u,1;p,1,1,1)} = 1 - \sum_{j\geq 1}\frac{(p-1)^{j-1}z^{j+d\binom{j}{2}}u^j}{\prod_{i=1}^{j}(1-z^i)},$$

as shown in equations (2.3) and (2.4) in [1].

Another application of Theorem 2.3 can be formulated as follows. A *Carlitz composition*[1] of n, introduced in [2], is a composition of n in which no adjacent parts are the same. In other words, a Carlitz composition π is a composition with $\operatorname{lev}(\pi) = 0$. If we substitute $s = 0$ in the generating function $F_d(z, u, 1; p, q, r, s)$, then Theorem 2.3 gives the generating function $CF_d(z, u; p, q, r)$ for the number of Carlitz compositions of n according to number of parts, ascents of size d or more, ascents of size less than d and descents.

Corollary 2.7.
The generating function $CF_d(z, u; p, q, r)$ is given by

$$\frac{1+\sum_{j\geq 1}\left(\frac{(1-r)z^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{-r+qz^i+(p-q)z^{id}}{1-z^i}\right)}{1-\sum_{j\geq 1}\left(\frac{rz^j u^j}{1-z^j}\prod_{i=1}^{j-1}\frac{-r+qz^i+(p-q)z^{id}}{1-z^i}\right)}.$$

3. Ascents of size less than d

In this section we determine the mean and variance of the number of ascents of size less than d in compositions of n. From Theorem 2.3 we obtain

$$\begin{aligned}
\frac{1}{G(z,q)} &= \frac{1}{F_d(z,1,1;1,q,1,1)} = 1-\sum_{j\geq 1}\frac{z^j}{1-z^j}\prod_{i=1}^{j-1}\frac{(q-1)\,z^i+(1-q)\,z^{id}}{1-z^i}\\
&= 1-\sum_{j\geq 1}(q-1)^{j-1}\frac{z^j}{1-z^j}\prod_{i=1}^{j-1}\frac{z^i-z^{id}}{1-z^i} = 1-\sum_{j\geq 1}(q-1)^{j-1}\frac{z^{1+2+\cdots+j}}{1-z^j}\prod_{i=1}^{j-1}\frac{1-z^{id-i}}{1-z^i}\\
&= 1-\sum_{j\geq 1}(q-1)^{j-1}\frac{z^{\binom{j+1}{2}}}{1-z^j}\frac{\prod_{i=1}^{j-1}\left(1-z^{i(d-1)}\right)}{\prod_{i=1}^{j-1}(1-z^i)} = 1-\sum_{j\geq 1}\frac{(q-1)^{j-1}z^{\binom{j+1}{2}}}{\prod_{i=1}^{j}(1-z^i)}\prod_{i=1}^{j-1}\left(1-z^{i(d-1)}\right).
\end{aligned}$$

This implies

$$G(z,1) = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = 1+\sum_{n\geq 1}2^{n-1}z^n,$$

confirming that the coefficient of z^n in $G(z,1)$ is given by 2^{n-1}, for all $n \geq 1$, since there are 2^{n-1} compositions of n. The expected value of the number of ascents of size less than d is

$$\frac{[z^n]\frac{\partial}{\partial q}\,G(z,q)|_{q=1}}{[z^n]\,G(z,1)} = \frac{[z^n]\frac{\partial}{\partial q}\,G(z,q)|_{q=1}}{2^{n-1}}.$$

In order to obtain an explicit expression, we write

$$\frac{1}{G(z,q)} = 1-\frac{z}{1-z}-(q-1)\,\frac{z^3(1-z^{d-1})}{(1-z)(1-z^2)}-\sum_{j\geq 3}\frac{(q-1)^{j-1}z^{\binom{j+1}{2}}}{\prod_{i=1}^{j}(1-z^i)}\prod_{i=1}^{j-1}\left(1-z^{i(d-1)}\right)$$

which gives

$$\frac{\partial}{\partial q}\,G(z,q)\Big|_{q=1} = G^2(z,1)\,\frac{z^3(1-z^{d-1})}{(1-z)(1-z^2)}.$$

[1] *Originally called waves by L. Carlitz in [2], and also known as Smirnov sequences (see [5], page 68). Named Carlitz compositions by Knopfmacher and Prodinger in [8] in honor of L. Carlitz.*

Hence,

$$\frac{\partial}{\partial q} G(z,q)\Big|_{q=1} = \frac{z^3(1-z^{d-1})}{(1-2z)^2(1+z)},$$

which implies

$$[z^n]\frac{\partial}{\partial q} G(z,q)\Big|_{q=1} = \frac{1}{9}\left((3n-5)2^{n-2} - (3n-3d-2)2^{n-d-1} + (-1)^{n-1} + (-1)^{n-1-d}\right),$$

for $n \geq d+2$. After dividing by 2^{n-1}, the total number of compositions of n, we obtain the following result.

Theorem 3.1.

The expected number of ascents of size less than d in the compositions of n is

$$\mathbb{E}(n) = \frac{1}{18}\left(3n-5-(3n-3d-2)\,2^{1-d}\right) + \frac{(-1)^{n-1}+(-1)^{n-1-d}}{9\cdot 2^{n-1}},$$

for $n \geq d$. Moreover, for fixed d, as $n \to \infty$,

$$\mathbb{E}(n) = \frac{n}{6}\left(1-2^{1-d}\right) + O(1).$$

Note that the above theorem for $d=1$ agrees with the fact that the number of ascents of size less than $d=1$ in the compositions of n is zero.

In order to find the variance we first need to compute $\frac{\partial^2}{\partial q^2} G(z,q)|_{q=1}$. From the definitions we have

$$\frac{\partial^2}{\partial q^2}\frac{1}{G(z,q)}\Big|_{q=1} = -2\,\frac{z^6(1-z^{d-1})(1-z^{2d-2})}{(1-z)(1-z^2)(1-z^3)},$$

which implies

$$\begin{aligned}
\frac{\partial^2}{\partial q^2} G(z,q)\Big|_{q=1} &= 2G^2(z,1)\,\frac{z^6(1-z^{d-1})(1-z^{2d-2})}{(1-z)(1-z^2)(1-z^3)} + 2\,\frac{\left(\frac{\partial}{\partial q}G(z,q)\right)^2\big|_{q=1}}{G(z,1)}\\
&= \frac{2z^6(1-z^{d-1})(1-z^{2d-2})}{(1+z)(1-z^3)(1-2z)^2} + 2\,\frac{\left(\frac{\partial}{\partial q}G(z,q)\right)^2\big|_{q=1}}{(1-z)/(1-2z)}\\
&= \frac{2z^6(1-z^{d-1})\left(2-z^2-z^{d-1}-z^d-z^{d+1}-z^{2d-2}+z^{2d-1}+2z^{2d}\right)}{(1-z^3)(1+z)^2(1-2z)^3}.
\end{aligned}$$

Then the coefficient of z^n, $n \geq 3d+5$ (the numerator of the generating function $\frac{\partial^2}{\partial q^2} G(z,q)|_{q=1}$ is a polynomial of degree $3d+5$), in $2^{1-n}\frac{\partial^2}{\partial q^2} G(z,q)|_{q=1}$ is given by

$$\begin{aligned}
b(n) = 6h\,(n-6-2d) - 2h\,(n-8) - 4h\,(n-5-3d) + 4h\,(n-5-2d)\\
- 2h\,(n-6-d) - 2h\,(n-4-3d) + 4h\,(n-6) - 6h\,(n-5-d) + 2h\,(n-3-3d),
\end{aligned}$$

where

$$h(n) = \frac{(-1)^n(37+i\sqrt{3})(1+i\sqrt{3})^n + (37-i\sqrt{3})(1-i\sqrt{3})^n}{1029\cdot 2^{2n}} + \frac{n(-1)^n}{27\cdot 2^n} - \frac{1-(-1)^n}{3\cdot 2^{n+1}} + \frac{32}{9261}\left(147n^2+511n+558\right)$$

and $i^2=-1$. Here $h(n)$ represents the coefficient of z^n in the series expansion of $2^{1-n}(1-z^3)^{-1}(1+z)^{-2}(1-2z)^{-3}$ and $b(n)$ represents the coefficient of z^n in the series expansion of $2^{1-n}\frac{\partial^2}{\partial q^2} G(z,q)|_{q=1}$. Hence, we can state the following result.

Theorem 3.2.
The variance of the expected number of ascents of size less than d in the compositions of n is given by

$$\mathbb{V}(n) = b(n) + \mathbb{E}(n) - (\mathbb{E}(n))^2,$$

for all $n \geq 3d + 5$. Moreover, for fixed d, as $n \to \infty$,

$$\mathbb{V}(n) \sim \frac{n}{756}\left(29 + 4(21d + 31)\, 2^{-d} - 4(127 + 42d)\, 4^{-d} + 288 \cdot 2^{-3d}\right).$$

Now we will find the limit distribution of our random variable. To do this, we refer the reader to Theorem IX.9 in [4]. A short version of this theorem is as follows: Let $H(z,t) = \frac{1}{A(z,t)}$ be a bivariate function that is bivariate analytic at $(z,t) = (0,0)$ and has nonnegative coefficients at $(0,0)$. Suppose that $H(z,1)$ is meromorphic in $z \leq r$ with only a simple pole at $z = \rho$ for some positive $\rho < r$. Then the distribution of the random variable with probability $P_n(t) = \frac{[z^n]H(z,t)}{[z^n]H(z,1)}$ converges to a Gaussian distribution with a speed of convergence of $O(n^{-1/2})$. To check our conditions, we denote $\frac{\partial^{i+j}}{\partial z^i \partial t^j} A(z,t)|_{(z,t)=(\rho,1)}$ by c_{ij}. From Theorem IX.9 of [4] we need to show that

$$c_{01}c_{10} \neq 0, \tag{4}$$
$$\rho c_{10}^2 c_{02} - \rho c_{10}c_{01}c_{11} + \rho c_{20}c_{01}^2 + c_{01}^2 c_{10} + \rho c_{01}c_{10}^2 \neq 0. \tag{5}$$

In our case, $H(z,q) = G(z,q) = \frac{1}{A(z,q)}$, where

$$A(z,q) = 1 - \sum_{j\geq 1} \frac{(q-1)^{j-1} z^{\binom{j+1}{2}}}{\prod_{i=1}^{j}(1-z^i)} \prod_{i=1}^{j-1}\left(1 - z^{i(d-1)}\right).$$

Since $A(z,1) = \frac{1-2z}{1-z}$, we have that $\rho = \frac{1}{2}$, which implies

$$\begin{aligned}
c_{01} &= \frac{1}{3}\left(\frac{1}{2^{d-1}} - 1\right),\\
c_{10} &= -4,\\
c_{02} &= -\frac{2}{21}\left(1 - \frac{1}{2^{d-1}}\right)\left(1 - \frac{1}{2^{2d-2}}\right),\\
c_{11} &= -\frac{28}{9} + \frac{28}{9 \cdot 2^{d-1}} + \frac{2(d-1)}{3 \cdot 2^{d-1}},\\
c_{20} &= -16.
\end{aligned}$$

Therefore, we have $c_{10}c_{01} = \frac{-4}{3}\left(\frac{1}{2^{d-1}} - 1\right)$ and

$$\rho c_{10}^2 c_{02} - \rho c_{10}c_{01}c_{11} + \rho c_{20}c_{01}^2 + c_{01}^2 c_{10} + \rho c_{01}c_{10}^2 = \frac{4}{189}\left(\frac{1}{2^{d-1}} - 1\right)\left(\frac{14}{2^{d-1}} + \frac{21d}{2^{d-1}} + 127 - \frac{36}{2^{2d-2}}\right) \neq 0,$$

which satisfy the conditions (4) and (5), for any positive integer d. Thus we deduce,

Theorem 3.3.
The distribution of the number of ascents of size less than d in compositions of n converges to a Gaussian distribution, with a speed of convergence of $O(n^{-1/2})$, with mean $\mathbb{E}(n)$ and variance $\mathbb{V}(n)$ given in Theorem 3.1 and 3.2.

Corollary 2.7 gives an explicit formula for the generating function for the number of Carlitz compositions according to the number of ascents of size d or more (respectively, less than d). Using similar techniques as described in this paper, one can obtain the mean, variance and asymptotic distribution of the number of ascents of size d or more (respectively, less than d) in Carlitz compositions.

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