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Operator-valued Feynman integral via conditional Feynman integrals on a function space

Research Article

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Abstract: Let $C_0^r[0,t]$ denote the analogue of the r-dimensional Wiener space, define $X_t : C^r[0,t] \to \mathbb{R}^{2r}$ by $X_t(x) = (x(0), x(t))$. In this paper, we introduce a simple formula for the conditional expectations with the conditioning function X_t. Using this formula, we evaluate the conditional analytic Feynman integral for the functional

$$\Gamma_t(x) = \exp\left\{\int_0^t \theta(s, x(s))\, d\eta(s)\right\}\phi(x(t)), \quad x \in C^r[0,t],$$

where η is a complex Borel measure on $[0,t]$, and $\theta(s,\cdot)$ and ϕ are the Fourier–Stieltjes transforms of the complex Borel measures on $\mathbb{R}^r$. We then introduce an integral transform as an analytic operator-valued Feynman integral over $C^r[0,t]$, and evaluate the integral transform for the function Γ_t via the conditional analytic Feynman integral as a kernel.

MSC: 28C20

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1. Introduction

Let r be a positive integer and let $C_0^r[0,t]$ denote the r-dimensional Wiener space. Cameron and Storvick [3] introduced a very general analytic operator-valued function space Feynman integral $J_q^{an}(F)$, on $C_0^r[0,t]$, which maps an $L_2(\mathbb{R}^r)$-function

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ψ into an $L_2(\mathbb{R}^r)$-function $J_q^{an}(F)\psi$. In [4, 15], the existence of the analytic operator-valued Feynman integral $J_q^{an}(F)$ as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ was studied. Chung, Park and Skoug [12] showed that it can be expressed by the formula

$$\big(J_q^{an}(F)\psi\big)(\vec{\xi}) = \left(\frac{q}{2\pi i t}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} E^{anf_q}[F|X](\vec{\xi})(\vec{\eta})\psi(\vec{\eta}) \exp\left\{\frac{qi}{2t}\|\vec{\eta}-\vec{\xi}\|_{\mathbb{R}^r}^2\right\} dm_L^r(\vec{\eta}), \tag{1}$$

where m_L^r is the r-dimensional Lebesgue measure on the Borel class of $\mathbb{R}^r$ and $E^{anf_q}[F|X]$ is the conditional analytic Feynman integral of F given X. Further work extending the above $\mathcal{L}(L_1, L_\infty)$-theory with the conditional analytic Feynman integrals was made by Cho [8], over the space $C_0(\mathbb{B})$ of Wiener paths in abstract Wiener space $\mathbb{B}$ which generalizes the space $C_0^r[0,t]$ [16, 18]. In fact, the conditional Wiener integral over $C_0(\mathbb{B})$ was introduced by Chang, Cho, Song and Yoo [6, 7] and they derived a simple formula for the conditional Wiener integral with the conditioning function $X : C_0(\mathbb{B}) \to \mathbb{B}$ defined by $X(x) = x(t)$, which calculates directly the conditional Wiener integral in terms of the Wiener integral. Applying this simple formula to a certain function F defined on $C_0(\mathbb{B})$, Cho [8] obtained the analytic operator-valued Feynman integral $J_q^{an}(F) : L_{p_1}(\mathbb{B}) \to L_{p_2}(\mathbb{B})$, $1 \le p_1, p_2 \le \infty$, using the formula

$$\big(J_q^{an}(F)\psi\big)(\xi) = \int_{\mathbb{B}} E^{anf_q}[F|X](\xi)(\eta)\psi(\eta)\, dm_{t^{1/2}}(\eta), \quad \xi \in \mathbb{B}, \tag{2}$$

where $m_{t^{1/2}}$ is the probability distribution of X on the Borel class of $\mathbb{B}$.

In this paper, we further develop the concepts of (1) and (2) on another generalized Wiener space $(C^r[0,t], w_\varphi^r)$, the analogue of the r-dimensional Wiener space associated with the probability measure φ on the Borel class of $\mathbb{R}$ [13, 19]. For the conditioning function $X_t : C^r[0,t] \to \mathbb{R}^{2r}$ defined by $X_t(x) = (x(0), x(t))$, we proceed to express the analytic $\mathcal{L}(L_1, L_\infty)$-operator valued Feynman integrals in terms of the conditional analytic Feynman integrals. In fact, with the conditioning function X_t, we introduce a simple formula for the conditional w_φ^r-integrals over $C^r[0,t]$ and using this formula, we then evaluate the conditional analytic Feynman w_φ^r-integral $E^{anf_q}[\Gamma_t|X_t]$ for the functional

$$\Gamma_t(x) = \exp\left\{\int_0^t \theta(s, x(s))\, d\eta(s)\right\} \phi(x(t)), \quad x \in C^r[0,t],$$

which is important in quantum mechanics and Feynman integration theories, where η is a complex Borel measure on $[0,t]$, and $\theta(s,\cdot)$ and ϕ are the Fourier–Stieltjes transforms of the complex Borel measures on $\mathbb{R}^r$. Finally, we establish that for a nonzero real q, the analytic operator-valued Feynman integral $J_q^{an}(\Gamma_t)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$, the space of the bounded linear operators from $L_1(\mathbb{R}^r)$ to $L_\infty(\mathbb{R}^r)$, and it is given by the formula

$$\big(J_q^{an}(\Gamma_t)\psi\big)(\vec{\xi}) = \left(\frac{q}{i\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} E^{anf_q}[\Gamma_t|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)\psi(\vec{\eta}_2)\Psi(-iq, \vec{\eta}_1 - \vec{\xi}) \exp\left\{\frac{iq\|\vec{\eta}_2 - \vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1, \vec{\eta}_2)$$

for $\vec{\xi} \in \mathbb{R}^r$ and $\psi \in L_1(\mathbb{R}^r)$, where Ψ is the analytic extension of the probability density of φ^r. Thus $J_q^{an}(\Gamma_t)$ can be interpreted as an integral transform with the kernel

$$\left(\frac{q}{i\sqrt{2\pi t}}\right)^r \exp\left\{\frac{iq\|\vec{\eta}_2 - \vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} E^{anf_q}[\Gamma_t|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)\Psi(-iq, \vec{\eta}_1 - \vec{\xi}).$$

2. An analogue of the r-dimensional Wiener space

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and the subset of complex numbers with positive real parts, respectively, and let $\mathbb{C}_+^\sim = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge 0\} \setminus \{0\}$. Further, let m_L denote the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.

Now, we introduce the probability measure w_φ^r on $\left(C^r[0,t], \mathcal{B}(C^r[0,t])\right)$, where $\mathcal{B}(C^r[0,t])$ denotes the Borel σ-algebra of $C^r[0,t]$.

For a positive real t, let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \dots, t_n)$ with $0 = t_0 < t_1 < \dots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \dots, x(t_n)).$$

For $B_j, j = 0, 1, \dots, n$, in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)$ of $C[0,t]$ is called an interval. Let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let

$$m_\varphi\left(J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)\right) = \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^n B_j} \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} dm_L^n(u_1, \dots, u_n)\, d\varphi(u_0).$$

Then $\mathcal{B}(C[0,t])$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $\left(C[0,t], \mathcal{B}(C[0,t])\right)$ such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [13, 19]. Let $C^r = C^r[0,t]$ be the product space of $C[0,t]$ with the product measure w_φ^r. Since $C[0,t]$ is a separable Banach space, we have $\mathcal{B}(C^r[0,t]) = \prod_{j=1}^r \mathcal{B}(C[0,t])$. This probability measure space $\left(C^r[0,t], \mathcal{B}(C^r[0,t]), w_\varphi^r\right)$ is called an analogue of the r-dimensional Wiener space.

Lemma 2.1 ([13, Lemma 2.1]).
If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then we have

$$\int_C f\big(x(t_0), x(t_1), \dots, x(t_n)\big)\, dw_\varphi(x)$$
$$\overset{*}{=} \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0, u_1, \dots, u_n) \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} dm_L^n(u_1, \dots, u_n)\, d\varphi(u_0),$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $\{e_k : k = 1, 2, \dots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_k is of bounded variation. For v in $L_2[0,t]$ and x in $C[0,t]$, we put

$$(v, x) = \lim_{n\to\infty} \sum_{k=1}^n \int_0^t \langle v, e_k\rangle e_k(s)\, dx(s)$$

if the limit exists. Here $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. (v,x) is called the *Paley–Wiener–Zygmund integral of v with respect to x*. Note that $\langle\cdot,\cdot\rangle$ also denotes the dot product over Euclidean space.

3. A simple formula for the conditional w_φ^r-integrals and the operator-valued function space integrals

In this section we derive a simple formula for the analogue of the conditional Wiener integrals over $C^r[0,t]$ with the vector-valued conditioning function X_t given by $X_t(x) = (x(0), x(t))$. First we define the conditional w_φ^r-integral over $C^r[0,t]$.

Definition 3.1.

Let $F : C^r[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C^r[0,t]$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X [17]. Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ^r-a.e. $x \in C^r[0,t]$, where P_X is the probability distribution of X. The function ψ is called the *conditional w_φ^r-integral of F given X* and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ be a partition of $[0,t]$. For any x in $C^r[0,t]$, define the polygonal function $[x]$ of x on $[0,t]$ by

$$[x](s) = x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})), \qquad t_{j-1} \le s \le t_j, \quad j = 1, \ldots, n.$$

Similarly, for $\vec{\xi}_{n+1} = (\vec{\xi}_0, \vec{\xi}_1, \ldots, \vec{\xi}_n) \in \mathbb{R}^{(n+1)r}$, we define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ by the right hand side of the above equality where $x(t_{j-1})$ and $x(t_j)$ are replaced by $\vec{\xi}_{j-1}$ and $\vec{\xi}_j$, respectively.

In the following theorem, we introduce a simple formula for conditional w_φ^r-integrals on $C^r[0,t]$. The proof of the theorem is given in [9–11].

Theorem 3.2.

Let $F : C^r[0,t] \to \mathbb{C}$ be integrable and $X_{n+1} : C^r[0,t] \to \mathbb{R}^{(n+1)r}$ be given by $X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. Then for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{(n+1)r}$, we have,

$$E[F|X_{n+1}](\vec{\xi}_{n+1}) = E[F(x - [x] + [\vec{\xi}_{n+1}])],$$

where $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on $\left(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r})\right)$.

Let $X_t : C^r[0,t] \to \mathbb{R}^{2r}$ be given by $X_t(x) = (x(0), x(t))$ and let $F : C^r[0,t] \to \mathbb{C}$ be a function. Further, let $X_t^{\lambda,\vec{\xi}}(x) = X_t(\lambda^{-\frac{1}{2}}x + \vec{\xi})$ and $F^{\lambda,\vec{\xi}}(x) = F(\lambda^{-\frac{1}{2}}x + \vec{\xi})$ for $\lambda > 0$ and for $\vec{\xi} \in \mathbb{R}^r$. Suppose that for $\lambda > 0$, $F^{\lambda,\vec{\xi}}$ is integrable over $C^r[0,t]$ for $\vec{\xi} \in \mathbb{R}^r$. Then for $\vec{\xi} \in \mathbb{R}^r$, we have, by Theorem 3.2,

$$E[F^{\lambda,\vec{\xi}}|X_t^{\lambda,\vec{\xi}}](\vec{\eta}_1, \vec{\eta}_2) = E\left[F\left(\lambda^{-\frac{1}{2}}\left(x(\cdot) - x(0) - \frac{\cdot}{t}(x(t) - x(0))\right) + \vec{\eta}_1 + \frac{\cdot}{t}(\vec{\eta}_2 - \vec{\eta}_1)\right)\right]$$

for $P_{X_t^{\lambda,\vec{\xi}}}$-a.e. $(\vec{\eta}_1, \vec{\eta}_2) \in \mathbb{R}^{2r}$, where $P_{X_t^{\lambda,\vec{\xi}}}$ is the probability distribution of $X_t^{\lambda,\vec{\xi}}$ on $\left(\mathbb{R}^{2r}, \mathcal{B}(\mathbb{R}^{2r})\right)$. Let

$$(K_\lambda(F))(\vec{\eta}_1, \vec{\eta}_2) = E\left[F\left(\lambda^{-\frac{1}{2}}\left(x(\cdot) - x(0) - \frac{\cdot}{t}(x(t) - x(0))\right) + \vec{\eta}_1 + \frac{\cdot}{t}(\vec{\eta}_2 - \vec{\eta}_1)\right)\right]. \tag{3}$$

If $(K_\lambda(F))(\vec{\eta}_1, \vec{\eta}_2)$ has the analytic extension $J_\lambda(\vec{\eta}_1, \vec{\eta}_2)$ on $\mathbb{C}_+$ as a function of λ, then it is called the *conditional analytic Wiener w_φ^r-integral of F over $C^r[0,t]$ given X_t with parameter λ* and denoted by

$$J_\lambda(\vec{\eta}_1, \vec{\eta}_2) = E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)$$

for $\lambda \in \mathbb{C}_+$. For a nonzero real q, if the limit

$$\lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)$$

exists, where λ approaches $-iq$ through $\mathbb{C}_+$, then it is called the *conditional analytic Feynman w_φ^r-integral of F over $C^r[0,t]$ given X_t with parameter q* and we write

$$\lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2) = E^{anf_q}[F|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2).$$

Next we define the analytic operator-valued function space integral.

Definition 3.3.

Let $F : C^r[0,t] \to \mathbb{C}$ be a function. For any $\lambda > 0$, ψ in $L_1(\mathbb{R}^r)$ and $\vec{\xi}$ in $\mathbb{R}^r$, let $\psi_t^{\lambda,\vec{\xi}}(x) = \psi(\lambda^{-\frac{1}{2}}x(t) + \vec{\xi})$ and

$$(I_\lambda(F)\psi)(\vec{\xi}) = \int_{C^r} F^{\lambda,\vec{\xi}}(x)\psi_t^{\lambda,\vec{\xi}}(x)\,dw_\varphi^r(x).$$

If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{R}^r)$ as a function of $\vec{\xi}$ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$, we say that the *operator-valued function space integral* $I_\lambda(F)$ exists. Next suppose that there exists an $\mathcal{L}$-valued function which is weakly analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0,\infty)$. Then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{an}(F)$ and is called the *analytic operator-valued Wiener integral of* F *associated with parameter* λ. Finally, for a nonzero real q suppose that there exists an operator $J_q^{an}(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{R}^r)$, $I_\lambda^{an}(F)\psi$ converges weakly to $J_q^{an}(F)\psi$, as λ approaches $-iq$ through $\mathbb{C}_+$. Then $J_q^{an}(F)$ is called the *analytic operator-valued Feynman integral of* F *with parameter* q.

Note that in Definition 3.3, the weak limit and the weak analyticity are based on the weak* topology on $L_\infty(\mathbb{R}^r)$ induced by its pre-dual $L_1(\mathbb{R}^r)$ [4, 15].

Lemma 3.4.

Let $\lambda > 0$ and $\vec{\xi} \in \mathbb{R}^r$. Suppose that φ^r is a continuous measure with respect to m_L^r. Then $P_{X_t^{\lambda,\vec{\xi}}} \ll (m_L^r)^2$ and

$$\frac{dP_{X_t^{\lambda,\vec{\xi}}}}{d(m_L^r)^2}(\vec{\eta}_1,\vec{\eta}_2) = \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\}\frac{d\varphi^r}{dm_L^r}(\lambda^{\frac{1}{2}}(\vec{\eta}_1-\vec{\xi}))$$

for $(m_L^r)^2$-a.e. $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$.

Proof. For a Borel subset B of $\mathbb{R}^{2r}$ we have by Lemma 2.1

$$P_{X_t^{\lambda,\vec{\xi}}}(B) = \left(\frac{1}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \chi_B\big(\lambda^{-\frac{1}{2}}\vec{u}_0+\vec{\xi},\lambda^{-\frac{1}{2}}\vec{u}_1+\vec{\xi}\big)\exp\left\{-\frac{\|\vec{u}_1-\vec{u}_0\|_{\mathbb{R}^r}^2}{2t}\right\}\frac{d\varphi^r}{dm_L^r}(\vec{u}_0)\,dm_L^r(\vec{u}_0)\,dm_L^r(\vec{u}_1)$$

since $\varphi^r \ll m_L^r$. Let $\vec{\eta}_1 = \lambda^{-\frac{1}{2}}\vec{u}_0 + \vec{\xi}$ and $\vec{\eta}_2 = \lambda^{-\frac{1}{2}}\vec{u}_1 + \vec{\xi}$. By the change of variable and Fubini theorems, we have,

$$P_{X_t^{\lambda,\vec{\xi}}}(B) = \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \chi_B(\vec{\eta}_1,\vec{\eta}_2)\exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\}\frac{d\varphi^r}{dm_L^r}(\lambda^{\frac{1}{2}}(\vec{\eta}_1-\vec{\xi}))\,dm_L^r(\vec{\eta}_1)\,dm_L^r(\vec{\eta}_2)$$

which completes the proof. □

Theorem 3.5.

Let the assumptions and notations be given as in Lemma 3.4. For $F : C^r[0,t] \to \mathbb{C}$, suppose that $E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ exists on $\mathbb{C}_+ \times \mathbb{R}^{3r}$, and for each bounded subset Ω of $\mathbb{C}_+$, there exists $M_\Omega \geq 0$ such that $|E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)| \leq M_\Omega$ for all $\lambda \in \Omega$ and all $(\vec{\xi},\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{3r}$. Further, suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying the following conditions:

(i) for each $\lambda > 0$ and $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta}) = \frac{d\varphi^r}{dm_L^r}(\lambda^{\frac{1}{2}}\vec{\eta})$,

(ii) for each $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta})$ is analytic on $\mathbb{C}_+$ as a function of λ,

(iii) for each bounded subset Ω of $\mathbb{C}_+$, Ψ is bounded on $\Omega \times \mathbb{R}^r$.

Then for $\lambda \in \mathbb{C}_+$, the analytic operator-valued Wiener integral $I_\lambda^{an}(F)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and is given by

$$(I_\lambda^{an}(F)\psi)(\vec{\xi}) = \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)\psi(\vec{\eta}_2)\Psi(\lambda,\vec{\eta}_1-\vec{\xi})\exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2) \tag{4}$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\vec{\xi} \in \mathbb{R}^r$. In addition, suppose that for a nonzero real q and $\vec{\xi} \in \mathbb{R}^r$, $E^{anf_q}[F|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ exists for $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$. Moreover, suppose that Ψ can be extended to $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ with the following two additional conditions:

(ii)′ for each $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta})$ is continuous at $\lambda = -iq$ as a function of λ,

(iii)′ there exists a function $\Phi_q \in L_1(\mathbb{R}^r)$ satisfying

$$|\Psi(\lambda,\vec{\eta})| \le |\Phi_q(\vec{\eta})| \text{ for all } (\lambda,\vec{\eta}) \in \Omega_\epsilon \times \mathbb{R}^r, \tag{5}$$

where $\Omega_\epsilon = \{\lambda \in \mathbb{C}_+ : |\lambda + iq| < \epsilon\}$ for some real $\epsilon > 0$.

Then the analytic operator-valued Feynman integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and it is given by (4) where λ and E^{anw_λ} are replaced by $-iq$ and E^{anf_q}, respectively.

Proof. Let $\lambda > 0$, $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} \in \mathbb{R}^r$. With the notation from Definition 3.3, we have, by Definition 3.1

$$(I_\lambda(F)\psi)(\vec{\xi}) = \int_{C^r} F^{\lambda,\vec{\xi}}(x)\psi_t^{\lambda,\vec{\xi}}(x)\,dw_\varphi^r(x) = \int_{C^r} E[F^{\lambda,\vec{\xi}}\psi_t^{\lambda,\vec{\xi}}|X_t^{\lambda,\vec{\xi}}](X_t^{\lambda,\vec{\xi}}(x))\,dw_\varphi^r(x).$$

For $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$, we have,

$$\psi\left(\lambda^{-\frac{1}{2}}\left(x(t)-x(0)-\frac{t}{t}(x(t)-x(0))\right) + \vec{\eta}_1 + \frac{t}{t}(\vec{\eta}_2-\vec{\eta}_1)\right) = \psi(\vec{\eta}_2)$$

which implies

$$(I_\lambda(F)\psi)(\vec{\xi}) = \int_{\mathbb{R}^{2r}} (K_\lambda(F))(\vec{\eta}_1,\vec{\eta}_2)\psi(\vec{\eta}_2)\,dP_{X_t^{\lambda,\vec{\xi}}}(\vec{\eta}_1,\vec{\eta}_2),$$

where $K_\lambda(F)$ is given by (3). Now suppose that Ψ satisfies (i), (ii) and (iii). By (i), Lemma 3.4, and the change of variable theorem we have

$$(I_\lambda(F)\psi)(\vec{\xi}) = \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} (K_\lambda(F))(\vec{\eta}_1,\vec{\eta}_2)\psi(\vec{\eta}_2)\Psi(\lambda,\vec{\eta}_1-\vec{\xi})\exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2).$$

Let Ω_λ be a bounded subset of $\mathbb{C}_+$ containing λ. Then for any $\vec{\xi} \in \mathbb{R}^r$ we have

$$\begin{aligned}|(I_\lambda(F)\psi)(\vec{\xi})| &\le M_{\Omega_\lambda}\|\Psi\|_{\Omega_\lambda,\infty}\left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} |\psi(\vec{\eta}_2)|\exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2)\\ &= \lambda^{\frac{r}{2}} M_{\Omega_\lambda}\|\Psi\|_{\Omega_\lambda,\infty}\|\psi\|_{L_1(\mathbb{R}^r)},\end{aligned} \tag{6}$$

where $\|\Psi\|_{\Omega_\lambda,\infty}$ denotes the essential supremum of Ψ on $\Omega_\lambda \times \mathbb{R}^r$ so that $I_\lambda(F)\psi \in L_\infty(\mathbb{R}^r)$ and $I_\lambda(F) \in \mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$. For $\psi \in L_1(\mathbb{R}^r)$ let $(Q_\lambda(F)\psi)(\vec{\xi})$ be the right hand side of (4) for $(\lambda,\vec{\xi}) \in \mathbb{C}_+ \times \mathbb{R}^r$ and let Ω be any bounded subset of $\mathbb{C}_+$. Using the same method we have for $\lambda \in \Omega$ and $\vec{\xi} \in \mathbb{R}^r$

$$|(Q_\lambda(F)\psi)(\vec{\xi})| \le M_\Omega\|\Psi\|_{\Omega,\infty}||\psi\|_{L_1(\mathbb{R}^r)}\left(\frac{|\lambda|^2}{\operatorname{Re}\lambda}\right)^{\frac{r}{2}} \tag{7}$$

so that $Q_\lambda(F)\psi \in L_\infty(\mathbb{R}^r)$ and $Q_\lambda(F) \in \mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ for $\lambda \in \mathbb{C}_+$. Furthermore, $Q_\lambda(F) = I_\lambda(F)$ on $(0,\infty)$, and $(Q_\lambda(F)\psi)(\vec{\xi})$ is an analytic function of $\lambda \in \mathbb{C}_+$ by (7) and Morera's theorem. Using (7), the ML-inequality, Fubini's theorem, and the dominated convergence theorem, we can easily show that $\int_{\mathbb{R}^r}(Q_\lambda(F)\psi)(\vec{\xi})\psi_1(\vec{\xi})\,dm_L^r(\vec{\xi})$ is analytic on $\mathbb{C}_+$ for any $\psi_1 \in L_1(\mathbb{R}^r)$ by Morera's theorem, which implies that $I_\lambda(F)$ is weakly analytic on $\mathbb{C}_+$. Hence $I_\lambda^{an}(F)$ exists and $I_\lambda^{an}(F) = Q_\lambda(F)$ for $\lambda \in \mathbb{C}_+$. Further, suppose that Ψ satisfies (ii)$'$ and (iii)$'$. For $\psi \in L_1(\mathbb{R}^r)$ let $(J_q^{an}(F)\psi)(\vec{\xi})$ be the right hand side of (4) for $\vec{\xi} \in \mathbb{R}^r$ where λ and E^{anw_λ} are replaced by $-iq$ and E^{anf_q}, respectively. Then we have by (ii)$'$ and (5)

$$|(J_q^{an}(F)\psi)(\vec{\xi})| \le M_{\Omega_\epsilon}\left(\frac{|q|}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} |\psi(\vec{\eta}_2)|\,|\Phi_q(\vec{\eta}_1-\vec{\xi})|\,d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2) < M_{\Omega_\epsilon}\|\psi\|_{L_1(\mathbb{R}^r)}\|\Phi_q\|_{L_1(\mathbb{R}^r)}\left(\frac{\epsilon+|q|}{\sqrt{2\pi t}}\right)^r \equiv C_{\Omega_\epsilon}$$

which implies $J_q^{an}(F)\psi \in L_\infty(\mathbb{R}^r)$ and $J_q^{an}(F) \in \mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$. Since both integrands in the representations of $(I_\lambda^{an}(F)\psi)(\vec{\xi})$ and $(J_q^{an}(F)\psi)(\vec{\xi})$ are bounded by $M_{\Omega_\epsilon}|\psi(\vec{\eta}_2)|\,|\Phi_q(\vec{\eta}_1-\vec{\xi})|$, which is independent of the complex numbers in $\Omega_\epsilon \cup \{-iq\}$ and integrable as a function of $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$, we have, by the dominated convergence theorem,

$$(I_\lambda^{an}(F)\psi)(\vec{\xi}) \to (J_q^{an}(F)\psi)(\vec{\xi})$$

as λ approaches $-iq$ through $\mathbb{C}_+$. By (5), $(I_\lambda^{an}(F)\psi)(\vec{\xi})$ is also bounded by C_{Ω_ϵ} for any $\lambda \in \Omega_\epsilon$ and $\vec{\xi} \in \mathbb{R}^r$ so for $\psi_1 \in L_1(\mathbb{R}^r)$, $(I_\lambda^{an}(F)\psi)(\vec{\xi})\psi_1(\vec{\xi})$ and $(J_q^{an}(F)\psi)(\vec{\xi})\psi_1(\vec{\xi})$ are bounded by $C_{\Omega_\epsilon}|\psi_1(\vec{\xi})|$. Now, we have by the dominated convergence theorem

$$\int_{\mathbb{R}^r}(I_\lambda^{an}(F)\psi)(\vec{\xi})\psi_1(\vec{\xi})\,dm_L^r(\vec{\xi}) \to \int_{\mathbb{R}^r}(J_q^{an}(F)\psi)(\vec{\xi})\psi_1(\vec{\xi})\,dm_L^r(\vec{\xi})$$

as $\lambda \to -iq$ through $\mathbb{C}_+$, that is, $I_\lambda^{an}(F)\psi$ converges weakly to $J_q^{an}(F)\psi$ as λ approaches $-iq$ through $\mathbb{C}_+$, which completes the proof. □

Next we give an example of φ^r which is not normally distributed.

Example 3.6.

For $\vec{\eta} = (\eta_1,\dots,\eta_r) \in \mathbb{R}^r$ and $\lambda \in \mathbb{C}_+^\sim$, let

$$\frac{d\varphi^r}{dm_L^r}(\vec{\eta}) = \left(\frac{1}{\pi}\right)^r \prod_{j=1}^r \frac{1}{1+\eta_j^2} \quad \text{and} \quad \Psi(\lambda,\vec{\eta}) = \left(\frac{1}{\pi}\right)^r \prod_{j=1}^r \frac{1}{1+\lambda\eta_j^2}.$$

Then φ^r is a probability measure on the Borel class of $\mathbb{R}^r$ and the condition (i) of Theorem 3.5 is satisfied. Further, for $\vec{\eta} \in \mathbb{R}^r$, $\Psi(\lambda,\vec{\eta})$ is analytic on $\mathbb{C}_+$ and continuous on $\mathbb{C}_+^\sim$ because $1+\lambda\eta_j^2 \ne 0$ for $\lambda \in \mathbb{C}_+^\sim$, satisfying conditions (ii) and (ii)$'$ of Theorem 3.5. Now we have for $(\lambda,\eta_j) \in \mathbb{C}_+^\sim \times \mathbb{R}$

$$\left|\frac{1}{1+\lambda\eta_j^2}\right|^2 = \frac{1}{(1+\eta_j^2\operatorname{Re}\lambda)^2 + (\eta_j^2\operatorname{Im}\lambda)^2} \le \min\left\{1, \frac{1}{1+\eta_j^4(\operatorname{Im}\lambda)^2}\right\}$$

which satisfies conditions (iii) and (iii)$'$ of Theorem 3.5. Let F be a function satisfying the assumptions of Theorem 3.5. Applying Theorem 3.5 to F, for any nonzero real q, we conclude that the analytic operator-valued Feynman integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and it is given by

$$(J_q^{an}(F)\psi)(\vec{\xi}) = \left(\frac{q}{\pi i\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} E^{anf_q}[F|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)\psi(\vec{\eta}_2)\left[\prod_{j=1}^r \frac{1}{1-iq(\eta_{1,j}-\xi_j)^2}\right]\exp\left\{\frac{iq\|\vec{\eta}_2-\vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2)$$

for $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} = (\xi_1,\dots,\xi_r) \in \mathbb{R}^r$, where $\vec{\eta}_1 = (\eta_{1,1},\dots,\eta_{1,r})$. Note that the probability distribution φ having the above density for $r=1$ is known as the Cauchy distribution [1, p. 211].

Using the same method as used in the proof of Theorem 3.5 but without (6) and (7), we can easily prove the following theorem.

Theorem 3.7.

If the conditions (iii) *and* (iii)$'$ *in Theorem 3.5 are replaced by the condition: for each bounded subset* Ω *of* $\mathbb{C}_+$ *there exists a function* $\Phi_\Omega \in L_1(\mathbb{R}^r)$ *satisfying*

$$|\Psi(\lambda,\vec{\eta})| \le |\Phi_\Omega(\vec{\eta})| \quad \text{for all} \quad (\lambda,\vec{\eta}) \in \Omega \times \mathbb{R}^r, \tag{8}$$

then the statement of Theorem 3.5 holds true.

4. The operator-valued function space integrals via the conditional analytic Feynman integrals

We begin this section with introducing the Banach algebra $\mathcal{S}_{w_\varphi^r}$ corresponding to Cameron and Storvick's [5] Banach algebra $\mathcal{S}$.

Now, let $\mathcal{M}(L_2^r[0,t])$ be the space of the complex Borel measures on $L_2^r \equiv L_2^r[0,t]$ and let $\mathcal{S}_{w_\varphi^r}$ be the space of the functions of the form:

$$F(x) = \int_{L_2^r[0,t]} \exp\left\{ i \sum_{j=1}^{r} (v_j, x_j) \right\} d\sigma(\vec{v}), \quad x = (x_1,\dots,x_r) \in C^r[0,t], \tag{9}$$

where $\sigma \in \mathcal{M}(L_2^r[0,t])$, $\vec{v} = (v_1,\dots,v_r) \in L_2^r[0,t]$. Using the same method as in [5], it can be shown that $\mathcal{S}_{w_\varphi^r}$ is a Banach algebra.

Using the well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\}\, dm_L(u) = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a}\right\} \tag{10}$$

for $a \in \mathbb{C}_+$ and any real b, we have the following theorem by Corollary 3.3 of [10].

Theorem 4.1.

Let $X_t : C^r[0,t] \to \mathbb{R}^{2r}$ *be given by* $X_t(x) = (x(0), x(t))$ *and let* $F \in \mathcal{S}_{w_\varphi^r}$ *be given by* (9). *Then we have for* $\lambda \in \mathbb{C}_+$ *and* $\vec{\xi} \in \mathbb{R}^r$

$$E^{anw_\lambda}[F|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2) = \int_{L_2^r[0,t]} \exp\left\{ -\frac{1}{2\lambda t}\left[t\|\vec{v}\|_{L_2^r}^2 - \|\vec{V}_t\|_{\mathbb{R}^r}^2\right] + \frac{i}{t}\langle \vec{\eta}_2 - \vec{\eta}_1, \vec{V}_t\rangle \right\} d\sigma(\vec{v}) \tag{11}$$

for $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$, *where* $\vec{V}_t = \left(\int_0^t v_1(s)ds, \dots, \int_0^t v_r(s)ds\right)$. *Moreover, for any nonzero real* q, $E^{anf_q}[F|X_t]$ *is given by the right hand side of* (11) *where* λ *is replaced by* $-iq$.

For $F \in \mathcal{S}_{w_\varphi^r}$ given by (9), we know from Theorem 4.1 that $E^{anw}[F|X_t]$ is bounded by $\|\sigma\|$, where $\|\sigma\|$ denotes the total variation of σ. Combining Lemma 3.4, Theorems 3.5, 3.7 and 4.1, we have the following theorem.

Theorem 4.2.

Let the assumptions be given as in Lemma 3.4 and let $F \in \mathcal{S}_{w_\varphi^r}$. *Suppose that for a nonzero real* q *there exists a function* Ψ *on* $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ *satisfying the conditions* (i), (ii), (ii)$'$ *of Theorem 3.5, and either* (iii) *and* (iii)$'$ *of Theorem 3.5 or* (8) *of Theorem 3.7. Then the analytic operator-valued Feynman integral* $J_q^{an}(F)$ *exists as an element of* $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ *and it is given by* (4) *where* λ *and* E^{anw_λ} *are replaced by* $-iq$ *and* E^{anf_q}, *respectively* ($E^{anf_q}[F|X_t]$ *is given by Theorem 4.1*).

Theorem 4.3.

Let the assumptions and notations be given as in Theorem 4.1. Furthermore, suppose that φ^r is normally distributed with the mean vector $\vec{0}$ and the variance-covariance matrix $\alpha^2 I_r$, where I_r is the r-dimensional identity matrix. Then for $\lambda \in \mathbb{C}_+$ the analytic operator-valued Wiener integral $I_\lambda^{an}(F)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and it is given by

$$(I_\lambda^{an}(F)\psi)(\vec{\xi}) = \left[\frac{\lambda}{2\pi(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}\big[t\|\vec{v}\|^2_{L_2^r} - \|\vec{V}_t\|^2_{\mathbb{R}^r}\big]\right\} \int_{\mathbb{R}^r} \psi(\vec{\eta}) H\left(\lambda, \vec{\xi}, \vec{\eta}, \frac{1}{t}\vec{V}\right) dm_L^r(\vec{\eta})\, d\sigma(\vec{v}) \tag{12}$$

for $\psi \in L_1(\mathbb{R}^r)$ and m_L^r-a.e. $\vec{\xi} \in \mathbb{R}^r$, where

$$H(\lambda, \vec{\xi}, \vec{\eta}, \vec{\zeta}) = \exp\left\{-\frac{\lambda}{2\alpha^2}\|\vec{\xi}-\vec{\eta}\|^2_{\mathbb{R}^r} - \frac{t\alpha^2}{2\lambda(t+\alpha^2)}\left\|\vec{\zeta} + \frac{\lambda i}{\alpha^2}(\vec{\xi}-\vec{\eta})\right\|^2_{\mathbb{R}^r}\right\} \tag{13}$$

for $\lambda \in \mathbb{C}_+^\sim$ and $\vec{\zeta} \in \mathbb{R}^r$. Furthermore, for any nonzero real q the analytic operator-valued Feynman integral $J_q^{an}(F)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and it is given by (12) where λ is replaced by $-iq$.

Proof. By the assumptions, the hypothesis of Lemma 3.4 is satisfied and we have $\frac{d\varphi^r}{dm_L^r}(\vec{\eta}) = \left(\frac{1}{\sqrt{2\pi\alpha}}\right)^r \exp\left\{-\frac{\|\vec{\eta}\|^2_{\mathbb{R}^r}}{2\alpha^2}\right\}$ for m_L^r-a.e. $\vec{\eta} \in \mathbb{R}^r$. We also know from Theorem 4.1 that $E^{anw}[F|X_t]$ is bounded. Let $\Psi(\lambda, \vec{\eta}) = \left(\frac{1}{\sqrt{2\pi\alpha}}\right)^r \exp\left\{-\frac{\lambda\|\vec{\eta}\|^2_{\mathbb{R}^r}}{2\alpha^2}\right\}$ for $(\lambda, \vec{\eta}) \in \mathbb{C}_+ \times \mathbb{R}^r$. Then Ψ satisfies the conditions (i), (ii) and (iii) of Theorem 3.5, so the existence of the analytic operator-valued Wiener integral $I_\lambda^{an}(F)$ follows. Now, for $\lambda \in \mathbb{C}_+$, $\vec{\xi} \in \mathbb{R}^r$ and $\psi \in L_1(\mathbb{R}^r)$ we have by (4), (10), Theorem 4.1, and Fubini's theorem,

$$\begin{aligned}
(I_\lambda^{an}(F)\psi)(\vec{\xi}) &= \left(\frac{\lambda}{2\pi\alpha\sqrt{t}}\right)^r \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}\big[t\|\vec{v}\|^2_{L_2^r} - \|\vec{V}_t\|^2_{\mathbb{R}^r}\big]\right\} \int_{\mathbb{R}^r} \psi(\vec{\eta}_2) \\
&\quad \times \int_{\mathbb{R}^r} \exp\left\{\frac{i}{t}\langle \vec{\eta}_2 - \vec{\eta}_1, \vec{V}_t\rangle - \frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|^2_{\mathbb{R}^r}}{2t} - \frac{\lambda\|\vec{\eta}_1-\vec{\xi}\|^2_{\mathbb{R}^r}}{2\alpha^2}\right\} dm_L^r(\vec{\eta}_1)\, dm_L^r(\vec{\eta}_2)\, d\sigma(\vec{v}) \\
&= \left(\frac{\lambda}{2\pi\alpha\sqrt{t}}\right)^r \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}\big[t\|\vec{v}\|^2_{L_2^r} - \|\vec{V}_t\|^2_{\mathbb{R}^r}\big]\right\} \int_{\mathbb{R}^r} \psi(\vec{\eta}_2) \exp\left\{-\frac{\lambda}{2\alpha^2}\|\vec{\xi}-\vec{\eta}_2\|^2_{\mathbb{R}^r}\right\} \\
&\quad \times \int_{\mathbb{R}^r} \exp\left\{-\frac{\lambda}{2}\frac{t+\alpha^2}{t\alpha^2}\|\vec{\eta}_2-\vec{\eta}_1\|^2_{\mathbb{R}^r} + i\left\langle \vec{\eta}_2-\vec{\eta}_1, \frac{1}{t}\vec{V}_t + \frac{\lambda i}{\alpha^2}(\vec{\xi}-\vec{\eta}_2)\right\rangle\right\} dm_L^r(\vec{\eta}_1)\, dm_L^r(\vec{\eta}_2)\, d\sigma(\vec{v}) \\
&= \left[\frac{\lambda}{2\pi(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{L_2^r[0,t]} \exp\left\{-\frac{1}{2\lambda t}\big[t\|\vec{v}\|^2_{L_2^r} - \|\vec{V}_t\|^2_{\mathbb{R}^r}\big]\right\} \int_{\mathbb{R}^r} \psi(\vec{\eta}_2)\, H\left(\lambda, \vec{\xi}, \vec{\eta}_2, \frac{1}{t}\vec{V}\right) dm_L^r(\vec{\eta}_2)\, d\sigma(\vec{v})
\end{aligned}$$

which proves the first part of the theorem. To complete the proof, we must prove the existence of $J_q^{an}(F)$. For $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} \in \mathbb{R}^r$, let $(J_q^{an}(F)\psi)(\vec{\xi})$ be the right hand side of (12) where λ is replaced by $-iq$. Since

$$\left|H\left(\lambda, \vec{\xi}, \vec{\eta}, \frac{1}{t}\vec{V}\right)\right| = \left|\exp\left\{-\frac{\lambda}{2(t+\alpha^2)}\|\vec{\xi}-\vec{\eta}\|^2_{\mathbb{R}^r} - \frac{\alpha^2}{2\lambda t(t+\alpha^2)}\|\vec{V}_t\|^2_{\mathbb{R}^r} - \frac{i}{t+\alpha^2}\langle \vec{V}_t, \vec{\xi}-\vec{\eta}\rangle\right\}\right| \le 1,$$

$I_\lambda^{an}(F)\psi$ converges weakly to $J_q^{an}(F)\psi$ as λ approaches $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This completes the proof. □

5. The series expansions for the function space integrals

Let $\mathcal{M}(\mathbb{R}^r)$ be the class of all complex Borel measures on $\mathbb{R}^r$ and $\mathcal{G}$ be the set of all $\mathbb{C}$-valued functions θ on $[0,\infty)\times\mathbb{R}^r$ which have the form

$$\theta(s, \vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{w}\rangle\}\, d\sigma_s(\vec{w}), \tag{14}$$

where $\{\sigma_s : s \in [0,\infty)\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the following conditions:

(1) for each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,t]$,

(2) $\|\sigma_s\| \in L_1[0,t]$.

We have the following theorem.

Theorem 5.1.
For a positive integer n, let

$$F_n(x) = \left[\int_0^t \theta(s, x(s))\, ds\right]^n \tag{15}$$

for $x \in C^r[0,t]$, where $\theta \in \mathcal{G}$ is given by (14). Then $F_n \in \mathcal{S}_{w_\varphi^r}$ which implies the existence of $E^{anf_q}[F_n|X_t]$ for any nonzero real q by Theorem 4.1, and for $\vec{\xi} \in \mathbb{R}^r$ it is given by

$$E^{anf_q}[F_n|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2) = n! \int_{\Delta_n} \int_{\mathbb{R}^{nr}} A(\vec{s}, \vec{v}, \vec{\eta}_1, \vec{\eta}_2)\, B(-iq, \vec{s}, \vec{v})\, d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s} \tag{16}$$

for $(\vec{\eta}_1, \vec{\eta}_2) \in \mathbb{R}^{2r}$, where $s_0 = 0$, $s_{n+1} = t$, $\vec{s} = (s_1, \dots, s_n)$, $\vec{v} = (\vec{v}_1, \dots, \vec{v}_n) \in \mathbb{R}^{nr}$, $\Delta_n = \{\vec{s} : 0 < s_1 < \dots < s_n < t\}$,

$$A(\vec{s}, \vec{v}, \vec{\eta}_1, \vec{\eta}_2) = \exp\left\{\frac{i}{t}\sum_{j=1}^n \left\langle (t - s_j)\vec{\eta}_1 + s_j \vec{\eta}_2, \vec{v}_j \right\rangle\right\}$$

and for $\lambda \in \mathbb{C}_+^\sim$,

$$B(\lambda, \vec{s}, \vec{v}) = \exp\left\{-\frac{1}{2\lambda t^2}\sum_{j=1}^{n+1}(s_j - s_{j-1})\left\|\sum_{l=j}^n (t - s_l)\vec{v}_l - \sum_{l=1}^{j-1} s_l \vec{v}_l\right\|_{\mathbb{R}^r}^2\right\}. \tag{17}$$

Proof. It is not difficult to show, using the same process as in [5], that $F_n \in \mathcal{S}_{w_\varphi^r}$. Using the simplex method [14], we have

$$F_n(x) = n! \int_{\Delta_n} \int_{\mathbb{R}^{nr}} \exp\left\{i \sum_{j=1}^n \langle x(s_j), \vec{v}_j \rangle\right\} d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s}.$$

Let $\Xi_n(\vec{s}) = n!\left[\prod_{j=1}^{n+1} \frac{1}{2\pi(s_j - s_{j-1})}\right]^{r/2}$ and let $(K_\lambda(F_n))(\vec{\eta}_1, \vec{\eta}_2)$ be given by (3) for $\lambda > 0$ and $(\vec{\eta}_1, \vec{\eta}_2) \in \mathbb{R}^{2r}$. Then we have, by Lemma 2.1,

$$\begin{aligned}
(K_\lambda(F_n))(\vec{\eta}_1, \vec{\eta}_2) &= n! \int_{C^r} \int_{\Delta_n} \int_{\mathbb{R}^{nr}} A(\vec{s}, \vec{v}, \vec{\eta}_1, \vec{\eta}_2) \\
&\quad \times \exp\left\{i\lambda^{-\frac{1}{2}} \sum_{j=1}^n \left\langle x(s_j) - x(0) - \frac{s_j}{t}(x(t) - x(0)), \vec{v}_j \right\rangle\right\} d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s}\, dw_\varphi^r(x) \\
&= \Xi_n(\vec{s}) \int_{\Delta_n} \int_{\mathbb{R}^{nr}} A(\vec{s}, \vec{v}, \vec{\eta}_1, \vec{\eta}_2) \\
&\quad \times \int_{\mathbb{R}^{(n+1)r}} \int_{\mathbb{R}^r} \exp\left\{i\lambda^{-\frac{1}{2}} \sum_{j=1}^n \left\langle \vec{u}_j - \vec{u}_0 - \frac{s_j}{t}(\vec{u}_{n+1} - \vec{u}_0), \vec{v}_j \right\rangle - \frac{1}{2}\sum_{j=1}^{n+1} \frac{\|\vec{u}_j - \vec{u}_{j-1}\|_{\mathbb{R}^r}^2}{s_j - s_{j-1}}\right\} \\
&\qquad d\varphi^r(\vec{u}_0)\, dm_L^{(n+1)r}(\vec{u}_1, \dots, \vec{u}_n, \vec{u}_{n+1})\, d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s}.
\end{aligned}$$

For $j=1,\dots,n+1$, let $\vec{\zeta}_j=\vec{u}_j-\vec{u}_{j-1}$. Then, by the change of variable theorem,

$$\begin{aligned}(K_\lambda(F_n))(\vec{\eta}_1,\vec{\eta}_2)&=\Xi_n(\vec{s})\int_{\Delta_n}\int_{\mathbb{R}^{nr}}A(\vec{s},\vec{v},\vec{\eta}_1,\vec{\eta}_2)\int_{\mathbb{R}^{(n+1)r}}\exp\left\{i\lambda^{-\frac{1}{2}}\sum_{j=1}^{n}\left\langle\sum_{l=1}^{j}\vec{\zeta}_l-\frac{s_j}{t}\sum_{l=1}^{n+1}\vec{\zeta}_l,\vec{v}_j\right\rangle-\frac{1}{2}\sum_{j=1}^{n+1}\frac{\|\vec{\zeta}_j\|^2_{\mathbb{R}^r}}{s_j-s_{j-1}}\right\}\\&\qquad dm_L^{(n+1)r}(\vec{\zeta}_1,\dots,\vec{\zeta}_n,\vec{\zeta}_{n+1})\,d\left(\prod_{j=1}^{n}\sigma_{s_j}\right)(\vec{v})\,d\vec{s}\\&=\Xi_n(\vec{s})\int_{\Delta_n}\int_{\mathbb{R}^{nr}}A(\vec{s},\vec{v},\vec{\eta}_1,\vec{\eta}_2)\int_{\mathbb{R}^{(n+1)r}}\exp\left\{\frac{i}{\sqrt{\lambda}t}\sum_{j=1}^{n+1}\left\langle\vec{\zeta}_j,\sum_{l=j}^{n}(t-s_l)\vec{v}_l-\sum_{l=1}^{j-1}s_l\vec{v}_l\right\rangle-\frac{1}{2}\sum_{j=1}^{n+1}\frac{\|\vec{\zeta}_j\|^2_{\mathbb{R}^r}}{s_j-s_{j-1}}\right\}\\&\qquad dm_L^{(n+1)r}(\vec{\zeta}_1,\dots,\vec{\zeta}_n,\vec{\zeta}_{n+1})\,d\left(\prod_{j=1}^{n}\sigma_{s_j}\right)(\vec{v})\,d\vec{s}\\&\overset{**}{=}n!\int_{\Delta_n}\int_{\mathbb{R}^{nr}}A(\vec{s},\vec{v},\vec{\eta}_1,\vec{\eta}_2)\,B(\lambda,\vec{s},\vec{v})\,d\left(\prod_{j=1}^{n}\sigma_{s_j}\right)(\vec{v})\,d\vec{s},\end{aligned}$$

where the last equality follows from (10). The theorem now follows from Morera's theorem and the dominated convergence theorem. □

Theorem 5.2.

Let

$$F_t(x)=\exp\left\{\int_0^t\theta(s,x(s))\,ds\right\}$$

for $x\in C^r[0,t]$, where $\theta\in\mathcal{G}$ is given by (14). Then $F_t\in\mathcal{S}_{w_\varphi^r}$ which implies, by Theorem 4.1, the existence of $E^{anf_q}[F_t|X_t]$ for any nonzero real q. For $\vec{\xi}\in\mathbb{R}^r$ it is given by

$$E^{anf_q}[F_t|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)=1+\sum_{n=1}^{\infty}\frac{1}{n!}E^{anf_q}[F_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$$

for $(\vec{\eta}_1,\vec{\eta}_2)\in\mathbb{R}^{2r}$, where $E^{anf_q}[F_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ is given by (16) in Theorem 5.1.

Proof. It is not difficult to show that $F_t\in\mathcal{S}_{w_\varphi^r}$ using the same process as used in [5]. By the Maclaurin series of the exponential function we have,

$$F_t(x)=1+\sum_{n=1}^{\infty}\frac{1}{n!}F_n(x)\quad\text{for}\quad x\in C^r[0,t],\tag{18}$$

where F_n is given by (15). Further, we have for any $x\in C^r[0,t]$

$$|F_t(x)|\le 1+\sum_{n=1}^{\infty}\frac{1}{n!}|F_n(x)|\le 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\int_0^t\|\sigma_s\|\,ds\right]^n=\exp\left\{\int_0^t\|\sigma_s\|\,ds\right\}$$

so that the convergence of (18) is uniform. Now the theorem follows. □

Let $\phi(\vec{u})=\int_{\mathbb{R}^r}\exp\{i\langle\vec{u},\vec{w}\rangle\}d\nu(\vec{w})$ for $\nu\in\mathcal{M}(\mathbb{R}^r)$. Then for $\lambda>0$, $x\in C^r[0,t]$ and $(\vec{\eta}_1,\vec{\eta}_2)\in\mathbb{R}^{2r}$, we have

$$\phi\left(\lambda^{-\frac{1}{2}}\left(x(t)-x(0)-\frac{t}{t}(x(t)-x(0))\right)+\vec{\eta}_1+\frac{t}{t}(\vec{\eta}_2-\vec{\eta}_1)\right)=\phi(\vec{\eta}_2).\tag{19}$$

Furthermore, it is not difficult to show that $G_n\in\mathcal{S}_{w_\varphi^r}$ using the same process as in [5], where $G_n(x)=F_n(x)\phi(x(t))$ for $x\in C^r[0,t]$ and F_n is given by (15), so that we have the following theorem by Theorem 5.2.

Theorem 5.3.

Let the assumptions and notations be given as in Theorem 5.2. For $x \in C^r[0,t]$ let

$$G_n(x) = F_n(x)\phi(x(t)) \quad \text{and} \quad G_t(x) = F_t(x)\phi(x(t)),$$

where $\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{w}\rangle\}\, d\nu(\vec{w})$ for $\nu \in \mathcal{M}(\mathbb{R}^r)$. Then $G_n, G_t \in \mathcal{S}_{w_\varphi^r}$ which implies, by Theorem 4.1, the existence of $E^{anf_q}[G_n|X_t]$ and $E^{anf_q}[G_t|X_t]$ for any nonzero real q. For $\vec{\xi} \in \mathbb{R}^r$, $E^{anf_q}[G_n|X_t]$ and $E^{anf_q}[G_t|X_t]$ are given by

$$E^{anf_q}[G_n|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2) = \phi(\vec{\eta}_2) E^{anf_q}[F_n|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)$$

and

$$E^{anf_q}[G_t|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2) = \phi(\vec{\eta}_2) E^{anf_q}[F_t|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2) = \phi(\vec{\eta}_2) + \sum_{n=1}^{\infty} \frac{1}{n!} E^{anf_q}[G_n|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)$$

for $(\vec{\eta}_1, \vec{\eta}_2) \in \mathbb{R}^{2r}$, where $E^{anf_q}[F_n|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)$ and $E^{anf_q}[F_t|X_t](\vec{\xi})(\vec{\eta}_1, \vec{\eta}_2)$ are given in Theorems 5.1 and 5.2, respectively.

Combining Lemma 3.4, Theorems 3.5, 3.7, 5.1, 5.2, and 5.3, we have the following theorem.

Theorem 5.4.

Let the assumptions and notations be as given in Lemma 3.4 and Theorems 5.1, 5.2 and 5.3.

(1) *Suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying the conditions* (i) *and* (ii) *of Theorem 3.5, and either* (iii) *of Theorem 3.5 or* (8) *of Theorem 3.7. Then for $\lambda \in \mathbb{C}_+$ the analytic operator-valued Wiener integrals $I_\lambda^{an}(F_n)$ and $I_\lambda^{an}(G_n)$ exist as elements of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and they are given by* (4) *where F is replaced by F_n and G_n, respectively (for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F_n|X_t]$ is given by the right hand side of the equality $\overset{**}{=}$ in the proof of Theorem 5.1). Furthermore, as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$, the analytic operator-valued Wiener integral $I_\lambda^{an}(F_t)$ is given by*

$$(I_\lambda^{an}(F_t)\psi)(\vec{\xi}) = (I_\lambda^{an}(1)\psi)(\vec{\xi}) + \sum_{n=1}^{\infty} \frac{1}{n!} (I_\lambda^{an}(F_n)\psi)(\vec{\xi}) \tag{20}$$

for $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} \in \mathbb{R}^r$, where

$$(I_\lambda^{an}(1)\psi)(\vec{\xi}) = \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} \psi(\vec{\eta}_2)\Psi(\lambda, \vec{\eta}_1 - \vec{\xi}) \exp\left\{-\frac{\lambda\|\vec{\eta}_2 - \vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1, \vec{\eta}_2).$$

The analytic operator-valued Wiener integral $I_\lambda^{an}(G_t)$ is also given by

$$\begin{aligned}(I_\lambda^{an}(G_t)\psi)(\vec{\xi}) = &\left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} \phi(\vec{\eta}_2)\psi(\vec{\eta}_2)\Psi(\lambda, \vec{\eta}_1 - \vec{\xi}) \exp\left\{-\frac{\lambda\|\vec{\eta}_2 - \vec{\eta}_1\|_{\mathbb{R}^r}^2}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1, \vec{\eta}_2) \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} (I_\lambda^{an}(G_n)\psi)(\vec{\xi}).\end{aligned} \tag{21}$$

(2) *Suppose that for a nonzero real q there exists a function Ψ on $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ satisfying the conditions* (i) *and* (ii) *of Theorem 3.5, and either* (iii) *and* (iii)′ *of Theorem 3.5 or* (8) *of Theorem 3.7. Then the analytic operator-valued Feynman integrals $J_q^{an}(F_n)$, $J_q^{an}(G_n)$, $J_q^{an}(F_t)$ and $J_q^{an}(G_t)$ exist as elements of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and each of them is given by the corresponding expression of the analytic operator-valued Wiener integral in* (1) *where λ is replaced by $-iq$, E^{anw_λ} is replaced by E^{anf_q} and I_λ^{an} is replaced by J_q^{an}.*

Theorem 5.5.

Suppose that φ^r is normally distributed with the mean vector $\vec{0}$ and the variance-covariance matrix $\alpha^2 I_r$. Let the assumptions and notations be as given in Theorems 5.1 and 5.3. Then for any nonzero real q, the analytic operator-valued Feynman integral $J_q^{an}(G_t)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and for $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} \in \mathbb{R}^r$, it is given by

$$(J_q^{an}(G_t)\psi)(\vec{\xi}) = \left[\frac{q}{2\pi i(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \psi(\vec{\eta})\phi(\vec{\eta}) \exp\left\{\frac{iq}{2(t+\alpha^2)}\|\vec{\eta}-\vec{\xi}\|^2_{\mathbb{R}^r}\right\} dm_L^r(\vec{\eta}) + \sum_{n=1}^{\infty} \frac{1}{n!}(J_q^{an}(G_n)\psi)(\vec{\xi}), \tag{22}$$

where

$$\begin{aligned}(J_q^{an}(G_n)\psi)(\vec{\xi}) = n!\left[\frac{q}{2\pi i(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{\Delta_n}\int_{\mathbb{R}^{nr}} B(-iq,\vec{s},\vec{v}) \int_{\mathbb{R}^r} \psi(\vec{\eta})\phi(\vec{\eta}) \exp\left\{i\sum_{j=1}^n \langle\vec{\eta},\vec{v}_j\rangle\right\} \\ \times H\left(-iq,\vec{\xi},\vec{\eta},\frac{1}{t}\sum_{j=1}^n (s_j-t)\vec{v}_j\right) dm_L^r(\vec{\eta})\, d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s}\,;\end{aligned}$$

$\vec{v} = (\vec{v}_1,\ldots,\vec{v}_n) \in \mathbb{R}^{nr}$, and H and B are given by (13) and (17), respectively.

Proof. The existence of $J_q^{an}(G_t)$ is guaranteed by Theorems 4.3 and 5.3. Let Ψ be given as in the proof of Theorem 4.3. Then, by (1) of Theorem 5.4, $(I_\lambda^{an}(G_t)\psi)(\vec{\xi})$ is given by (21) for $\lambda \in \mathbb{C}_+$, $\vec{\xi} \in \mathbb{R}^r$ and $\psi \in L_1(\mathbb{R}^r)$. Furthermore, we have by (4), (10), Theorems 5.1, 5.3, and Fubini's theorem

$$\begin{aligned}(I_\lambda^{an}(G_n)\psi)(\vec{\xi}) &= \left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} E^{anw_\lambda}[F_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)\psi(\vec{\eta}_2)\phi(\vec{\eta}_2)\Psi(\lambda,\vec{\eta}_1-\vec{\xi}) \exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|^2_{\mathbb{R}^r}}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2) \\ &= n!\left(\frac{\lambda}{2\pi\alpha\sqrt{t}}\right)^r \int_{\Delta_n}\int_{\mathbb{R}^{nr}} B(\lambda,\vec{s},\vec{v}) \int_{\mathbb{R}^r} \psi(\vec{\eta}_2)\phi(\vec{\eta}_2) \exp\left\{i\sum_{j=1}^n \langle\vec{\eta}_2,\vec{v}_j\rangle\right\} \\ &\quad \times \int_{\mathbb{R}^r} \exp\left\{\frac{i}{t}\left\langle \vec{\eta}_2-\vec{\eta}_1, \sum_{j=1}^n (s_j-t)\vec{v}_j\right\rangle - \frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|^2_{\mathbb{R}^r}}{2t} - \frac{\lambda\|\vec{\eta}_1-\vec{\xi}\|^2_{\mathbb{R}^r}}{2\alpha^2}\right\} dm_L^r(\vec{\eta}_1)\, dm_L^r(\vec{\eta}_2)\, d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s}.\end{aligned}$$

Using the same method as used in the proof of Theorem 4.3, we get

$$\begin{aligned}(I_\lambda^{an}(G_n)\psi)(\vec{\xi}) = n!\left[\frac{\lambda}{2\pi(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{\Delta_n}\int_{\mathbb{R}^{nr}} B(\lambda,\vec{s},\vec{v}) \int_{\mathbb{R}^r} \psi(\vec{\eta}_2)\phi(\vec{\eta}_2) \exp\left\{i\sum_{j=1}^n \langle\vec{\eta}_2,\vec{v}_j\rangle\right\} \\ \times H\left(\lambda,\vec{\xi},\vec{\eta}_2,\frac{1}{t}\sum_{j=1}^n (s_j-t)\vec{v}_j\right) dm_L^r(\vec{\eta}_2)\, d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{v})\, d\vec{s}.\end{aligned}$$

Further, a simple calculation shows that

$$\begin{aligned}&\left(\frac{\lambda}{\sqrt{2\pi t}}\right)^r \int_{\mathbb{R}^{2r}} \phi(\vec{\eta}_2)\psi(\vec{\eta}_2)\Psi(\lambda,\vec{\eta}_1-\vec{\xi}) \exp\left\{-\frac{\lambda\|\vec{\eta}_2-\vec{\eta}_1\|^2_{\mathbb{R}^r}}{2t}\right\} d(m_L^r)^2(\vec{\eta}_1,\vec{\eta}_2) \\ &\qquad = \left[\frac{\lambda}{2\pi(t+\alpha^2)}\right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \phi(\vec{\eta}_2)\psi(\vec{\eta}_2) \exp\left\{-\frac{\lambda}{2(t+\alpha^2)}\|\vec{\eta}_2-\vec{\xi}\|^2_{\mathbb{R}^r}\right\} dm_L^r(\vec{\eta}_2).\end{aligned}$$

The theorem now follows by use of the dominated convergence theorem. □

6. The stability theories

In the previous sections, we established the analytic operator-valued Feynman integrals for the functions which belong to $\mathcal{S}_{w_\varphi^r}$, via the conditional analytic Feynman w_φ^r-integrals over the generalized Wiener paths. In this section, which is the main section of the present paper, we do the same things for functionals that need not belong to the Banach algebra $\mathcal{S}_{w_\varphi^r}$.

Let η be a complex Borel measure on $[0,t]$. Then $\eta=\mu+\nu$ can be decomposed uniquely into the sum of a continuous measure μ and a discrete measure ν. Further, let δ_{p_j} denote the Dirac measure with total mass 1 concentrated at p_j.

Let $\mathcal{G}^*$ be the set of all $\mathbb{C}$-valued functions θ on $[0,\infty)\times\mathbb{R}^r$ which are of the form

$$\theta(s,\vec{u})=\int_{\mathbb{R}^r}\exp\{i\langle\vec{u},\vec{w}\rangle\}\,d\sigma_s(\vec{w}), \tag{23}$$

where $\{\sigma_s : s\in[0,\infty)\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the conditions:

(1) for each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,t]$,

(2) $\|\sigma_s\|\in L_1\big([0,t],\mathcal{B}([0,t]),|\eta|\big)$.

Theorem 6.1.
Let m, n be two positive integers and let $\eta=\mu+\sum_{j=1}^m w_j\delta_{p_j}$, where $0<p_1<\ldots<p_m<t$ and the w_j's are in $\mathbb{C}$. Let $\theta\in\mathcal{G}^*$ be given by (23) and $\Lambda_n(x)=\big[\int_0^t\theta(s,x(s))\,d\eta(s)\big]^n$ for $x\in C^r[0,t]$. Then for $\lambda\in\mathbb{C}_+$ and $\vec{\xi}\in\mathbb{R}^r$ we have

$$\begin{aligned} E^{anw_\lambda}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)&=n!\sum_{q_0+q_1+\ldots+q_m=n}\left(\prod_{j=1}^m\frac{w_j^{q_j}}{q_j!}\right)\sum_{j_0+j_1+\ldots+j_m=q_0}\\ &\int_{\Delta_{q_0;j_0,\ldots,j_m}}\int_{\mathbb{R}^{nr}}A_1\big(j_0,\ldots,j_m;\vec{\eta}_1,\vec{\eta}_2,\vec{s},\vec{v},\vec{h}\big)\,B_1\big(j_0,\ldots,j_m;\lambda,\vec{s},\vec{v},\vec{h}\big)\,d\left(\prod_{u=0}^m\prod_{v=1}^{j_m}\sigma_{s_{u,v}}\times\prod_{u=1}^m\sigma_{p_u}^{q_u}\right)(\vec{v},\vec{h})\,d\mu^{q_0}(\vec{s}) \end{aligned} \tag{24}$$

for $(\vec{\eta}_1,\vec{\eta}_2)\in\mathbb{R}^{2r}$, where

$$A_1\big(j_0,\ldots,j_m;\vec{\eta}_1,\vec{\eta}_2,\vec{s},\vec{v},\vec{h}\big)=\exp\left\{\frac{i}{t}\sum_{u=0}^m\sum_{v=1}^{j_u+1}\big\langle(t-s_{u,v})\vec{\eta}_1+s_{u,v}\vec{\eta}_2,\vec{v}_{u,v}\big\rangle\right\}$$

and

$$\begin{aligned} B_1\big(j_0,\ldots,j_m;\lambda,\vec{s},\vec{v},\vec{h}\big)=\exp\Bigg\{-\frac{1}{2\lambda t^2}\sum_{u=0}^m\sum_{v=1}^{j_u+1}(s_{u,v}-s_{u,v-1})\Bigg\|&\sum_{\beta=u+1}^m\sum_{\gamma=1}^{j_\beta+1}(t-s_{\beta,\gamma})\vec{v}_{\beta,\gamma}\\ &+\sum_{\gamma=v}^{j_u+1}(t-s_{u,\gamma})\vec{v}_{u,\gamma}-\sum_{\gamma=1}^{v-1}s_{u,\gamma}\vec{v}_{u,\gamma}-\sum_{\beta=0}^{u-1}\sum_{\gamma=1}^{j_\beta+1}s_{\beta,\gamma}\vec{v}_{\beta,\gamma}\Bigg\|_{\mathbb{R}^r}^2\Bigg\} \end{aligned}$$

with the convention that $\vec{s}=\big(s_{0,1},\ldots,s_{0,j_0},s_{1,1},\ldots,s_{1,j_1},\ldots,s_{m,1},\ldots,s_{m,j_m}\big)$ for $j_0+j_1+\ldots+j_m=q_0$, $\Delta_{q_0;j_0,\ldots,j_m}=\{\vec{s}:0<s_{0,1}<\ldots<s_{0,j_0}<p_1<s_{1,1}<\ldots<s_{1,j_1}<p_2<\ldots<p_m<s_{m,1}<\ldots<s_{m,j_m}<t\}$, $\vec{v}=(\vec{v}_{0,1},\ldots,\vec{v}_{0,j_0},\vec{v}_{1,1},\ldots,\vec{v}_{1,j_1},\ldots,\vec{v}_{m,1},\ldots,\vec{v}_{m,j_m})$, $\vec{h}=(\vec{h}_{1,1},\ldots,\vec{h}_{1,q_1},\vec{h}_{2,1},\ldots,\vec{h}_{2,q_2},\ldots,\vec{h}_{m,1},\ldots,\vec{h}_{m,q_m})$, $\vec{v}_{m,j_m+1}=\vec{0}$, $s_{0,0}=0$, $s_{m,j_m+1}=t$, $s_{u-1,j_{u-1}+1}=s_{u,0}=p_u$ and $\vec{v}_{u-1,j_{u-1}+1}=\sum_{v=1}^{q_u}\vec{h}_{u,v}$ for $u=1,\ldots,m$. Moreover, for any nonzero real q, $E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ is given by the right hand side of (24) where λ is replaced by $-iq$.

Proof. For $\lambda > 0$ and $(\vec{\eta}_1, \vec{\eta}_2) \in \mathbb{R}^{2r}$, let $(K_\lambda(\Lambda_n))(\vec{\eta}_1, \vec{\eta}_2)$ be given by (3) where F is replaced by Λ_n. Then we have, by the binomial expansion and the simplex method [14],

$$(K_\lambda(\Lambda_n))(\vec{\eta}_1, \vec{\eta}_2) = n! \sum_{q_0+q_1+\ldots+q_m=n} \left(\prod_{j=1}^{m} \frac{w_j^{q_j}}{q_j!} \right)$$
$$\times \int_{C^r} \int_{\Delta_{q_0}} \left[\prod_{j=1}^{q_0} \theta\left(s_j, \lambda^{-\frac{1}{2}} \left(x(s_j) - x(0) - \frac{s_j}{t}(x(t) - x(0)) \right) + \vec{\eta}_1 + \frac{s_j}{t}(\vec{\eta}_2 - \vec{\eta}_1) \right) d\mu^{q_0}(s_1, \ldots, s_{q_0}) \right]$$
$$\times \left[\prod_{j=1}^{m} \left[\theta\left(p_j, \lambda^{-\frac{1}{2}} \left(x(p_j) - x(0) - \frac{p_j}{t}(x(t) - x(0)) \right) + \vec{\eta}_1 + \frac{p_j}{t}(\vec{\eta}_2 - \vec{\eta}_1) \right) \right]^{q_j} \right] dw_\varphi^r(x),$$

where $\Delta_{q_0} = \{(s_1, \ldots, s_{q_0}) : 0 < s_1 < \ldots < s_{q_0} < t\}$. Let $\vec{s}$, $s_{0,0}$, $s_{u,0}$, s_{m,j_m+1} and $\Delta_{q_0;j_0,\ldots,j_m}$ be as in the assumptions. Then we have, by (23),

$$(K_\lambda(\Lambda_n))(\vec{\eta}_1, \vec{\eta}_2) = n! \sum_{q_0+q_1+\ldots+q_m=n} \left(\prod_{j=1}^{m} \frac{w_j^{q_j}}{q_j!} \right) \sum_{j_0+j_1+\ldots+j_m=q_0}$$
$$\int_{\Delta_{q_0;j_0,\ldots,j_m}} \int_{C^r} \left[\int_{\mathbb{R}^{q_0 r}} \exp\left\{ i \sum_{u=0}^{m} \sum_{v=1}^{j_u} \left\langle \lambda^{-\frac{1}{2}} \left(x(s_{u,v}) - x(s_{0,0}) - \frac{s_{u,v}}{t}(x(s_{m,j_m+1}) - x(s_{0,0})) \right) \right.\right.\right.$$
$$\left.\left. + \vec{\eta}_1 + \frac{s_{u,v}}{t}(\vec{\eta}_2 - \vec{\eta}_1), \vec{v}_{u,v} \right\rangle \right\} d\left(\prod_{u=0}^{m} \prod_{v=1}^{j_m} \sigma_{s_{u,v}} \right)(\vec{v}) \right]$$
$$\times \left[\int_{\mathbb{R}^{(q_1+\ldots+q_m)r}} \exp\left\{ i \sum_{u=1}^{m} \sum_{v=1}^{q_u} \left\langle \lambda^{-\frac{1}{2}} \left(x(s_{u,0}) - x(s_{0,0}) - \frac{s_{u,0}}{t}(x(s_{m,j_m+1}) - x(s_{0,0})) \right) \right.\right.\right.$$
$$\left.\left.\left. + \vec{\eta}_1 + \frac{s_{u,0}}{t}(\vec{\eta}_2 - \vec{\eta}_1), \vec{h}_{u,v} \right\rangle \right\} d\left(\prod_{u=1}^{m} \sigma_{p_u}^{q_u} \right)(\vec{h}) \right] dw_\varphi^r(x)\, d\mu^{q_0}(\vec{s}),$$

where $\vec{v}$ and $\vec{h}$ are given by the assumptions. For $u = 1, \ldots, m$, let $s_{u-1,j_{u-1}+1} = s_{u,0}$, $\vec{v}_{u-1,j_{u-1}+1} = \sum_{v=1}^{q_u} \vec{h}_{u,v}$ and let $\vec{v}_{m,j_m+1} = \vec{0}$. Then we have, by Fubini's theorem

$$(K_\lambda(\Lambda_n))(\vec{\eta}_1, \vec{\eta}_2) = n! \sum_{q_0+q_1+\ldots+q_m=n} \left(\prod_{j=1}^{m} \frac{w_j^{q_j}}{q_j!} \right) \sum_{j_0+j_1+\ldots+j_m=q_0}$$
$$\int_{\Delta_{q_0;j_0,\ldots,j_m}} \int_{\mathbb{R}^{nr}} A_1\left(j_0, \ldots, j_m; \vec{\eta}_1, \vec{\eta}_2, \vec{s}, \vec{v}, \vec{h} \right)$$
$$\times \int_{C^r} \exp\left\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{m} \sum_{v=1}^{j_u+1} \left\langle x(s_{u,v}) - x(s_{0,0}) - \frac{s_{u,v}}{t}(x(s_{m,j_m+1}) - x(s_{0,0})), \vec{v}_{u,v} \right\rangle \right\} dw_\varphi^r(x)$$
$$d\left(\prod_{u=0}^{m} \prod_{v=1}^{j_m} \sigma_{s_{u,v}} \times \prod_{u=1}^{m} \sigma_{p_u}^{q_u} \right)(\vec{v}, \vec{h})\, d\mu^{q_0}(\vec{s}).$$

Let

$$S(j_0, \ldots, j_m; \vec{s}) = n! \left[\prod_{u=0}^{m} \prod_{v=1}^{j_u+1} \frac{1}{2\pi(s_{u,v} - s_{u,v-1})} \right]^{\frac{r}{2}}.$$

Then we have, by an application of Lemma 2.1,

$$(K_\lambda(\Lambda_n))(\vec{\eta}_1,\vec{\eta}_2)=\sum_{q_0+q_1+\ldots+q_m=n}\left(\prod_{j=1}^{m}\frac{w_j^{q_j}}{q_j!}\right)\sum_{j_0+j_1+\ldots+j_m=q_0}$$
$$\int_{\Delta_{q_0;j_0,\ldots,j_m}}\int_{\mathbb{R}^{nr}}A_1\left(j_0,\ldots,j_m;\vec{\eta}_1,\vec{\eta}_2,\vec{s},\vec{v},\vec{h}\right)S(j_0,\ldots,j_m;\vec{s})$$
$$\times\left[\int_{\mathbb{R}^{(q_0+m+1)r}}\int_{\mathbb{R}^r}\exp\left\{i\lambda^{-\frac{1}{2}}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\left\langle\vec{\zeta}_{u,v}-\vec{\zeta}_{0,0}-\frac{s_{u,v}}{t}(\vec{\zeta}_{m,j_m+1}-\vec{\zeta}_{0,0}),\vec{v}_{u,v}\right\rangle\right.\right.$$
$$\left.\left.-\frac{1}{2}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\frac{\|\vec{\zeta}_{u,v}-\vec{\zeta}_{u,v-1}\|^2_{\mathbb{R}^r}}{s_{u,v}-s_{u,v-1}}\right\}d\varphi^r(\vec{\zeta}_{0,0})\,dm_L^{(q_0+m+1)r}(\vec{\zeta})\right]$$
$$d\left(\prod_{u=0}^{m}\prod_{v=1}^{j_m}\sigma_{s_{u,v}}\times\prod_{u=1}^{m}\sigma_{p_u}^{q_u}\right)(\vec{v},\vec{h})\,d\mu^{q_0}(\vec{s}),$$

where $\vec{\zeta}_{u-1,j_{u-1}+1}=\vec{\zeta}_{u,0}$ for $u=1,\ldots,m$ and $\vec{\zeta}=\left(\vec{\zeta}_{0,1},\ldots,\vec{\zeta}_{0,j_0+1},\vec{\zeta}_{1,1},\ldots,\vec{\zeta}_{1,j_1+1},\ldots,\vec{\zeta}_{m,1},\ldots,\vec{\zeta}_{m,j_m+1}\right)$. Let $\vec{z}_{u,v}=\vec{\zeta}_{u,v}-\vec{\zeta}_{u,v-1}$ for $u=0,\ldots,m$; $v=1,\ldots,j_u+1$. By the change of variable theorem,

$$(K_\lambda(\Lambda_n))(\vec{\eta}_1,\vec{\eta}_2)=\sum_{q_0+q_1+\ldots+q_m=n}\left(\prod_{j=1}^{m}\frac{w_j^{q_j}}{q_j!}\right)\sum_{j_0+j_1+\ldots+j_m=q_0}$$
$$\int_{\Delta_{q_0;j_0,\ldots,j_m}}\int_{\mathbb{R}^{nr}}A_1\left(j_0,\ldots,j_m;\vec{\eta}_1,\vec{\eta}_2,\vec{s},\vec{v},\vec{h}\right)S(j_0,\ldots,j_m;\vec{s})$$
$$\times\left[\int_{\mathbb{R}^{(q_0+m+1)r}}\exp\left\{i\lambda^{-\frac{1}{2}}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\left\langle\sum_{\beta=0}^{u-1}\sum_{\gamma=1}^{j_\beta+1}\vec{z}_{\beta,\gamma}+\sum_{\gamma=1}^{v}\vec{z}_{u,\gamma}-\frac{s_{u,v}}{t}\sum_{\beta=0}^{m}\sum_{\gamma=1}^{j_\beta+1}\vec{z}_{\beta,\gamma},\vec{v}_{u,v}\right\rangle\right.\right.$$
$$\left.\left.-\frac{1}{2}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\frac{\|\vec{z}_{u,v}\|^2_{\mathbb{R}^r}}{s_{u,v}-s_{u,v-1}}\right\}dm_L^{(q_0+m+1)r}(\vec{z})\right]$$
$$d\left(\prod_{u=0}^{m}\prod_{v=1}^{j_m}\sigma_{s_{u,v}}\times\prod_{u=1}^{m}\sigma_{p_u}^{q_u}\right)(\vec{v},\vec{h})\,d\mu^{q_0}(\vec{s}),$$

where $\vec{z}=(\vec{z}_{0,1},\ldots,\vec{z}_{0,j_0+1},\vec{z}_{1,1},\ldots,\vec{z}_{1,j_1+1},\ldots,\vec{z}_{m,1},\ldots,\vec{z}_{m,j_m+1})$. Now,

$$(K_\lambda(\Lambda_n))(\vec{\eta}_1,\vec{\eta}_2)=\sum_{q_0+q_1+\ldots+q_m=n}\left(\prod_{j=1}^{m}\frac{w_j^{q_j}}{q_j!}\right)\sum_{j_0+j_1+\ldots+j_m=q_0}$$
$$\int\limits_{\Delta_{q_0;j_0,\ldots,j_m}}\int\limits_{\mathbb{R}^{nr}}A_1\left(j_0,\ldots,j_m;\vec{\eta}_1,\vec{\eta}_2,\vec{s},\vec{v},\vec{h}\right)S(j_0,\ldots,j_m;\vec{s})$$
$$\times\left[\int_{\mathbb{R}^{(q_0+m+1)r}}\exp\left\{\frac{i}{\sqrt{\lambda}t}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\left\langle\vec{z}_{u,v},\sum_{\beta=u+1}^{m}\sum_{\gamma=1}^{j_\beta+1}(t-s_{\beta,\gamma})\vec{v}_{\beta,\gamma}+\sum_{\gamma=v}^{j_u+1}(t-s_{u,\gamma})\vec{v}_{u,\gamma}\right.\right.\right.$$
$$\left.\left.\left.-\sum_{\gamma=1}^{v-1}s_{u,\gamma}\vec{v}_{u,\gamma}-\sum_{\beta=0}^{u-1}\sum_{\gamma=1}^{j_\beta+1}s_{\beta,\gamma}\vec{v}_{\beta,\gamma}\right\rangle-\frac{1}{2}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\frac{\|\vec{z}_{u,v}\|^2_{\mathbb{R}^r}}{s_{u,v}-s_{u,v-1}}\right\}dm_L^{(q_0+m+1)r}(\vec{z})\right]$$
$$d\left(\prod_{u=0}^{m}\prod_{v=1}^{j_m}\sigma_{s_{u,v}}\times\prod_{u=1}^{m}\sigma_{p_u}^{q_u}\right)(\vec{v},\vec{h})\,d\mu^{q_0}(\vec{s})$$

$$= n! \sum_{q_0+q_1+\ldots+q_m=n} \left(\prod_{j=1}^{m} \frac{w_j^{q_j}}{q_j!}\right) \sum_{j_0+j_1+\ldots+j_m=q_0}$$

$$\int_{\Delta_{q_0;j_0,\ldots,j_m}} \int_{\mathbb{R}^{nr}} A_1\left(j_0,\ldots,j_m;\vec{\eta}_1,\vec{\eta}_2,\vec{s},\vec{v},\vec{h}\right) B_1(j_0,\ldots,j_m;\lambda,\vec{s},\vec{v},\vec{h})\, d\left(\prod_{u=0}^{m}\prod_{v=1}^{j_m}\sigma_{s_{u,v}} \times \prod_{u=1}^{m}\sigma_{p_u}^{q_u}\right)(\vec{v},\vec{h})\, d\mu^{q_0}(\vec{s}),$$

where the last equality follows from (10). The results now follow from Morera's theorem and the dominated convergence theorem. □

Using the same method as used in the proof of Theorem 5.1 above, we obtain the following corollaries.

Corollary 6.2.

Under the assumptions in Theorem 6.1, with the exception that $\eta = \mu$, that is, η being a continuous measure, $E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ is given by the right hand side of (16) in Theorem 5.1 where $d\vec{s}$ is replaced by $d\mu^n(\vec{s})$.

Corollary 6.3.

Under the assumptions in Theorem 6.1, with the exception $\eta = \sum_{j=1}^{m} w_j\delta_{p_j}$, that is, η is a discrete measure, $E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ is given by

$$E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2) = n! \sum_{q_1+\ldots+q_m=n} \left(\prod_{j=1}^{m}\frac{w_j^{q_j}}{q_j!}\right) \int_{\mathbb{R}^{nr}} \exp\left\{\frac{i}{t}\sum_{u=1}^{m}\sum_{v=1}^{q_u}\left\langle (t-p_u)\vec{\eta}_1 + p_u\vec{\eta}_2, \vec{h}_{u,v}\right\rangle \right.$$
$$\left. + \frac{1}{2qit^2}\sum_{u=1}^{m+1}(p_u - p_{u-1})\left\| \sum_{\beta=u}^{m}\sum_{v=1}^{q_\beta}(t-p_\beta)\vec{h}_{\beta,v} - \sum_{\beta=1}^{u-1}\sum_{v=1}^{q_\beta} p_\beta \vec{h}_{\beta,v}\right\|_{\mathbb{R}^r}^2 \right\} d\left(\prod_{u=1}^{m}\sigma_{p_u}^{q_u}\right)(\vec{h})$$

where $p_0 = 0$, $p_{m+1} = t$ and $\vec{h} = \left(\vec{h}_{1,1},\ldots,\vec{h}_{1,q_1},\vec{h}_{2,1},\ldots,\vec{h}_{2,q_2},\ldots,\vec{h}_{m,1},\ldots,\vec{h}_{m,q_m}\right)$.

Using the same methods as used in the proof of Theorem 5.2, we can easily prove the following theorem.

Theorem 6.4.

Let

$$\Lambda_t(x) = \exp\left\{\int_0^t \theta(s,x(s))\, d\eta(s)\right\}$$

for $x \in C^r[0,t]$, where $\theta \in \mathcal{G}^*$ is given by (23). Then $E^{anf_q}[\Lambda_t|X_t]$ exists for any nonzero real q and for $\vec{\xi} \in \mathbb{R}^r$ it is given by

$$E^{anf_q}[\Lambda_t|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$$

for $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$, where $E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ is given by (24) in Theorem 6.1 where λ is replaced by $-iq$.

By (19) and Theorem 6.4, we have the following theorem.

Theorem 6.5.

Let the assumptions and notations be given as in Theorem 6.4. For $x \in C^r[0,t]$ let

$$\Gamma_n(x) = \Lambda_n(x)\phi(x(t)) \quad \text{and} \quad \Gamma_t(x) = \Lambda_t(x)\phi(x(t))$$

where $\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle\vec{u},\vec{w}\rangle\}\,d\nu(\vec{w})$ for $\nu \in \mathcal{M}(\mathbb{R}^r)$. Then $E^{anf_q}[\Gamma_n|X_t]$ and $E^{anf_q}[\Gamma_t|X_t]$ exist for any nonzero real q, and for $\vec{\xi} \in \mathbb{R}^r$ they are given by

$$E^{anf_q}[\Gamma_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2) = \phi(\vec{\eta}_2)E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$$

and

$$E^{anf_q}[\Gamma_t|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2) = \phi(\vec{\eta}_2)E^{anf_q}[\Lambda_t|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2) = \phi(\vec{\eta}_2) + \sum_{n=1}^{\infty}\frac{1}{n!}E^{anf_q}[\Gamma_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$$

for $(\vec{\eta}_1,\vec{\eta}_2) \in \mathbb{R}^{2r}$, where $E^{anf_q}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ and $E^{anf_q}[\Lambda_t|X_t](\vec{\xi})(\vec{\eta}_1,\vec{\eta}_2)$ are given in Theorems 6.1 and 6.4, respectively.

Combining Lemma 3.4, Theorems 3.5, 3.7, 6.1, 6.4, and 6.5, we have the following theorem.

Theorem 6.6.

Let the assumptions and notations be as given in Lemma 3.4 and Theorems 6.1, 6.4 and 6.5 above.

(1) *Suppose that there exists a function Ψ on $\mathbb{C}_+ \times \mathbb{R}^r$ satisfying conditions* (i) *and* (ii) *of Theorem 3.5, and either* (iii) *of Theorem 3.5 or* (8) *of Theorem 3.7. Then for $\lambda \in \mathbb{C}_+$ the analytic operator-valued Wiener integrals $I_\lambda^{an}(\Lambda_n)$ and $I_\lambda^{an}(\Gamma_n)$ exist as elements of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and they are given by* (4) *where F is replaced by Λ_n and Γ_n, respectively ($E^{anw_\lambda}[\Lambda_n|X_t](\vec{\xi})(\vec{\eta}_1,\,\vec{\eta}_2)$ is given in Theorem 6.1). Furthermore, as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$, the analytic operator-valued Wiener integral $I_\lambda^{an}(\Lambda_t)$ exists and is given by* (20) *in Theorem 5.4 where F_t and F_n are replaced by Λ_t and Λ_n, respectively. The analytic operator-valued Wiener integral $I_\lambda^{an}(\Gamma_t)$ also exists and is given by* (21) *in Theorem 5.4 where G_t and G_n are replaced by Γ_t and Γ_n, respectively.*

(2) *Suppose that for a nonzero real q there exists a function Ψ on $(\mathbb{C}_+ \cup \{-iq\}) \times \mathbb{R}^r$ satisfying conditions* (i) *and* (ii) *of Theorem 3.5, and either* (iii) *and* (iii)′ *of Theorem 3.5 or* (8) *of Theorem 3.7. Then the analytic operator-valued Feynman integrals $J_q^{an}(\Lambda_n)$, $J_q^{an}(\Gamma_n)$, $J_q^{an}(\Lambda_t)$ and $J_q^{an}(\Gamma_t)$ exist as elements of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and each of them is given by the corresponding expression of the analytic operator-valued Wiener integral in* (1) *where λ is replaced by $-iq$, E^{anw_λ} is replaced by E^{anf_q} and I_λ^{an} is replaced by J_q^{an}.*

Combining the methods used in the proofs of Theorems 4.3 and 5.5 above, our final theorem follows readily.

Theorem 6.7.

Suppose that φ^r is normally distributed with the mean vector $\vec{0}$ and the variance-covariance matrix $\alpha^2 I_r$. Let the assumptions and notations be as given in Theorems 6.1 and 6.5. Then for any nonzero real q the analytic operator-valued Feynman integral $J_q^{an}(\Gamma_t)$ exists as an element of $\mathcal{L}\big(L_1(\mathbb{R}^r), L_\infty(\mathbb{R}^r)\big)$ and is given by (22) *in Theorem 5.5 where G_t and G_n are replaced by Γ_t and Γ_n, respectively, at that*

$$\begin{aligned}
\big(J_q^{an}(\Gamma_n)\psi\big)(\vec{\xi}) &= n!\left[\frac{q}{2\pi i(t+\alpha^2)}\right]^{\frac{r}{2}} \sum_{q_0+q_1+\ldots+q_m=n}\left(\prod_{j=1}^{m}\frac{w_j^{q_j}}{q_j!}\right)\sum_{j_0+j_1+\ldots+j_m=q_0} \\
&\int_{\Delta_{q_0;j_0,\ldots,j_m}}\int_{\mathbb{R}^{nr}} B_1\big(j_0,\ldots,j_m;-iq,\vec{s},\vec{v},\vec{h}\big) \\
&\quad\times\left[\int_{\mathbb{R}^r}\psi(\vec{\eta})\phi(\vec{\eta})\exp\left\{i\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}\langle\vec{\eta},\vec{v}_{u,v}\rangle\right\}H\left(-iq,\vec{\xi},\vec{\eta},\frac{1}{t}\sum_{u=0}^{m}\sum_{v=1}^{j_u+1}(s_{u,v}-t)\vec{v}_{u,v}\right)dm_L^r(\vec{\eta})\right] \\
&\qquad d\left(\prod_{u=0}^{m}\prod_{v=1}^{j_m}\sigma_{s_{u,v}}\times\prod_{u=1}^{m}\sigma_{p_u}^{q_u}\right)(\vec{v},\vec{h})\,d\mu^{q_0}(\vec{s})
\end{aligned}$$

for $\psi \in L_1(\mathbb{R}^r)$ and $\vec{\xi} \in \mathbb{R}^r$, where H and B_1 are given in Theorems 4.3 and 6.1, respectively.

Remark 6.8.

- The conditions of Theorems 4.3, 5.5 and 6.7 are independent of those in Lemma 3.4, Theorems 3.5 and 3.7 if φ^r is normally distributed.
- If $\eta = \mu + \sum_{j=1}^{m} w_j \delta_{p_j}$, where $0 \le p_1 < \ldots < p_m \le t$, we can obtain all the results in the present section with minor modifications.
- If $\eta = \mu + \sum_{j=1}^{\infty} w_j \delta_{p_j}$, then using the following version of the $\aleph_0$-nomial formula [14, p. 41]

$$\left(\sum_{p=0}^{\infty} b_p\right)^n = \sum_{h=0}^{\infty} \sum_{q_0+q_1+\ldots+q_h=n, q_h\neq 0} \frac{n!}{q_0!q_1!\ldots q_h!} b_0^{q_0} b_1^{q_1} \ldots b_h^{q_h},$$

we can obtain the results of Theorems 6.1, 6.4, 6.5, 6.6 and 6.7 with minor modifications.

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