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On stability and robust stability of positive linear Volterra equations in Banach lattices

Research Article

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Abstract: We study positive linear Volterra integro-differential equations in Banach lattices. A characterization of positive equations is given. Furthermore, an explicit spectral criterion for uniformly asymptotic stability of positive equations is presented. Finally, we deal with problems of robust stability of positive systems under structured perturbations. Some explicit stability bounds with respect to these perturbations are given.

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1. Introduction

Let $X = (X, \|\cdot\|)$ be a complex Banach lattice with the real part $X_{\mathbb{R}}$ and the positive convex cone X_+ (cf. [15], [8, Chapter C], [7]) and let $\mathcal{L}(X)$ be the Banach space of all bounded linear operators on X. We are concerned with abstract linear Volterra integro-differential equations in the Banach lattice X of the form

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \tag{1}$$

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where A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B(\cdot):\mathbb{R}_+ = [0,+\infty)\to\mathcal{L}(X)$ is continuous at t with respect to the operator norm. In addition, we assume that

$$(T(t))_{t\geq 0} \quad \text{is a compact semigroup and} \quad \int_0^{+\infty} \|B(t)\|\,dt < +\infty. \tag{2}$$

In [5], Hino and Murakami gave primary criteria for the uniform asymptotic stability of the zero solution of (1) in terms of invertibility of the characteristic operator

$$zI_X - A - \int_0^{+\infty} e^{-zt}B(t)\,dt \qquad (I_X \text{ is the identical operator on } X)$$

on the closed right half plane as well as integrability of the resolvent of (1).

In a very recent paper, for X being a finite dimensional space, P.H.A. Ngoc et al. [12] studied positivity of the equation (1) (which is characterized by $(e^{At})_{t\geq 0}$ being a positive matrix semigroup on $\mathbb{R}^{n\times n}$ and $B(t)\in\mathbb{R}_+^{n\times n}$ for all $t\geq 0$) and showed that for positive equations, the invertibility of the characteristic matrix on the closed right half plane reduces to that of $zI_n - \left(A + \int_0^{+\infty} B(t)\,dt\right)$; here I_n denotes the $n\times n$ identical matrix. Consequently, such a positive equation is uniformly asymptotically stable if and only if the spectral bound of the matrix $A + \int_0^{+\infty} B(t)\,dt$ is negative, or equivalently, the associated linear ordinary differential equation

$$\dot{x}(t) = \left(A + \int_0^{+\infty} B(s)\,ds\right)x(t), \qquad t\geq 0, \tag{3}$$

is asymptotically stable, a surprising result.

In the present paper, we first introduce the notion of positive linear Volterra integro-differential equations in Banach lattices. Then, we give a characterization of positive linear Volterra equations of the form (1) in terms of positivity of the C_0-semigroup generated by A and positivity of the kernel function $B(\cdot)$. Furthermore, we prove that under the assumptions of positivity of the C_0-semigroup generated by A and positivity of the kernel function $B(\cdot)$, the uniform asymptotic stability of (1) is still determined by the spectral bound of the operator $A + \int_0^{+\infty} B(t)\,dt$. Finally, we deal with problems of robust stability of (1) under structured perturbations. Some explicit stability bounds with respect to these perturbations are given. An example is given to illustrate the obtained results. Our analysis is based on the theory of positive C_0-semigroups on Banach lattices, see e.g. [1], [8].

2. Preliminaries

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup (or shortly, C_0-semigroup) of bounded linear operators on the complex Banach space $(X,\|\cdot\|)$. Denote by A the generator of the semigroup $(T(t))_{t\geq 0}$ and by $\mathcal{D}(A)$ its domain. That is,

$$\mathcal{D}(A) = \left\{x\in X \,:\, \lim_{t\to 0}\frac{T(t)x - x}{t} \in X\right\}$$

and

$$Ax = \lim_{t\to 0}\frac{T(t)x - x}{t}, \qquad x\in\mathcal{D}(A).$$

Since A is a closed operator, $\mathcal{D}(A)$ is a Banach space with the graph norm

$$\|x\|_{\mathcal{D}(A)} = \|x\| + \|Ax\|, \qquad x\in\mathcal{D}(A). \tag{4}$$

The resolvent set $\rho(A)$, by definition, consists of all $\lambda\in\mathbb{C}$ for which $(\lambda I_X - A)$ has a bounded linear inverse in X. The complement of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of A and denoted by $\sigma(A)$. In general, by the same way as in the above, one can define the resolvent set $\rho(A)$ and the spectral set $\sigma(A)$ for an arbitrary linear operator

$$A:\ \mathcal{D}(A)\subset X\to X.$$

With the C_0-semigroup $(T(t))_{t\geq 0}$, we associate the following quantities:

(1) the *spectral bound* $s(A)$,

$$s(A) = \sup\left\{\Re\lambda \,:\, \lambda \in \sigma(A)\right\},$$

where $\sigma(A)$ is the spectrum of the linear operator A, and $\Re\lambda$ denotes the real part of $\lambda \in \mathbb{C}$;

(2) the *growth bound* $\omega(A)$,

$$\omega(A) = \inf\left\{\omega \in \mathbb{R} \,:\, \text{there exists } M > 0 \text{ such that } \|T(t)\| \le Me^{\omega t} \text{ for all } t \ge 0\right\}.$$

It is well known that

$$-\infty \le s(A) \le \omega(A) < +\infty, \tag{5}$$

see, e.g. [1], [8].

Next, the C_0-semigroup $(T(t))_{t\ge 0}$ is called

(1) *Hurwitz stable* if $\sigma(A) \subset \mathbb{C}^- = \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$,

(2) *strictly Hurwitz stable* if $s(A) < 0$,

(3) *uniformly exponentially stable* if $\omega(A) < 0$.

It is well known that for an eventually norm continuous semigroup, that is,

$$\lim_{t\to t_0} \|T(t) - T(t_0)\| = 0 \quad \text{for some} \quad t_0 \ge 0,$$

we have $s(A) = \omega(A)$, see e.g. [8]. So, the strict Hurwitz stability and the uniform exponential stability of eventually norm continuous semigroups coincide.

To make the presentation self-contained, we give some basic facts on Banach lattices which will be used in the sequel (see, e.g. [15]). Let $X \ne \{0\}$ be a *real* vector space endowed with an order relation $\le$. Then X is called an *ordered vector space*. Denote the *positive* elements of X by $X_+ = \{x \in X \,:\, x \ge 0\}$. If furthermore the *lattice property* holds, that is, if $x \vee y = \sup\{x, y\} \in X$ for $x, y \in X$, then X is called a *vector lattice*. It is important to note that X_+ is *generating*, that is,

$$X = X_+ - X_+ = \{x - y : x, y \in X_+\}.$$

The modulus $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$. If $\|\cdot\|$ is a norm on the vector lattice X satisfying the *lattice norm property*, that is, if

$$|x| \le |y| \;\Rightarrow\; \|x\| \le \|y\|, \qquad x, y \in X, \tag{6}$$

then X is called a *normed vector lattice*. If, in addition, $(X, \|\cdot\|)$ is a Banach space then X is called a (*real*) *Banach lattice*.

We now extend the notion of Banach lattices to the complex case. For this extension all underlying vector lattices X are assumed to be *relatively uniformly complete,* that is, if for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} |\lambda_n| < +\infty$, for every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in X it holds that

$$0 \le x_n \le \lambda_n x \quad \Rightarrow \quad \sup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n} x_i\right) \in X.$$

Now let X be a relatively uniformly complete vector lattice. The *complexification* of X is defined by $X_{\mathbb{C}} = X + \imath X$, where $\imath = \sqrt{-1}$. The modulus of $z = x + \imath y \in X_{\mathbb{C}}$ is defined by

$$|z| = \sup_{0\le\phi\le 2\pi} |(\cos\phi)x + (\sin\phi)y| \in X. \tag{7}$$

A *complex vector lattice* is defined as the complexification of a relatively uniformly complete vector lattice endowed with the modulus (7). If X is normed then

$$\|x\| = \||x|\|, \qquad x \in X_{\mathbb{C}}, \tag{8}$$

defines a norm on $X_{\mathbb{C}}$ satisfying the lattice norm property. If X is a Banach lattice then $X_{\mathbb{C}}$ endowed with the modulus (7) and the norm (8) is called a *complex Banach lattice*. Throughout this paper, for simplicity of presentation, we write $X, X_{\mathbb{R}}$ instead of $X_{\mathbb{C}}, X$, respectively. Let $E_{\mathbb{R}}, F_{\mathbb{R}}$ be real Banach lattices and $T \in \mathcal{L}(E_{\mathbb{R}}, F_{\mathbb{R}})$. Then T is called *positive*, denoted by $T \geq 0$, if $T(E_+) \subset F_+$. By $S \leq T$ we mean $T - S \geq 0$, for $T, S \in \mathcal{L}(E_{\mathbb{R}}, F_{\mathbb{R}})$.

An operator $T \in \mathcal{L}(E, F)$ is called *real* if $T(E_{\mathbb{R}}) \subset F_{\mathbb{R}}$. An operator $T \in \mathcal{L}(E, F)$ is called *positive*, denoted by $T \geq 0$, if T is real and $T(E_+) \subset F_+$. We introduce the notation

$$\mathcal{L}_+(E, F) = \big\{T \in \mathcal{L}(E, F): \ T \geq 0\big\}. \tag{9}$$

For $T \in \mathcal{L}_+(E, F)$, we emphasize a simple but important fact that

$$\|T\| = \sup_{x \in E_+, \|x\|=1} \|Tx\|, \tag{10}$$

see e.g. [15, p. 230].

3. Characterization of positive linear Volterra integro-differential equations in Banach lattices

Let X be a complex Banach lattice endowed with the real part $X_{\mathbb{R}}$ and the positive convex cone X_+ and let $\mathcal{L}(X)$ be the Banach space of all bounded linear operators on X. In what follows, $C([0, \sigma], X)$ denotes the Banach space of all X-valued continuous functions on $[0, \sigma]$, equipped with the supremum norm.

Consider an abstract Volterra integro-differential equation in X defined by (1), where A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B(\cdot) : \mathbb{R}_+ \to \mathcal{L}(X)$ is continuous with respect to the operator norm. In addition, we assume that (2) holds true.

For any $(\sigma, \phi) \in \mathbb{R}_+ \times C([0, \sigma], X)$, there exists a unique continuous function $x : \mathbb{R}_+ \to X$ such that $x \equiv \phi$ on $[0, \sigma]$ and the following relation holds:

$$x(t) = T(t-\sigma)\phi(\sigma) + \int_\sigma^t T(t-s) \left\{ \int_0^s B(s-\tau)x(\tau)\, d\tau \right\} ds, \qquad t \geq \sigma, \tag{11}$$

see e.g. [3]. The function x is called a (*mild*) *solution* of the equation (1) on $[\sigma, +\infty)$, and denoted by $x(\,\cdot\,; \sigma, \phi)$.

Definition 3.1.

We say that (1) is *positive* if $x(t; \sigma, \phi) \in X_+$ for all $t \in [\sigma, +\infty)$, whenever $(\sigma, \phi) \in \mathbb{R}_+ \times C([0, \sigma], X_+)$.

We are now in the position to state and prove the first main result of this paper.

Theorem 3.2.

If A generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \geq 0$ for every $t \geq 0$ then (1) is positive. Conversely, if (1) is positive and A is the infinitesimal generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X then $B(t) \geq 0$ for each $t \geq 0$.

Proof. Suppose A generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \geq 0$ for every $t \geq 0$. Fix $(\sigma, \phi) \in \mathbb{R}_+ \times C([0,\sigma], X_+)$ and $x(t) = x(t;\sigma,\phi)$, $t \geq \sigma$. By (11), we have

$$x(t+\sigma) = T(t)\phi(\sigma) + \int_\sigma^{t+\sigma} T(t+\sigma-s)\left\{\int_0^s B(s-\tau)x(\tau)\,d\tau\right\} ds, \qquad t \geq 0.$$

This implies that

$$x(t+\sigma) = T(t)\phi(\sigma) + \int_0^t T(t-s)\left\{\int_0^{s+\sigma} B(s+\sigma-\tau)x(\tau)\,d\tau\right\} ds, \qquad t \geq 0.$$

Thus,

$$\begin{aligned} x(t+\sigma) &= T(t)\phi(\sigma) + \int_0^t T(t-s)\left\{\int_0^\sigma B(s+\sigma-\tau)\phi(\tau)\,d\tau + \int_\sigma^{s+\sigma} B(s+\sigma-\tau)x(\tau)\,d\tau\right\} ds \\ &= f(t) + \int_0^t T(t-s)\left\{\int_0^s B(s-\tau)x(\tau+\sigma)\,d\tau\right\} ds, \qquad t \geq 0, \end{aligned}$$

where

$$f(t) = T(t)\phi(\sigma) + \int_0^t T(t-s)\left\{\int_0^\sigma B(s+\sigma-\tau)\phi(\tau)\,d\tau\right\} ds, \qquad t \geq 0. \tag{12}$$

Set $y(t) = x(t+\sigma)$, $t \geq 0$. Then, $y(\cdot)$ satisfies

$$y(t) = f(t) + \int_0^t T(t-s)\left\{\int_0^s B(s-\tau)y(\tau)\,d\tau\right\} ds, \qquad t \geq 0. \tag{13}$$

Fix $t_0 > 0$. Consider the operator L defined by

$$\begin{aligned} L:\ & C([0,t_0],X) \to C([0,t_0],X) \\ \psi \mapsto L\psi(t) &= f(t) + \int_0^t T(t-s)\left\{\int_0^s B(s-\tau)\psi(\tau)d\tau\right\} ds, \qquad t \in [0,t_0], \end{aligned}$$

where $f(\cdot)$ is defined as in (12). By induction, it is easy to show that for $\psi_1, \psi_2 \in C([0,t_0],X)$ and $k \in \mathbb{N}$, we have

$$\left\|L^k\psi_2(t) - L^k\psi_1(t)\right\| \leq \frac{M^k t^k}{k!}\|\psi_2 - \psi_1\|_{C([0,t_0],X)} \quad \text{for all} \quad t \in [0,t_0],$$

where $M = M_1M_2$ and $M_1 = \max_{s\in[0,t_0]}\|T(s)\|$, $M_2 = \int_0^{t_0}\|B(s)\|\,ds$. Thus, L^k is a contraction for $k \in \mathbb{N}$ sufficiently large. Fix a $k_0 \in \mathbb{N}$ sufficiently large and set $S = L^{k_0}$. By the contraction mapping principle, there exists a unique solution of the equation $y = Ly$ in $C([0,t_0],X)$. Moreover, it is well known that the sequence $(S^m\psi_0)_{m\in\mathbb{N}} = (L^{mk_0}\psi_0)_{m\in\mathbb{N}}$, with an arbitrary $\psi_0 \in C([0,t_0],X)$, converges to this solution in the space $C([0,t_0],X)$. Choose $\psi_0 \in C([0,t_0],X_+)$. Since $(T(t))_{t\geq 0}$ is a positive semigroup, $B(t) \geq 0$ and $f(t) \in X_+$ for all $t \geq 0$, it follows that $L^{mk_0}\psi_0 \in C([0,t_0],X_+)$ for all $m \in \mathbb{N}$. Taking (13) into account, we derive that

$$L^{mk_0}\psi_0 \to y(\cdot) \in C([0,t_0],X_+) \quad \text{as} \quad m \to +\infty.$$

Thus, $y(t) = x(t+\sigma) \in X_+$ for all $t \in [0,t_0]$. Recall that t_0 is an arbitrary fixed positive number. Hence, letting $t_0 \to \infty$, we get $x(t) \in X_+$ for all $t \geq \sigma$.

Conversely, assume that the equation (1) is positive and A is the infinitesimal generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X. We first show that $B(t)$ is real for each $t \geq 0$. Let $\sigma > 0$ and $a \in X_+$ be given. For each integer n such that $1/n < \sigma$, consider a function $\phi_n \in C([0,\sigma], X_+)$ defined by $\phi_n(t) = a$ if $t \in [0, \sigma - 1/n]$ and $\phi_n(t) = n(\sigma - t)a$ if $t \in (\sigma - 1/n, \sigma]$. By the positivity of (1) we have $x(t;\sigma,\phi_n) \geq 0$ for any $t \geq \sigma$, and hence

$$\begin{aligned}\frac{1}{h}x(h+\sigma,\sigma,\phi_n) &= \frac{1}{h}\left(T(h)\phi_n(\sigma) + \int_\sigma^{\sigma+h} T(h+\sigma-s)\left(\int_0^s B(s-\tau)x(\tau;\sigma,\phi_n)\,d\tau\right)\,ds\right)\\ &= \frac{1}{h}\int_\sigma^{\sigma+h} T(h+\sigma-s)\left(\int_0^s B(s-\tau)x(\tau;\sigma,\phi_n)\,d\tau\right)\,ds \geq 0\end{aligned}$$

for any $h > 0$. Observe that

$$\lim_{h\to+0}\left[\frac{1}{h}\int_\sigma^{\sigma+h} T(h+\sigma-s)\left(\int_0^s B(s-\tau)x(\tau;\sigma,\phi_n)\,d\tau\right)\,ds\right] = \int_0^\sigma B(\sigma-\tau)x(\tau;\sigma,\phi_n)\,d\tau = \int_0^\sigma B(\sigma-\tau)\phi_n(\tau)\,d\tau.$$

Thus, $\int_0^\sigma B(\sigma-\tau)\phi_n(\tau)\,d\tau \geq 0$ and by letting $n\to\infty$, we get $\int_0^\sigma B(s)a\,ds \geq 0$ for any $\sigma \geq 0$. Therefore,

$$\int_t^{t+h} B(s)a\,ds = \int_0^{t+h} B(s)a\ ds - \int_0^t B(s)a\,ds \in X_+ - X_+ = X_{\mathbb{R}}$$

for any $t \geq 0$ and $h > 0$. Consequently,

$$B(t)a = \lim_{h\to+0}\left(\frac{1}{h}\int_t^{t+h} B(s)a\,ds\right) \in X_{\mathbb{R}}, \qquad a \in X_+.$$

This yields, $B(t)X_{\mathbb{R}} \subset X_{\mathbb{R}}$, which means that $B(t)$ is real for each $t \geq 0$.

Next, we show that $B(t) \geq 0$ for each $t \geq 0$. Let $(\sigma,\phi) \in \mathbb{R}_+ \times C([0,\sigma], X_+)$ with $\phi(\sigma) = 0$ be given. By the positivity of (1), we have $y(t) = x(t+\sigma;\sigma,\phi) \geq 0$ on $[0,\infty)$. Note that y satisfies

$$\begin{aligned}y(t) &= T(t)\phi(\sigma) + \int_\sigma^{t+\sigma} T(t+\sigma-s)\left\{\int_0^s B(s-\tau)x(\tau)\,d\tau\right\}\,ds\\ &= \int_0^t T(t-u)\left\{\int_0^{\sigma+u} B(\sigma+u-\tau)x(\tau)\,d\tau\right\}\,du = \int_0^t T(t-u)p(u)\,du, \qquad t \geq 0,\end{aligned}$$

where

$$p(u) = \int_0^{\sigma+u} B(\sigma+u-\tau)x(\tau)\,d\tau.$$

Let $\lambda \in \mathbb{R}$ be sufficiently large so that $\sup_{t\geq 0}\left(e^{(-\lambda+1)t}\|T(t)\|\right) < \infty$. It follows that $\lambda \in \rho(A)$ and

$$R(\lambda,A)x = \int_0^{+\infty} e^{-\lambda t}T(t)x\,dt, \qquad x \in X.$$

In particular, by [4, Theorem 2.16.5] we see that $\lambda \in \rho(A^*)$ and $R(\lambda,A^*) = R(\lambda,A)^*$ because of $\lambda \in \rho(A)$. Let v_+^* be an arbitrary element in $(X^*)_+$, the space of all positive bounded linear functionals on X. Set $v^* = R(\lambda,A^*)v_+^*$. Then, we have $v^* \in \mathcal{D}(A^*)$ and

$$\langle v^*, y(t)\rangle = \left\langle v^*, \int_0^t T(t-u)p(u)du\right\rangle, \qquad t \geq 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical duality pairing of X^* and X. Since $y(t) \geq 0$, the positivity of $(T(t))_{t\geq 0}$ implies that

$$R(\lambda,A)\,y(t) = \int_0^{+\infty} e^{-\lambda u}T(u)y(t)du \geq 0,$$

and hence $\langle v^*, y(t)\rangle = \langle v_+^*, R(\lambda, A)y(t)\rangle \geq 0$ since $v_+^* \geq 0$. Consequently, $(d^+/dt)\langle v^*, y(t)\rangle|_{t=0} \geq 0$ since $\langle v^*, y(0)\rangle = v^*(0) = 0$. Notice that $AR(\lambda, A) = -I_X + \lambda R(\lambda, A)$. It follows that

$$(AR(\lambda, A))^* = -I_{X^*} + \lambda R(\lambda, A)^* = -I_{X^*} + \lambda R(\lambda, A^*) = A^*R(\lambda, A^*),$$

and we thus get

$$\begin{aligned}
\frac{d^+}{dt}\left\langle v^*, \int_0^t T(t-u)p(u)\,du \right\rangle &= \frac{d^+}{dt}\left\langle v_+^*, R(\lambda, A)\int_0^t T(t-u)p(u)\,du \right\rangle \\
&= \lim_{h\to+0}\frac{1}{h}\left\{\left\langle v_+^*, R(\lambda, A)\int_0^{t+h} T(t+h-u)p(u)\,du - R(\lambda, A)\int_0^t T(t-u)p(u)\,du \right\rangle\right\} \\
&= \lim_{h\to+0}\left\{\left\langle v^*, \frac{1}{h}\int_t^{t+h} T(t+h-u)p(u)\,du \right\rangle + \left\langle v_+^*, R(\lambda, A)\frac{T(h)-I_X}{h}\int_0^t T(t-u)p(u)\,du \right\rangle\right\} \\
&= \langle v^*, p(t)\rangle + \left\langle v_+^*, AR(\lambda, A)\int_0^t T(t-u)p(u)\,du \right\rangle = \langle v^*, p(t)\rangle + \big\langle (AR(\lambda, A))^* v_+^*, y(t)\big\rangle \\
&= \langle v^*, p(t)\rangle + \big\langle A^*R(\lambda, A^*)v_+^*, y(t)\big\rangle = \langle v^*, p(t)\rangle + \langle A^*v^*, y(t)\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d^+}{dt}\langle v^*, y(t)\rangle|_{t=0} = \langle v^*, p(0)\rangle + \langle A^*v^*, y(0)\rangle &= \left\langle v^*, \int_0^\sigma B(\sigma-\tau)x(\tau)\,d\tau \right\rangle \\
&= \left\langle R(\lambda, A)^*v_+^*, \int_0^\sigma B(\sigma-\tau)\phi(\tau)\,d\tau \right\rangle = \left\langle v_+^*, R(\lambda, A)\int_0^\sigma B(\sigma-\tau)\phi(\tau)\,d\tau \right\rangle,
\end{aligned}$$

and, consequently, $\left\langle v_+^*, R(\lambda, A)\int_0^\sigma B(\sigma-\tau)\phi(\tau)\,d\tau \right\rangle \geq 0$. Rewriting $\phi(\sigma-\tau)$ as $\psi(\tau)$, we have

$$\left\langle v_+^*, R(\lambda, A)\int_0^\sigma B(u)\psi(u)\,du \right\rangle \geq 0 \tag{14}$$

for any $v_+^* \in (X^*)_+$ and any $\psi \in C([0,\sigma]; X_+)$ with $\psi(0) = 0$. We claim that

$$R(\lambda, A)B(t)a \geq 0 \quad \text{for all} \quad t \in (0,\sigma],\ a \in X_+. \tag{15}$$

Seeking a contradiction, we assume that there are $t_1 \in (0,\sigma]$ and $a \in X_+$ such that $R(\lambda, A)B(t_1)a \notin X_+$. Notice that $R(\lambda, A)B(t_1)a \in X_{\mathbb{R}}$ by $R(\lambda, A) \geq 0$ and $B(t)a \in X_{\mathbb{R}}$. Since X_+ is a closed convex cone and $R(\lambda, A)B(t_1)a \notin X_+$, there exists a $v_+^* \in X^*$ such that $\big\langle v_+^*, R(\lambda, A)B(t_1)a\big\rangle < \inf\{\langle v_+^*, x\rangle \mid x \in X_+\} = l$, see e.g. [6, Chapter 3, Theorem 6]. Note that for any $x \in X_+$ and $n = 1, 2, \ldots$ we have $l \leq \langle v_+^*, nx\rangle = n\langle v_+^*, x\rangle$, or equivalently, $l/n \leq \langle v_+^*, x\rangle$. This yields $\langle v_+^*, x\rangle \geq 0$ for any $x \in X_+$, and consequently $l \geq 0$ as well as $l \leq \langle v_+^*, 0\rangle = 0$. It follows that $l = 0$ and $\big\langle v_+^*, R(\lambda, A)B(t_1)a\big\rangle > 0$. Hence $v_+^* \in (X^*)_+$, and moreover there exists an interval $[c, d] \subset (0, \sigma)$ satisfying $\big\langle v_+^*, R(\lambda, A)B(t)a\big\rangle < 0$ for all $t \in [c, d]$. Then one can choose a nonnegative scalar continuous function χ so that $\chi(0) = 0$ and

$$\left\langle v_+^*, \int_0^\sigma R(\lambda, A)B(t)\chi(t)a\,dt \right\rangle = \int_0^\sigma \big\langle v_+^*, R(\lambda, A)B(t)a\big\rangle \chi(t)\,dt < 0,$$

which leads to a contradiction by considering $\chi(t)a$ as $\psi(t)$ in (14).

Finally, it follows from (15) and the fact that $\lim_{\lambda\to+\infty}\lambda R(\lambda, A)x = x$ for any $x \in X$, that $B(t) \geq 0$ for $t \in [0,\sigma]$. Since $\sigma > 0$ is arbitrary, $B(t) \geq 0$ for all $t \geq 0$. This completes the proof. $\square$

Remark 3.3.
In the study of linear Volterra equations of type (1), the resolvent $R(t)$ which is introduced as the inverse Laplace transform of $\left[\lambda - A - B(\lambda)\right]^{-1}$ plays a crucial role; see e.g. [2, 14]. Observe that the resolvent does not appear explicitly in the proof of Theorem 3.2. But the solution $y(t)$ of (13) with $T(t)x$ in place of $f(t)$ is identical with $R(t)x$, and hence the former part in the proof of Theorem 3.2 indeed proves the positivity of the operator $R(t)$. Thus one can also establish the former part of Theorem 3.2 by applying the expression formula (in terms of the resolvent) (e.g. [5, Proposition 2.4]) for solutions of nonhomogeneous equations.

In particular case of $X = \mathbb{R}^{n\times n}$, it has been shown in [12] (Theorem 3.7) that the equation (1) is positive if and only if A generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times n}_{+}$ for every $t \geq 0$. However, for equations in infinite dimensional spaces, it is still an open question whether the positivity of (1) implies that A generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ in X? If this is true then as in the case of positive equations in finite dimensional spaces, (1) is positive if and only if A generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \in \mathcal{L}_+(X)$ for every $t \geq 0$, by Theorem 3.2.

Finally, it is worth noticing that the proof of Theorem 3.2 is much more difficult than that of Theorem 3.7 in [12].

4. Stability and robust stability of positive linear Volterra integro-differential equations in Banach lattices

4.1. An explicit criterion for uniform asymptotic stability of positive equations in Banach lattices

In this subsection, by exploiting positivity of equations, we give an explicit criterion for the uniform asymptotic stability of positive equations. We recall here the notion of the uniform asymptotic stability of equation (1). For more details and further information, we refer readers to [5].

Definition 4.1.
The zero solution of (1) is said to be *uniformly asymptotically stable* (shortly, UAS) if and only if

(a) for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $(\sigma, \phi) \in \mathbb{R}_+ \times C([0,\sigma]; X)$, $\|\phi\|_{[0,\sigma]} = \sup_{0\leq s\leq \sigma} \|\phi(s)\| < \delta(\epsilon)$ implies that $\|x(t;\sigma,\phi)\| < \epsilon$ for all $t \geq \sigma$;

(b) there is $\delta_0 > 0$ such that for each $\epsilon_1 > 0$ there exists $l(\epsilon_1) > 0$ such that for any $(\sigma, \phi) \in \mathbb{R}_+ \times C([0,\sigma]; X)$, $\|\phi\|_{[0,\sigma]} < \delta_0$ implies that $\|x(t;\sigma,\phi)\| < \epsilon_1$ for all $t \geq \sigma + l(\epsilon_1)$.

Note that we continue to assume that (2) holds true.

Theorem 4.2 ([5]).
Assume that A generates a compact semigroup. Then the following statements are equivalent:

(i) the zero solution of (1) is UAS.

(ii) $\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds$ is invertible in $\mathcal{L}(X)$ for all $\lambda \in \mathbb{C}$, $\Re\lambda \geq 0$.

Before stating and proving the main result of this subsection, we prove an auxiliary lemma.

Lemma 4.3.
Assume that A generates a positive compact semigroup $(T(t))_{t\geq 0}$ on X and $P \in \mathcal{L}(X)$, $Q \in \mathcal{L}_+(X)$. If

$$|Px| \leq Q|x| \quad \text{for all} \quad x \in X,$$

then

$$\omega(A+P) = s(A+P) \leq s(A+Q) = \omega(A+Q).$$

Proof. Let $(G(t))_{t\geq 0}$ and $(H(t))_{t\geq 0}$ be the C_0-semigroups with the infinitesimal generators $A+P$ and $A+Q$, respectively. Since A generates the compact semigroup $(T(t))_{t\geq 0}$, so do $A+P$ and $A+Q$, see e.g. [1, 8]. This implies that $s(A+P)=\omega(A+P)$ and $s(A+Q)=\omega(A+Q)$, see e.g. [1, 8]. By the well-known property of compact C_0-semigroups, we get $e^{\sigma(C)}=\sigma(M(1))\backslash\{0\}$, where C is the infinitesimal generator of any compact C_0-semigroup $(M(t))_{t\geq 0}$ on X; see e.g. [1, Corollary IV.3.11]. Hence we have $e^{\omega(C)}=r(M(1))$, where $r(M(1))$ is the spectral radius of the operator $M(1)$. Thus, it remains to show that

$$r(G(1))\leq r(H(1)).$$

Note that $(G(t))_{t\geq 0}$ and $(H(t))_{t\geq 0}$ are defined respectively by

$$G(t)x=\lim_{n\to+\infty}(T(t/n)e^{(t/n)P})^n x,\qquad H(t)x=\lim_{n\to+\infty}(T(t/n)e^{(t/n)Q})^n x,\qquad x\in X,$$

for each $t\geq 0$; see e.g. [8, p. 44] and see also [1, Theorem III.5.2]. By the positivity of $(T(t))_{t\geq 0}$ and the hypothesis of $|Px|\leq Q|x|,\ x\in X$, it is easy to see that

$$|G(1)x|\leq H(1)|x|,\qquad x\in X.$$

Then, by induction

$$|G(1)^k x|\leq H(1)^k|x|,\qquad x\in X,\quad k\in\mathbb{N}. \tag{16}$$

From the property of a norm on Banach lattices (6), it follows from (16) and (10) that $\|G(1)^k\|\leq\|H(1)^k\|$. By the well-known Gelfand's formula, we have $r(G(1))\leq r(H(1))$, which completes the proof. $\square$

We are now in the position to prove the main result of this section.

Theorem 4.4.
Assume that A generates a positive compact semigroup $(T(t))_{t\geq 0}$ on X and $B(t)\geq 0$ for all $t\geq 0$. Then the following statements are equivalent:

(i) the zero solution of (1) is UAS;

(ii) $s\big(A+\int_0^{+\infty}B(\tau)\,d\tau\big)<0$.

Proof. (ii)⇒(i) Assume that the zero solution of (1) is not UAS. By Theorem 4.2, $\lambda I_X-A-\int_0^{+\infty}e^{-\lambda s}B(s)\,ds$ is not invertible for some $\lambda\in\mathbb{C}$, $\Re\lambda\geq 0$. This implies that $\lambda\in\sigma\big(A+\int_0^{+\infty}e^{-\lambda s}B(s)\,ds\big)$. Hence,

$$0\leq\Re\lambda\leq s\left(A+\int_0^{+\infty}e^{-\lambda s}B(s)\,ds\right).$$

On the other hand, it is easy to see that

$$\left|\int_0^{+\infty}e^{-\lambda s}B(s)\,ds\cdot x\right|\leq\int_0^{+\infty}B(s)\,ds\,|x|,$$

by the hypothesis of $B(t)\geq 0$ for all $t\geq 0$. Thus,

$$0\leq s\left(A+\int_0^{+\infty}e^{-\lambda s}B(s)\,ds\right)\leq s\left(A+\int_0^{+\infty}B(s)\,ds\right)$$

by Lemma 4.3.

(i)⇒(ii) For every $\lambda \geq 0$, we put $\Phi_\lambda = \int_0^{+\infty} B(t)e^{-\lambda t}\,dt$ and $f(\lambda) = s(A+\Phi_\lambda)$. Consider the real function defined by $g(\lambda) = \lambda - f(\lambda)$, $\lambda \geq 0$. We show that $g(0) = -s(A+\Phi_0) > 0$. Since $B(\cdot)$ is positive, by almost the same argument as in [1, Proposition VI.6.13] one can see that $f(\lambda)$ is non-increasing and left continuous at $\lambda > 0$. Hence $g(\lambda)$ is increasing and left continuous at λ with $\lim_{\lambda\to+\infty} g(\lambda) = +\infty$. We assert that the function $g(\lambda)$ is right continuous at $\lambda \geq 0$. Indeed, if this assertion is false, then there is a $\lambda_0 \geq 0$ such that $s^+ = \lim_{\varepsilon\to+0} f(\lambda_0+\varepsilon) < s_0 = f(\lambda_0)$. Notice that $s_0 = s(A+\Phi_{\lambda_0})$ and $\tilde{A} = A+\Phi_{\lambda_0}$ generates a positive and compact C_0-semigroup $(e^{\tilde{A}t})_{t\geq 0}$. It follows that $s_0 = s(\tilde{A}) \in \sigma(\tilde{A})$ by [1, Theorem VI.1.10]. Take a $t_0 \in \rho(\tilde{A})$. Since

$$\sigma(R(t_0,\tilde{A})) \setminus \{0\} = \left\{ \frac{1}{t_0-\mu} \;:\; \mu \in \sigma(\tilde{A}) \right\}$$

by [1, Theorem IV.1.13], we get $1/(t_0-s_0) \in \sigma(R(t_0,\tilde{A}))$. Observe that $1/(t_0-s_0)$ is isolated in the spectrum $\sigma(R(t_0,\tilde{A}))$ of the compact operator $R(t_0,\tilde{A})$. Therefore, if s_1 is sufficiently close to s_0 and $s_1 \neq s_0$, then $1/(t_0-s_1)$ is sufficiently close to $1/(t_0-s_0)$; hence $1/(t_0-s_1) \notin \sigma(R(t_0,\tilde{A}))$, in particular, $s_1 \notin \sigma(\tilde{A})$. Therefore one can choose an $s_1 \in (s^+, s_0)$ so that $s_1 \in \rho(\tilde{A})$, that is, $s_1 I_X - A - \Phi_{\lambda_0}$ has a bounded inverse $(s_1 I_X - A - \Phi_{\lambda_0})^{-1}$ in $\mathcal{L}(X)$. In the following, we will show that $(s_1 I_X - A - \Phi_{\lambda_0})^{-1} \geq 0$. Since $s^+ < s_1$, it follows that $s(A+\Phi_{\lambda_0+\varepsilon}) < s_1$ for small $\varepsilon > 0$. Then [1, Lemma VI.1.9] implies that $(s_1 I_X - A - \Phi_{\lambda_0+\varepsilon})^{-1} \geq 0$ and

$$\left(s_1 I_X - A - \Phi_{\lambda_0+\varepsilon}\right)^{-1} x = \int_0^{+\infty} e^{-s_1 t} \exp\left((A+\Phi_{\lambda_0+\varepsilon})t\right) x\,dt, \qquad x \in X.$$

Note that

$$s_1 I_X - A - \Phi_{\lambda_0+\varepsilon} = s_1 I_X - A - \Phi_{\lambda_0} + (\Phi_{\lambda_0} - \Phi_{\lambda_0+\varepsilon}) = \left(I_X - (\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right)(s_1 I_X - \tilde{A})$$

and

$$\|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\| \leq \int_0^{+\infty} \|B(\tau)e^{-\lambda_0\tau}(1-e^{-\varepsilon\tau})\|\,d\tau\,\|R(s_1,\tilde{A})\| \leq \int_0^{+\infty} \|B(\tau)\|(1-e^{-\varepsilon\tau})\,d\tau\,\|R(s_1,\tilde{A})\| \to 0$$

as $\varepsilon \to +0$. Therefore, if $\varepsilon > 0$ is small then $\left\|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right\| < 1/2$. Hence $I_X - (\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})$ is invertible and

$$\left(I_X - (\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right)^{-1} = \sum_{n=0}^{+\infty} \left\{(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right\}^n.$$

It follows that

$$\left(s_1 I_X - A - \Phi_{\lambda_0+\varepsilon}\right)^{-1} = R(s_1,\tilde{A}) \sum_{n=0}^{+\infty} \left\{(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right\}^n.$$

We thus get

$$\begin{aligned}
\left\|(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1} - (s_1 I - A - \Phi_{\lambda_0})^{-1}\right\| &= \left\| R(s_1,\tilde{A}) \sum_{n=1}^{+\infty} \left\{(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right\}^n \right\| \\
&\leq \|R(s_1,\tilde{A})\| \sum_{n=1}^{+\infty} \left\|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\right\|^n = \|R(s_1,\tilde{A})\| \frac{\|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\|}{1 - \|(\Phi_{\lambda_0+\varepsilon} - \Phi_{\lambda_0})R(s_1,\tilde{A})\|} \\
&\leq 2\|R(s_1,\tilde{A})\|^2 \int_0^{+\infty} \|B(\tau)\|(1-e^{-\varepsilon\tau})\,d\tau \to 0, \qquad \varepsilon \to +0.
\end{aligned}$$

Then the positivity of $(s_1 I - A - \Phi_{\lambda_0+\varepsilon})^{-1}$ follows from the positivity of $(s_1 I - A - \Phi_{\lambda_0})^{-1}$, as desired. Applying [1, Lemma VI.1.9] again, we get $s_1 > s(A+\Phi_{\lambda_0}) = s_0$, a contradiction to the fact that $s_1 < s_0$. Thus, $f(\lambda)$ and $g(\lambda)$ must be right continuous at $\lambda \geq 0$.

Assume on the contrary that $g(0) \leq 0$. Since the function g is continuous on $[0,+\infty)$ and $\lim_{\lambda\to+\infty} g(\lambda) = +\infty$, there is a $\lambda_1 \geq 0$ such that $g(\lambda_1) = 0$; that is, $\lambda_1 = s(A+\Phi_{\lambda_1})$.

Since $A+\Phi_{\lambda_1}$ generates a positive semigroup and $s(A+\Phi_{\lambda_1}) > -\infty$, by virtue of [1, Theorem VI.1.10] $\lambda_1 = s(A+\Phi_{\lambda_1}) \in \sigma(A+\Phi_{\lambda_1})$. Since $A+\Phi_{\lambda_1}$ generates a compact C_0-semigroup, it follows from [1, Corollary IV.1.19] that $\sigma(A+\Phi_{\lambda_1})$ is identical with $P_\sigma(A+\Phi_{\lambda_1})$, the point spectrum of $A+\Phi_{\lambda_1}$. Thus, there exists a nonzero $x_1 \in X$ such that $(A+\Phi_{\lambda_1})x_1 = \lambda_1 x_1$; that is, $Ax_1 + \int_0^{+\infty} B(\tau)e^{-\lambda_1\tau}x_1 d\tau = \lambda_1 x_1$. Put $x(t) = e^{\lambda_1 t}x_1$ for $t \in \mathbb{R}$. Then, it is easy to see that

$$\dot{x}(t) = Ax(t) + \int_0^{+\infty} B(\tau)x(t-\tau)d\tau, \qquad t \in \mathbb{R};$$

hence x satisfies the "limit" variant of (1). By virtue of [5, Proposition 2.3], the zero solution of this limit equation is UAS because of the uniform asymptotic stability of (1). Hence we must get $\lim_{t\to+\infty}\|x(t)\| = 0$. However, $\|x(t)\| = e^{\lambda_1 t}\|x_1\| \geq \|x_1\| > 0$ for $t \geq 0$, a contradiction. This completes the proof of the implication (i)⇒(ii). □

Remark 4.5.

Throughout this paper, the strong assumption on continuity of B in the operator norm is imposed. It may be expected that this assumption may be replaced by the weaker assumption that $B(t)$ is strongly continuous at t. In fact, the norm continuity of B is needed only to apply Theorem 4.2 which is essentially used in the proof of Theorem 4.4. Therefore, if Theorem 4.2 ([5, Theorem 3.3]) holds true under the weaker condition on $B(t)$, then one would be able to replace the strong assumption by the weaker one. Unfortunately, the authors have not succeeded in proving Theorem 4.2 under the weaker assumption.

As will be shown in the example of the last section, there are some Volterra integro-differential equations with a kernel function of bounded linear operators, which are derived from partial integro-differential equations as abstract equations on some Banach lattices. In [1, Section IV.7.c] and [14], however, Volterra integro-differential equations with the kernel function which is of the form $B(t) = a(t)A$, where $a \in W^{1,1}(\mathbb{R}_+, \mathbb{C})$, are treated. We point out that $B(t)$ in this paper is restricted to bounded linear operators; hence our result is not applicable to the equations with the kernel function of the form $a(t)A$, and further improvements so as to cover the wider class of equations must be done.

4.2. Robust stability of positive linear Volterra equations in Banach lattices

Let A generate a positive semigroup $(T(t))_{t\geq 0}$ on X and let $B(t) \geq 0$ for all $t \geq 0$. Assume that (2) holds true and the equation (1) is UAS. We now consider a perturbed equation of the form

$$\dot{x}(t) = (A + F\Delta C)x(t) + \int_0^t \big(B(t-s) + D\Gamma(t-s)E\big)\, x(s)\, ds, \qquad t \geq 0, \tag{17}$$

where $F \in \mathcal{L}(Y,X)$, $C \in \mathcal{L}(X,Z)$, $D \in \mathcal{L}(U,X)$, $E \in \mathcal{L}(X,V)$ are given operators and $\Delta \in \mathcal{L}(Z,Y)$, $\Gamma(\cdot) \in L^1(\mathbb{R}_+, \mathcal{L}(V,U)) \cap C(\mathbb{R}_+, \mathcal{L}(V,U))$ are unknown disturbances. Here and hereafter $X, Y, Z, U, V, \ldots$ are assumed to be complex Banach lattices.

We shall measure the size of a pair of perturbation $(\Delta, \Gamma(\cdot)) \in \mathcal{L}(Z,Y) \times [L^1(\mathbb{R}_+, \mathcal{L}(V,U)) \cap C(\mathbb{R}_+, \mathcal{L}(V,U))]$ by

$$\|(\Delta, \Gamma(\cdot))\| = \|\Delta\| + \int_0^{+\infty} \|\Gamma(s)\|\, ds.$$

The main problem here is to find a positive number α such that (17) remains UAS whenever

$$\|(\Delta, \Gamma(\cdot))\| = \|\Delta\| + \int_0^{+\infty} \|\Gamma(s)\|\, ds < \alpha.$$

Theorem 4.6.

Let A generate a positive compact semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \geq 0$ for all $t \geq 0$. Suppose the equation (1) is UAS and $F \in \mathcal{L}_+(Y,X)$, $C \in \mathcal{L}_+(X,Z)$, $D \in \mathcal{L}_+(U,X)$, $E \in \mathcal{L}_+(X,V)$. Then (17) is still UAS whenever

$$\|(\Delta, \Gamma(\cdot))\| < \frac{1}{\max\limits_{P\in\{F,D\}, Q\in\{C,E\}} \big\|Q\big(-A - \int_0^{+\infty} B(s)\, ds\big)^{-1} P\big\|}.$$

To prove the above theorem, we need the following auxiliary lemma.

Lemma 4.7.

Let A generate a positive compact semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \geq 0$ for all $t \geq 0$, and let $P \in \mathcal{L}_+(U,X)$, $Q \in \mathcal{L}_+(X,Z)$. If (1) is UAS then

$$\sup_{\lambda\in\mathbb{C},\Re\lambda\geq 0} \left\| Q\left(\lambda I - A - \int_0^{+\infty} e^{-\lambda s}B(s)ds\right)^{-1} P\right\| = \left\| Q\left(-A - \int_0^{+\infty} B(s)ds\right)^{-1} P\right\|.$$

Proof. For a fixed $\lambda \in \mathbb{C}$, $\Re\lambda \geq 0$, we set $W(\lambda) = \int_0^{+\infty} e^{-\lambda s}B(s)\,ds$. It is well known that $A + W(\lambda)$ with the domain $\mathcal{D}(A + W(\lambda)) = \mathcal{D}(A)$ is the generator of a compact C_0-semigroup $(V_\lambda(t))_{t\geq 0}$ satisfying

$$V_\lambda(t)x = \lim_{n\to\infty}\left(T\left(\frac{t}{n}\right)e^{\frac{t}{n}W(\lambda)}\right)^n x \qquad \text{for} \quad t \geq 0,\ x \in X, \tag{18}$$

see e.g. [8, p. 44]. Since $B(s) \geq 0$ for all $s \geq 0$, it follows that

$$|W(\lambda)x| = \left|\left(\int_0^{+\infty} e^{-\lambda s}B(s)\,ds\right)x\right| \leq \int_0^{+\infty} B(s)\,ds\,|x| = W(0)|x|, \qquad x \in X.$$

By Lemma 4.3, we get

$$s(A + W(\lambda)) = \omega(A + W(\lambda)) \leq \omega(A + W(0)) = s(A + W(0)), \qquad \lambda \in \mathbb{C}, \quad \Re\lambda \geq 0.$$

Since (1) is UAS, we have

$$\omega(A + W(\lambda)) \leq s(A + W(0)) < 0$$

by Theorem 4.4. For $\lambda \in \mathbb{C}$, $\Re\lambda \geq 0$, we can represent

$$\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s}B(s)\,ds\right)^{-1} x = \int_0^{+\infty} e^{-\lambda t}V_\lambda(t)\,x\,dt, \qquad x \in X, \tag{19}$$

By (18)–(19) and the positivity of $(T(t))_{t\geq 0}$ and $B(t) \geq 0$ for all $t \geq 0$, we get

$$\left|\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s}B(s)\,ds\right)^{-1} x\right| \leq \int_0^{+\infty} V_0(t)|x|\,dt = \left(-A - \int_0^{+\infty} B(s)\,ds\right)^{-1} |x|,$$

for every $\lambda \in \mathbb{C}$, $\Re\lambda \geq 0$. Furthermore, since $P \in \mathcal{L}_+(U,X)$ and $Q \in \mathcal{L}_+(X,Z)$, it follows that

$$\left|Q\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s}B(s)\,ds\right)^{-1} Pu\right| \leq Q\left(-A - \int_0^{+\infty} B(s)\,ds\right)^{-1} P|u| \quad \text{for all} \quad u \in U,$$

for every $\lambda \in \mathbb{C}$, $\Re\lambda \geq 0$. Therefore, by (6) we get

$$\left\|Q\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s}B(s)\,ds\right)^{-1} Pu\right\| \leq \left\|Q\left(-A - \int_0^{+\infty} B(s)ds\right)^{-1} P|u|\right\| \quad \text{for all} \quad u \in U.$$

This completes the proof. □

Proof of Theorem 4.6. Assume that the perturbed equation (17) is not UAS for some $(\Delta, \Gamma(\cdot)) \in \mathcal{L}(Y, Z) \times L^1(\mathbb{R}_+, \mathcal{L}(V, U)) \cap C(\mathbb{R}_+, \mathcal{L}(V, U))$. It follows from Theorem 4.2 that

$$\lambda I_X - (A + F\Delta C) - \int_0^{+\infty} e^{-\lambda s}(B(s) + D\Gamma(s)E)\, ds,$$

is not invertible for some $\lambda \in \mathbb{C}, \Re\lambda \geq 0$. Thus,

$$\lambda \in \sigma\left(A + F\Delta C + \int_0^{+\infty} e^{-\lambda s}(B(s) + D\Gamma(s)E)\, ds\right).$$

Since A is the generator of a compact semigroup, so is $A + F\Delta C + \int_0^{+\infty} e^{-\lambda s}(B(s) + D\Gamma(s)E)\, ds$. Therefore, λ is an eigenvalue of this operator by [1, Corollary IV.1.19]. This implies that

$$\left(A + F\Delta C + \int_0^{+\infty} e^{-\lambda s}(B(s) + D\Gamma(s)E)\, ds\right) x = \lambda x,$$

for some $x \in X$, $x \neq 0$. Since (1) is UAS, $\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds$ is invertible. We thus get

$$\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds\right)^{-1} \left(F\Delta C x + \int_0^{+\infty} e^{-\lambda s} D\Gamma(s)Ex\, ds\right) = x.$$

From $x \neq 0$ it follows that $\max\{\|Cx\|, \|Ex\|\} > 0$. Let $Q \in \{C, E\}$, that is, $Q = C$ or $Q = E$. Multiplying the last equation by Q from the left, we get

$$Q\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds\right)^{-1} F\Delta C x + Q\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds\right)^{-1} D\int_0^{+\infty} e^{-\lambda s}\Gamma(s)\, ds\, Ex = Qx.$$

This yields

$$\begin{aligned}\left\| Q\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds\right)^{-1} F\right\| \|\Delta\|\, \|Cx\| + \\ \left\| Q\left(\lambda I_X - A - \int_0^{+\infty} e^{-\lambda s} B(s)\, ds\right)^{-1} D\right\| \int_0^{+\infty} |e^{-\lambda s}|\, \|\Gamma(s)\|\, ds\, \|Ex\| \geq \|Qx\|.\end{aligned}$$

By Lemma 4.7, we derive that

$$\left\| Q\left(-A - \int_0^{+\infty} B(s)\, ds\right)^{-1} F\right\| \|\Delta\|\, \|Cx\| + \left\| Q\left(-A - \int_0^{+\infty} B(s)\, ds\right)^{-1} D\right\| \int_0^{+\infty} \|\Gamma(s)\|\, ds\, \|Ex\| \geq \|Qx\|.$$

Therefore,

$$\max_{P\in\{F,D\},\, Q\in\{C,E\}} \left\| Q\left(-A - \int_0^{+\infty} B(s)\, ds\right)^{-1} P\right\| \left(\|\Delta\| + \int_0^{+\infty} \|\Gamma(s)\|\, ds\right) \geq \frac{\|Qx\|}{\max\{\|Cx\|, \|Ex\|\}}.$$

Choose $Q \in \{C, E\}$ such that $\|Qx\| = \max\{\|Cx\|, \|Ex\|\}$. Then we obtain

$$\max_{P\in\{F,D\},\, Q\in\{C,E\}} \left\| Q\left(-A - \int_0^{+\infty} B(s)\, ds\right)^{-1} P\right\| \left(\|\Delta\| + \int_0^{+\infty} \|\Gamma(s)\|\, ds\right) \geq 1,$$

which is equivalent to

$$\|(\Delta, \Gamma(\cdot))\| = \|\Delta\| + \int_0^{+\infty} \|\Gamma(s)\|\, ds \geq \frac{1}{\max\limits_{P\in\{F,D\},\, Q\in\{C,E\}} \left\|Q\big(-A - \int_0^{+\infty} B(s)\, ds\big)^{-1} P\right\|}.$$

This ends the proof. □

Remark 4.8.

It is important to note that the problem of finding the maximal $\alpha > 0$ such that any perturbed equation of the form (17) remains UAS whenever $\|(\Delta, \Gamma(\cdot))\| < \alpha$, is still open even for Volterra equations in finite dimensional spaces. This is the problem of computing stability radii of linear equations which has attracted a lot of attention from researchers during the last twenty years, see e.g. [9]–[12] and the references therein.

We now present two results of the problem of computing stability radii of equation (1) in some special cases of perturbation. More precisely, we now deal with perturbed equations of the form

$$\dot{x}(t) = (A + D_0\Delta E)x(t) + \int_0^t \Big(B(t-s) + D_1\Gamma(t-s)E\Big)\, x(s)\, ds, \qquad t \geq 0, \tag{20}$$

where $D_0 \in \mathcal{L}(Y_0, X)$, $E \in \mathcal{L}(X, Z)$, $D_1 \in \mathcal{L}(Y_1, X)$ are given and $\Delta \in \mathcal{L}(Z, Y_0)$; $\Gamma(\cdot) \in L^1\big(\mathbb{R}_+, \mathcal{L}(Z, Y_1)\big) \cap C\big(\mathbb{R}_+, \mathcal{L}(Z, Y_1)\big)$ are unknown disturbances.

Clearly, (20) is a particular case of (17) with $C = E$, $F = D_0$ and $D = D_1$. We introduce classes of perturbations defined as

$$\begin{aligned}
\mathcal{D}_{\mathbb{C}} &= \Big\{(\Delta, \Gamma) \,:\, \Delta \in \mathcal{L}(Z, Y_0),\ \Gamma(\cdot) \in L^1\big(\mathbb{R}_+, \mathcal{L}(Z, Y_1)\big) \cap C\big(\mathbb{R}_+, \mathcal{L}(Z, Y_1)\big)\Big\},\\
\mathcal{D}_{\mathbb{R}} &= \Big\{(\Delta, \Gamma) \,:\, \Delta \in \mathcal{L}_{\mathbb{R}}(Z, Y_0),\ \Gamma(\cdot) \in L^1\big(\mathbb{R}_+, \mathcal{L}_{\mathbb{R}}(Z, Y_1)\big) \cap C\big(\mathbb{R}_+, \mathcal{L}_{\mathbb{R}}(Z, Y_1)\big)\Big\},\\
\mathcal{D}_{+} &= \Big\{(\Delta, \Gamma) \,:\, \Delta \in \mathcal{L}_{+}(Z, Y_0),\ \Gamma(\cdot) \in L^1\big(\mathbb{R}_+, \mathcal{L}_{+}(Z, Y_1)\big) \cap C\big(\mathbb{R}_+, \mathcal{L}_{+}(Z, Y_1)\big)\Big\}.
\end{aligned}$$

Then, the complex, real and positive stability radius of (1) under perturbations of the form

$$A \rightsquigarrow A + D_0\Delta E, \qquad F(\cdot) \rightsquigarrow F(\cdot) + D_1\Gamma(\cdot)E,$$

are defined, respectively, by

$$\begin{aligned}
r_{\mathbb{C}} &= \inf\Big\{\|(\Delta, \Gamma)\| \,:\, (\Delta, \Gamma) \in \mathcal{D}_{\mathbb{C}},\ (20) \text{ is not UAS}\Big\},\\
r_{\mathbb{R}} &= \inf\Big\{\|(\Delta, \Gamma)\| \,:\, (\Delta, \Gamma) \in \mathcal{D}_{\mathbb{R}},\ (20) \text{ is not UAS}\Big\},\\
r_{+} &= \inf\Big\{\|(\Delta, \Gamma)\| \,:\, (\Delta, \Gamma) \in \mathcal{D}_{+},\ (20) \text{ is not UAS}\Big\}.
\end{aligned}$$

Here and in what follows, by convention, we define $\inf \emptyset = +\infty$ and $1/0 = +\infty$. By the definition, it is easy to see that

$$r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_{+}.$$

Theorem 4.9.

Let A generate a positive compact semigroup $(T(t))_{t\geq 0}$ on X, $B(t) \geq 0$ for all $t \geq 0$, $E \in \mathcal{L}_+(X, Z)$, and $D_i \in \mathcal{L}_+(Y_i, X)$, $i = 0, 1$. If (1) is UAS then

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_{+} = \frac{1}{\max\limits_{i=0,1} \Big\|E\Big(-A - \int_0^{+\infty} B(s)\, ds\Big)^{-1} D_i\Big\|}. \tag{21}$$

Proof. Observe that

$$r_{\mathbb{C}} \geq \frac{1}{\max\limits_{i=0,1} \Big\|E\Big(-A - \int_0^{+\infty} B(s)\, ds\Big)^{-1} D_i\Big\|}$$

by Theorem 4.6. Since $r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_+$, it remains to show that

$$r_{+} \leq \frac{1}{\max\limits_{i=0,1} \Big\|E\Big(-A - \int_0^{+\infty} B(s)\, ds\Big)^{-1} D_i\Big\|}. \tag{22}$$

Assume that

$$\max_{i=0,1}\left\|E\left(-A-\int_0^{+\infty}B(s)\,ds\right)^{-1}D_i\right\|=\left\|E\left(-A-\int_0^{+\infty}B(s)\,ds\right)^{-1}D_{i_0}\right\|>0$$

for some $i_0\in\{0,1\}$. Note that $A+\int_0^{+\infty}B(\tau)\,d\tau$ generates a positive C_0-semigroup and since (1) is UAS, $s\big(A+\int_0^{+\infty}B(\tau)\,d\tau\big)<0$ by Theorem 4.4. This implies that $R\big(0,A+\int_0^{+\infty}B(\tau)\,d\tau\big)=\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}\geq 0$ by [1, Lemma VI.1.9]. Therefore $E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}\in\mathcal{L}_+(Y_{i_0},Z)$. Let $0<\varepsilon<\big\|E\big(-A-\int_0^{+\infty}B(s)\,ds\big)^{-1}D_{i_0}\big\|$. By (10), one can choose $u\in(Y_{i_0})_+$, $\|u\|=1$, so that

$$\left\|E\left(-A-\int_0^{+\infty}B(\tau)\,d\tau\right)^{-1}D_{i_0}u\right\|>\left\|E\left(-A-\int_0^{+\infty}B(\tau)\,d\tau\right)^{-1}D_{i_0}\right\|-\varepsilon.$$

Since $z_0=E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}u\in Z_+$, there exists a positive $f\in Z^*$, $\|f\|=1$, satisfying $f(z_0)=\|z_0\|=\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}u\big\|$ (cf. [7, Proposition 1.5.7], [17, p. 249]). We now consider the operator $\Delta:Z\to Y_{i_0}$ defined by

$$z\mapsto\Delta z=\frac{f(z)}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}u\big\|}u.$$

It is clear that $\Delta\in\mathcal{L}_+(Z,Y_{i_0})$ and $\|\Delta\|=1/\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}u\big\|$. Set $x_0=\big(-A-\int_0^{+\infty}B(s)\,ds\big)^{-1}D_{i_0}u$. Then $Ex_0=E\big(-A-\int_0^{+\infty}B(s)\,ds\big)^{-1}D_{i_0}u=z_0$, and hence

$$\Delta Ex_0=\frac{f(z_0)}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}u\big\|}u=\frac{\|z_0\|}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau)^{-1}D_{i_0}u\big\|}u=u.$$

Then $x_0\neq 0$ because of $u\neq 0$. Moreover, we have

$$x_0=\left(-A-\int_0^{+\infty}B(s)\,ds\right)^{-1}D_{i_0}(\Delta Ex_0),$$

or equivalently,

$$\left(A+D_{i_0}\Delta E+\int_0^{+\infty}B(s)\,ds\right)x_0=0.$$

Consider the case of $i_0=0$. Then $\Delta\in\mathcal{L}_+(Z,Y_0)$ and $\big(A+D_0\Delta E+\int_0^{+\infty}B(s)\,ds\big)x_0=0$, which implies $0\in\sigma\big(A+D_0\Delta E+\int_0^{+\infty}B(s)\,ds\big)$. Hence

$$r_+\leq\|(\Delta,0)\|=\|\Delta\|=\frac{1}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_0u\big\|}<\frac{1}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)d\tau\big)^{-1}D_{i_0}\big\|-\varepsilon}.$$

We next consider the case of $i_0=1$. Then $\Delta\in\mathcal{L}_+(Z,Y_1)$ and $\big(A+D_1\Delta E+\int_0^{+\infty}B(s)\,ds\big)x_0=0$. Define $\Gamma(t)=e^{-t}\Delta$ for all $t\geq 0$. Then $\Gamma(\cdot)\in L^1\big(\mathbb{R}_+,\mathcal{L}_+(Z,Y_1)\big)\cap C\big(\mathbb{R}_+,\mathcal{L}(Z,Y_1)\big)$, and it satisfies $\big(A+\int_0^{+\infty}(B(s)+D_1\Gamma(s)E)\,ds\big)x_0=\big(A+\int_0^{+\infty}B(s)\,ds+D_1\Delta E\big)x_0=0$, whence $0\in\sigma\big(A+\int_0^{+\infty}(B(s)+D_1\Gamma(s)E)\,ds\big)$. Therefore,

$$r_+\leq\|(0,\Gamma)\|=\|\Delta\|=\frac{1}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_1u\big\|}<\frac{1}{\big\|E\big(-A-\int_0^{+\infty}B(\tau)\,d\tau\big)^{-1}D_{i_0}\big\|-\varepsilon}.$$

Since ε can be arbitrarily small, we thus get (22).

Finally, it is worth noticing that from the above argument and that of the proof of Theorem 4.6, $r_{\mathbb{C}}=r_{\mathbb{R}}=r_+=+\infty$ if and only if $\max_{i=0,1}\big\|E\big(-A-\int_0^{+\infty}B(s)\,ds\big)^{-1}D_i\big\|=0$. So (21) is obvious in this case. This completes the proof. □

Finally, we will treat perturbed equations of the form

$$\dot{x}(t) = (A + D\Delta E_0)x(t) + \int_0^t \big(B(t-s) + D\Gamma(t-s)E_1\big)x(s)ds, \qquad t \geq 0, \tag{23}$$

where $D \in \mathcal{L}(Y, X)$, $E_0 \in \mathcal{L}(X, Z_0)$, $E_1 \in \mathcal{L}(X, Z_1)$ are given and $\Delta \in \mathcal{L}(Z_0, Y)$, $\Gamma(\cdot) \in L^1\big(\mathbb{R}_+, \mathcal{L}(Z_1, Y)\big) \cap C\big(\mathbb{R}_+, \mathcal{L}(Z_1, Y)\big)$ are unknown disturbances.

In other words, A and $F(\cdot)$ are now subjected to perturbations of the form:

$$A \rightsquigarrow A + D\Delta E_0, \qquad F(\cdot) \rightsquigarrow F(\cdot) + D\Gamma(\cdot)E_1.$$

With an appropriate modification for the definition of stability radii, by the similar way as the above, we can get the following

Theorem 4.10.

Let A generate a positive compact semigroup $(T(t))_{t\geq 0}$ on X, $B(t) \geq 0$ for all $t \geq 0$, $E_i \in \mathcal{L}_+(X, Z_i)$, $i = 0, 1$, and $D \in \mathcal{L}_+(Y, X)$. If the equation (1) is UAS, then

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max\limits_{i=0,1} \big\| E_i\big(-A - \int_0^{+\infty} B(s)\, ds\big)^{-1} D \big\|}.$$

5. An example

In this section we give an example which shows how our results (especially Theorems 4.4 and 4.6) are applicable in the stability analysis of concrete equations.

We consider the partial integro-differential equation

$$\frac{\partial x(t,\xi)}{\partial t} = \frac{\partial^2 x(t,\xi)}{\partial \xi^2} + d(\xi)x(t,\xi) + \int_0^t k(t-s,\xi)x(s,\xi)ds, \qquad t \geq 0, \quad \xi \in [0,1], \tag{24}$$

subject to the boundary condition

$$\frac{\partial x(t,0)}{\partial \xi} = 0 = \frac{\partial x(t,1)}{\partial \xi}, \qquad t \geq 0, \tag{25}$$

where $d : [0,1] \to \mathbb{R}$ is a given continuous function with $\alpha = -\sup_{0\leq\xi\leq 1} d(\xi) > 0$ and $k : [0,\infty) \times [0,1] \to \mathbb{R}$ is a nonnegative continuous function satisfying $\sup_{0\leq\xi\leq 1} k(t,\xi) \leq K(t)$ for all $t \geq 0$, where K is given and $\int_0^{+\infty} K(t)dt < \infty$.

We first set up (24)–(25) as an abstract equation on a Banach lattice. To do this, we take $X = C([0,1], \mathbb{C})$, the Banach lattice of all continuous complex valued functions on $[0,1]$, equipped with the supremum norm, and consider a linear operator A defined by

$$(Af)(\xi) = f''(\xi) + d(\xi)f(\xi), \qquad \xi \in [0,1],$$

where

$$\mathcal{D}(A) = \big\{ f \in C^2([0,1]) \,:\, f'(0) = f'(1) = 0 \big\},$$

together with the operators $B(t)$, $t \geq 0$, defined by

$$(B(t)h)(\xi) = k(t,\xi)h(\xi), \qquad \xi \in [0,1], \quad h \in X.$$

Observe that $B(t)$ is a positive bounded linear operator on X with operator norm $\|B(t)\| = \sup_{0\leq\xi\leq 1} k(t,\xi)$ $(\leq K(t))$, together with the estimate $\|B(t) - B(\bar{t})\| = \sup_{0\leq\xi\leq 1} |k(t,\xi) - k(\bar{t},\xi)|$; consequently, the operator $B(\cdot)$ fulfils the condition

$B(\cdot) \in L^1\big(\mathbb{R}_+, \mathcal{L}_+(X)\big) \cap C\big(\mathbb{R}_+, \mathcal{L}_+(X)\big)$ because of $\int_0^{+\infty} K(t)\,dt < \infty$. It remains to verify that A generates a positive compact semigroup. As it is well known (e.g. [1, Ex. II.4.34–(1)]), $(d^2/d\xi^2, \mathcal{D}(A))$ generates a compact (analytic) positive contraction semigroup (one dimensional diffusion semigroup), say $(T_0(t))_{t\geq 0}$. Introducing a bounded linear operator M on X defined by

$$(Mh)(\xi) = d(\xi)h(\xi), \qquad \xi \in [0,1], \quad h \in X,$$

which generates a uniformly continuous semigroup $(e^{Mt})_{t\geq 0}$, we see that A is a bounded perturbation of $d^2/d\xi^2$, that is, $A = d^2/d\xi^2 + M$; consequently, by virtue of [1, Theorem II.4.29, Proposition III.1.12], A generates a compact (analytic) semigroup, say $(T(t))_{t\geq 0}$. Notice that $(e^{Mt})_{t\geq 0}$ is positive because of $(e^{Mt}h)(\xi) = e^{td(\xi)}h(\xi)$, $\xi \in [0,1]$. Therefore, $(T(t))_{t\geq 0}$ is positive, since

$$T(t)h = \lim_{n\to\infty} \left[T_0\left(\frac{t}{n}\right) e^{\frac{t}{n}M} \right]^n h, \qquad h \in X,$$

for each $t \geq 0$; see e.g. [8, p. 44].

Observe that $\int_0^\infty B(t)dt$ is a positive bounded linear operator defined by

$$\left[\left(\int_0^{+\infty} B(t)\,dt \right) h \right](\xi) = a(\xi)h(\xi), \qquad \xi \in [0,1], \quad h \in X,$$

where $a(\xi) = \int_0^{+\infty} k(t,\xi)dt$ $(\leq \int_0^{+\infty} K(t)dt < \infty)$. In what follows, we assume that

$$\sup_{0\leq\xi\leq 1} (d(\xi) + a(\xi)) = -\delta < 0 \tag{26}$$

for a constant δ. Under this assumption, we will next show that the zero solution of the equation (1) set up in the foregoing paragraph is UAS. We claim that the semigroup $(U(t))_{t\geq 0}$ generated by the operator $A + \int_0^{+\infty} B(t)\,dt$ satisfies the estimate

$$\|U(t)\| \leq e^{-\delta t}, \qquad t \geq 0. \tag{27}$$

Indeed, if the claim holds true, then it follows from the well-known result (e.g. [1, Theorem II.1.10]) that $s\big(A + \int_0^{+\infty} B(t)\,dt\big) \leq -\delta$ together with the estimate $\big\|\big(\lambda - A - \int_0^{+\infty} B(t)\,dt\big)^{-1}\big\| \leq 1/(\Re\lambda + \delta)$ for any $\lambda \in \mathbb{C}$ with $\Re\lambda > -\delta$; hence we conclude by Theorem 4.4 that the zero solution of (1) is UAS.

Now we will prove (27). Let $h \in D(A)$ be any element such that $\|h\| < 1$, and set $u(t,\xi) = (U(t)h)(\xi)$, $\xi \in [0,1]$, $t \geq 0$. Then u is a classical solution of the partial differential equation

$$\frac{\partial u(t,\xi)}{\partial t} = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + b(\xi)u(t,\xi), \qquad t \geq 0, \quad \xi \in [0,1],$$

subject to the boundary condition

$$\frac{\partial u(t,0)}{\partial \xi} = 0 = \frac{\partial u(t,1)}{\partial \xi}, \qquad t \geq 0,$$

where $b(t) = d(t) + a(t)$ $(\leq -\delta)$. Notice that $-1 < u(0,\xi) < 1$ for any $\xi \in [0,1]$. We will verify that $e^{\delta t}u(t,\xi) < 1$ for any $(t,\xi) \in [0,\infty) \times [0,1]$ by applying the strong maximum principle (e.g. [13, Theorems 3.6 and 3.7]). Indeed, if this is false, then there is a $(t_1,\xi_1) \in (0,\infty) \times [0,1]$ such that $e^{\delta t_1}u(t_1,\xi_1) = 1$ and $e^{\delta t}u(t,\xi) < 1$ for any $t < t_1$ and $\xi \in [0,1]$. Set $v(t,\xi) = e^{\delta t}u(t,\xi) - 1$ for $(t,\xi) \in [0,t_1] \times [0,1]$. On $(0,t_1] \times (0,1)$ we get

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial v}{\partial t} = e^{\delta t}\frac{\partial^2 u}{\partial \xi^2} - e^{\delta t}\left(\delta u + \frac{\partial u}{\partial t}\right) = e^{\delta t}(-b(\xi)u - \delta u) = -(v+1)(b(\xi)+\delta),$$

or

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial v}{\partial t} + (b(\xi)+\delta)v = -(b(\xi)+\delta) \geq 0,$$

together with the boundary condition

$$\frac{\partial v(t,0)}{\partial \xi} = 0 = \frac{\partial v(t,1)}{\partial \xi}, \qquad t \geq 0.$$

Since $b(\xi) + \delta \leq 0$ by the assumption, one can apply the strong maximum principle. Consequently, we get $\xi_1 = 0$, or $\xi_1 = 1$ and $v(t,\xi) < 0$ for any $(t,\xi) \in [0,t_1] \times (0,1)$. Since $v(t_1,\xi_1) = 0$, we get by the strong maximum principle again that $\frac{\partial v}{\partial \xi} < 0$ at (t_1,ξ_1) if $\xi_1 = 0$, and $\frac{\partial v}{\partial \xi} > 0$ at (t_1,ξ_1) if $\xi_1 = 1$; a contradiction to the boundary condition. Thus we must have that $e^{\delta t}u(t,\xi) < 1$ for any $(t,\xi) \in [0,\infty) \times [0,1]$. In a similar way, one can deduce that $e^{\delta t}u(t,\xi) > -1$ for any $(t,\xi) \in [0,\infty) \times [0,1]$. Thus we get $e^{\delta t}|u(t,\xi)| < 1$ on $[0,\infty) \times [0,1]$; in other words, $\|U(t)h\| \leq e^{-\delta t}$ for any $h \in D(A)$ with $\|h\| < 1$. Since $D(A)$ is dense in X, we get the desired estimate $\|U(t)\| \leq e^{-\delta t}$.

Next we will discuss the stability of the perturbed equation (17) under the same conditions as above. Since $\big\|R\big(0, A + \int_0^{+\infty} B(s)\,ds\big)\big\| \leq 1/\delta$, it follows that

$$\left\| Q \left(-A - \int_0^{+\infty} B(s)\,ds \right)^{-1} P \right\| \leq \|Q\|\,\|P\|/\delta.$$

Therefore, if a pair of perturbation $(\Delta, \Gamma(\cdot))$ satisfies

$$\|(\Delta, \Gamma(\cdot))\| < \frac{\delta}{\max\big\{\|P\|\|Q\| \,:\, P \in \{F, D\},\, Q \in \{C, E\}\big\}},$$

then it satisfies the condition in Theorem 4.6; hence the perturbed equation (17) is still UAS by Theorem 4.6.

Summarizing these facts we get:

Proposition 5.1.

Under the prescribed conditions on the functions d and k in (24)–(25), the zero solution of the abstract equation (1) on the Banach lattice $X = C([0,1],\mathbb{C})$ is UAS whenever

$$\sup_{0 \leq \xi \leq 1} \left(d(\xi) + \int_0^{+\infty} k(t,\xi)\,dt \right) = -\delta < 0.$$

Furthermore, the zero solution of the perturbed equation (17) is UAS under the additional conditions on a pair of perturbation $(\Delta, \Gamma(\cdot))$

$$\|(\Delta, \Gamma(\cdot))\| < \frac{\delta}{\max\big\{\|P\|\|Q\| \,:\, P \in \{F, D\},\, Q \in \{C, E\}\big\}}.$$

Remark 5.2.

We emphasize that for the above result it is advantageous to apply Theorem 4.4 rather than Theorem 4.2. Indeed, the verification of (ii) in Theorem 4.4 is rather easy as seen above; but that of the condition (ii) in Theorem 4.2 is not.

Finally we remark that the method employed in the stability analysis for (24)–(25) with one dimensional diffusion term is valid also in the stability analysis for the partial integro-differential equation with multi-dimensional diffusion term

$$\frac{\partial x(t,\xi)}{\partial t} = \sum_{i=1}^{l} \frac{\partial^2 x(t,\xi)}{\partial \xi_i^2} + d(\xi)x(t,\xi) + \int_0^t k(t-s,\xi)x(s,\xi)ds, \qquad t \geq 0, \quad \xi \in \Omega,$$

subject to the Neumann-boundary condition, where $\Omega \subset \mathbb{R}^l$ is a bounded domain with smooth boundary $\partial\Omega$ (e.g. $C^{2+\mu}$ for a $\mu \in (0,1)$). Indeed, we know by virtue of [16, Theorem 2] that the Laplacian operator $\sum_{i=1}^{l} \partial^2/\partial \xi_i^2$ with the domain $\mathcal{D} = \{f \in C^2(\bar{\Omega}) \,:\, \partial f/\partial n = 0 \text{ on } \partial\Omega\}$ (here $\partial/\partial n$ denotes the exterior normal derivative at $\partial\Omega$) generates a compact analytic (positive) semigroup on the Banach lattice $C(\bar{\Omega})$; hence one can accomplish the stability analysis for multi-dimensional case, repeating the argument employed for one dimensional case.

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