

Generalizations of Jensen–Steffensen and related integral inequalities for superquadratic functions

Research Article

Shoshana Abramovich^{1*}, Slavica Ivelić^{2†}, Josip Pečarić^{3‡}

¹ Department of Mathematics, University of Haifa, Haifa, Israel

² Faculty of Civil Technology and Architecture, University of Split, Split, Croatia

³ Faculty of Textile Technology, University of Zagreb, Zagreb, Croatia

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Abstract: We present integral versions of some recently proved results which improve the Jensen–Steffensen and related inequalities for superquadratic functions. For superquadratic functions which are not convex we get inequalities analogous to the integral Jensen–Steffensen inequality for convex functions. Therefore, we get refinements of all the results which use only the convexity of these functions. One of the inequalities that we obtain for a superquadratic function φ is

$$\bar{y} \geq \varphi(\bar{x}) + \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - \bar{x}|) d\lambda(t),$$

where $\bar{x} = \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} f(t) d\lambda(t)$ and $\bar{y} = \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(f(t)) d\lambda(t)$, which under suitable conditions like those satisfied by functions of power equal or more than 2, is a refinement of the Jensen–Steffensen–Boas inequality. We also prove related results of Mercer's type.

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1. Introduction

In this paper we present integral versions of the main results in [1]. We obtain new results for functions whose derivatives are superadditive. As a consequence, we get under suitable conditions, refinements of the Jensen–Steffensen–Boas

* E-mail: abramos@math.haifa.ac.il

† E-mail: sivelic@gradst.hr

‡ E-mail: pecaric@element.hr

inequality and also refinements of relevant results in [5] and [6] which use only the convexity of these functions. We also present Mercer's type variants of those results.

Let I be an interval in $\mathbb{R}$ and $\varphi : I \rightarrow \mathbb{R}$ a convex function on I . If $\mathbf{x} = (x_1, \dots, x_n)$ is any n -tuple in I^n and $\mathbf{a} = (a_1, \dots, a_n)$ any nonnegative n -tuple such that $A_n = \sum_{i=1}^n a_i > 0$, then the well known Jensen's inequality

$$\varphi \left(\frac{1}{A_n} \sum_{i=1}^n a_i x_i \right) \leq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) \quad (1)$$

holds (see [8, p. 43]).

The assumption " $\mathbf{a} = (a_1, \dots, a_n)$ any nonnegative n -tuple" can be relaxed at the expense of more restrictions on the n -tuple $\mathbf{x}$.

If $\mathbf{a} = (a_1, \dots, a_n)$ is a real n -tuple that satisfies

$$0 \leq A_j = \sum_{i=1}^j a_i \leq A_n, \quad j = 1, \dots, n, \quad A_n > 0, \quad (2)$$

and $\mathbf{x} = (x_1, \dots, x_n)$ is any monotonic n -tuple in I^n , then the inequality (1) is still valid. Inequality (1) considered under conditions (2) is known as the Jensen–Steffensen inequality for convex functions (see [8, p. 57]).

A.McD. Mercer [7] gave the following variant of Jensen's inequality which we refer to as Jensen's inequality of Mercer's type.

Theorem A.

If φ is a convex function on an interval containing an n -tuple $\mathbf{x} = (x_1, \dots, x_n)$ such that $0 < x_1 \leq x_2 \leq \dots \leq x_n$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a positive n -tuple with $\sum_{i=1}^n a_i = 1$, then

$$\varphi \left(x_1 + x_n - \sum_{i=1}^n a_i x_i \right) \leq \varphi(x_1) + \varphi(x_n) - \sum_{i=1}^n a_i \varphi(x_i). \quad (3)$$

An integral version of the Jensen–Steffensen inequality was also proved by Steffensen, but here we consider a variant given by R.P. Boas [4].

Theorem B (Jensen–Steffensen–Boas).

Let $f : [\alpha, \beta] \rightarrow (a, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$ and $-\infty \leq a < b \leq +\infty$, and let $\varphi : (a, b) \rightarrow \mathbb{R}$ be a convex function. If $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ is either continuous or of bounded variation satisfying

$$\lambda(\alpha) \leq \lambda(x) \leq \lambda(\beta) \quad \text{for all } x \in [\alpha, \beta], \quad \lambda(\beta) - \lambda(\alpha) > 0,$$

then

$$\varphi \left(\frac{\int_{\alpha}^{\beta} f(t) d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)} \right) \leq \frac{\int_{\alpha}^{\beta} \varphi(f(t)) d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}. \quad (4)$$

Superquadratic functions were introduced in [2] and [3] and dealt with in numerous papers. We quote the definition and some basic properties of superquadratic functions.

Definition C ([2, Definition 2.1]).

A function $\varphi : [0, b) \rightarrow \mathbb{R}$ is *superquadratic* provided that for all $0 \leq x < b$, there exists a constant $C(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C(x)(y - x) \quad (5)$$

for all $y \in [0, b)$.

There are many functions which are convex and also superquadratic. For example, the function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ defined by $\varphi(x) = x^p$ for $p \geq 2$. Therefore, in such cases we get refinements of all the results which use only the convexity of these functions.

Lemma D ([3, Lemma 2.3]).

Suppose that φ is superquadratic on $[0, b)$, $x_i \in [0, b)$, $i = 1, \dots, n$ and $a_i \geq 0$, $i = 1, \dots, n$, are such that $A_n = \sum_{i=1}^n a_i > 0$. Then

$$\frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \varphi(\bar{x}) \geq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|x_i - \bar{x}|), \quad (6)$$

where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$.

Lemma E ([2, Lemma 2.1]).

Let φ be a superquadratic function with $C(x)$ as in Definition C. Then:

- (i) $\varphi(0) \leq 0$,
- (ii) if $\varphi(0) = \varphi'(0) = 0$, then $C(x) = \varphi'(x)$ whenever φ is differentiable at $0 < x < b$,
- (iii) if $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.

Lemma F ([2, Lemma 3.1]).

Suppose that $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.

Lemma G ([3, Lemma 2.1]).

Suppose that φ is superquadratic and nonnegative. Then φ is convex and increasing. Also, if $C(x)$ is as in Definition C, then $C(x) \geq 0$.

The next theorem is the main theorem of [1]. In this paper we present its integral version.

Theorem H ([1, Theorem 1]).

Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be continuously differentiable and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be a superadditive function. Let $\mathbf{a}$ be a real n -tuple satisfying (2) and $\mathbf{x}$ be a monotonic n -tuple in $[0, b)^n$. Then

(a) the inequality

$$\varphi(c) - \varphi(0) + \varphi'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|x_i - c|) \leq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) \quad (7)$$

holds for all $c \in [0, b)$ where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$. Inserting $c = \bar{x}$ in (7) we get

$$\frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \varphi(\bar{x}) + \varphi(0) \geq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|x_i - \bar{x}|). \quad (8)$$

(b) If in addition $\varphi(0) \leq 0$, then φ is superquadratic and

$$\frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) \geq \varphi(c) + \varphi'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|x_i - c|). \quad (9)$$

Inserting $c = \bar{x}$ in (9) we get

$$\frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \varphi(\bar{x}) \geq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|x_i - \bar{x}|). \quad (10)$$

(c) If in addition $\varphi \geq 0$ and $\varphi(0) = \varphi'(0) = 0$, then φ is superquadratic and convex increasing and

$$\frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \varphi(c) - \varphi'(c)(\bar{x} - c) \geq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|x_i - c|) \geq 0. \quad (11)$$

We see that the inequality (10) extends Lemma D to coefficients satisfying (2) where $\mathbf{x}$ is a monotonic n -tuple and the inequality (11) refines Lemma A.

In this paper we prove analogs of Theorems A and B, its extensions and refinements.

2. The integral Jensen–Steffensen inequality for superquadratic functions

The results of this section are analogous to and in many cases refinements of Theorem B, the Jensen–Steffensen–Boas theorem, and also of relevant results in paper [6].

To prove our main results we need the following four lemmas.

Lemma 2.1 ([1, Lemma 1]).

Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be continuously differentiable and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be a superadditive function. Then the function $\psi : [0, b) \rightarrow \mathbb{R}$, defined by

$$\psi(y) = \varphi(y) - \varphi(z) - \varphi'(z)(y - z) - \varphi(|y - z|) + \varphi(0), \quad (12)$$

is nonnegative on $[0, b)$, nonincreasing on $[0, z]$, and nondecreasing on $[z, b)$, for $0 \leq z < b$.

Lemma 2.2 ([6, Lemma 2]).

Let $f : [\alpha, \beta] \rightarrow \mathbb{R}$ be a nonnegative function and let $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ be either continuous or of bounded variation. Suppose that f and λ have no common discontinuity points.

(a) If f is nondecreasing on $[\alpha, \beta]$, then

$$f(\beta) \inf_{\alpha \leq \tau \leq \beta} \int_{\tau}^{\beta} d\lambda(t) \leq \int_{\alpha}^{\beta} f(t) d\lambda(t) \leq f(\beta) \sup_{\alpha \leq \tau \leq \beta} \int_{\tau}^{\beta} d\lambda(t). \quad (13)$$

(b) If f is nonincreasing on $[\alpha, \beta]$, then

$$f(\alpha) \inf_{\alpha \leq \tau \leq \beta} \int_{\alpha}^{\tau} d\lambda(t) \leq \int_{\alpha}^{\beta} f(t) d\lambda(t) \leq f(\alpha) \sup_{\alpha \leq \tau \leq \beta} \int_{\alpha}^{\tau} d\lambda(t). \quad (14)$$

Lemma 2.3.

Let $f : [\alpha, \beta] \rightarrow [0, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$, and $\varphi : [0, b) \rightarrow \mathbb{R}$ be a continuously differentiable function. Let $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ be either continuous or of bounded variation satisfying

$$\lambda(\alpha) \leq \lambda(x) \leq \lambda(\beta) \quad \text{for all } x \in [\alpha, \beta], \quad \lambda(\beta) - \lambda(\alpha) > 0. \quad (15)$$

Then $\bar{x}$ and $\bar{y}$, given by

$$\bar{x} = \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} f(t) d\lambda(t), \quad \bar{y} = \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(f(t)) d\lambda(t), \quad (16)$$

are well defined and $\bar{x} \in [0, b)$.

Proof. See the proof of [6, Theorem 5]. □

Lemma 2.4.

Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be continuously differentiable and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be a superadditive function. Let $\psi : [0, b) \rightarrow \mathbb{R}$ be defined by

$$\psi(y) = \varphi(y) - \varphi(c) - \varphi'(c)(y - c) - \varphi(|y - c|) + \varphi(0), \quad c \in [0, b). \quad (17)$$

Let $f : [\alpha, \beta] \rightarrow [0, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$, and $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ be either continuous or of bounded variation, satisfying (15). If $g : [\alpha, \beta] \rightarrow \mathbb{R}$ is defined by $g(t) = \psi(f(t))$, then

$$\int_{\alpha}^{\beta} g(t) d\lambda(t) \geq 0.$$

Proof. Suppose that f is nondecreasing. First we conclude that $f([\alpha, \beta]) = [f(\alpha), f(\beta)] \subseteq [0, b)$. From Lemma 2.1 it follows that the function g is nonnegative. Since the functions f and $\varphi(f)$ are continuous, then the integral $\int_{\alpha}^{\beta} g(t) d\lambda(t)$ is well defined.

We must consider three cases.

Case 1: $f(\beta) \leq c$. Since f is nondecreasing on $[\alpha, \beta]$ and ψ is, according to Lemma 2.1, nonincreasing on $[0, c)$, then $g = \psi(f)$ is nonincreasing on $[\alpha, \beta]$. Applying Lemma 2.2(b) we get

$$\int_{\alpha}^{\beta} g(t) d\lambda(t) \geq g(\alpha) \inf_{\alpha \leq \tau \leq \beta} [\lambda(\tau) - \lambda(\alpha)] = 0.$$

Case 2: $c \leq f(\alpha)$. Since f is nondecreasing on $[\alpha, \beta]$ and ψ is, according to Lemma 2.1, nondecreasing on $[c, b)$, then $g = \psi(f)$ is nondecreasing on $[\alpha, \beta]$. Applying Lemma 2.2(a) we get

$$\int_{\alpha}^{\beta} g(t) d\lambda(t) \geq g(\beta) \inf_{\alpha \leq \tau \leq \beta} [\lambda(\beta) - \lambda(\tau)] = 0.$$

Case 3: $f(\alpha) < c < f(\beta)$. Since f is continuous on $[\alpha, \beta]$, there exists at least one point $\xi \in (\alpha, \beta)$ such that $f(\xi) = c$. Then

$$f([\alpha, \xi]) = [f(\alpha), c] \subseteq [0, c).$$

By Lemma 2.1 we conclude that g is nonincreasing on $[\alpha, \xi]$. On the other hand,

$$f([\xi, \beta]) = [c, f(\beta)] \subseteq (c, b),$$

hence, by Lemma 2.1, g is nondecreasing on $[\xi, \beta]$.

Using the same argument as in the previous cases we get

$$\int_{\alpha}^{\beta} g(t) d\lambda(t) = \int_{\alpha}^{\xi} g(t) d\lambda(t) + \int_{\xi}^{\beta} g(t) d\lambda(t) \geq g(\alpha) \inf_{\alpha \leq \tau \leq \beta} [\lambda(\tau) - \lambda(\alpha)] + g(\beta) \inf_{\alpha \leq \tau \leq \beta} [\lambda(\beta) - \lambda(\tau)] = 0.$$

Hence, in all three cases we get

$$\int_{\alpha}^{\beta} g(t) d\lambda(t) \geq 0.$$

If f is nonincreasing the proof is similar, and also here we need to consider three cases. That is, by similar reasoning we get that in the first case when $f(\alpha) < c$ the function g is nondecreasing, in the second case when $c \leq f(\beta)$, g is nonincreasing and in the third case when $f(\alpha) < c < f(\beta)$, g is nonincreasing on $[\alpha, \xi]$ and nondecreasing on $[\xi, \beta]$. In all three cases we have $\int_{\alpha}^{\beta} g(t) d\lambda(t) \geq 0$. □

Now we state and prove our main result.

Theorem 2.5.

Let $f : [\alpha, \beta] \rightarrow [0, b)$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$. Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be continuously differentiable and let $\varphi' : [0, b) \rightarrow \mathbb{R}$ be a superadditive function. Let $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ be either continuous or of bounded variation satisfying (15). Then

(a) the inequality

$$\varphi(c) - \varphi(0) + \varphi'(c)(\bar{x} - c) + \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t) \leq \bar{y} \quad (18)$$

holds for each $c \in [0, b)$, where $\bar{x}$ and $\bar{y}$ are defined as in (16). Inserting $c = \bar{x}$ in (18) we get

$$\varphi(\bar{x}) - \varphi(0) + \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - \bar{x}|) d\lambda(t) \leq \bar{y}. \quad (19)$$

(b) If in addition $\varphi(0) \leq 0$, the function φ is superquadratic and

$$\bar{y} \geq \varphi(c) + \varphi'(c)(\bar{x} - c) + \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t). \quad (20)$$

Inserting $c = \bar{x}$ in (20) we get

$$\bar{y} \geq \varphi(\bar{x}) + \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - \bar{x}|) d\lambda(t). \quad (21)$$

(c) If in addition $\varphi \geq 0$ and $\varphi(0) = \varphi'(0) = 0$, the function φ is superquadratic and convex increasing and

$$\bar{y} - \varphi(c) - \varphi'(c)(\bar{x} - c) \geq \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t) \geq 0. \quad (22)$$

Proof. (a) Suppose that f is nondecreasing (for f nonincreasing the proof is analogous).

We define the function $g : [\alpha, \beta] \rightarrow \mathbb{R}$ as in Lemma 2.4, that is

$$g(t) := \psi(f(t)) = \varphi(f(t)) - \varphi(c) - \varphi'(c)(f(t) - c) - \varphi(|f(t) - c|) + \varphi(0), \quad c \in [0, b).$$

Then by Lemma 2.4 it follows that the integral $\int_{\alpha}^{\beta} g(t) d\lambda(t)$ is well defined and nonnegative, i.e.,

$$\int_{\alpha}^{\beta} g(t) d\lambda(t) = \int_{\alpha}^{\beta} [\varphi(f(t)) - \varphi(c) - \varphi'(c)(f(t) - c) - \varphi(|f(t) - c|) + \varphi(0)] d\lambda(t) \geq 0. \quad (23)$$

From (23) follows

$$\begin{aligned} & \int_{\alpha}^{\beta} \varphi(f(t)) d\lambda(t) - \varphi(c) \cdot [\lambda(\beta) - \lambda(\alpha)] - \varphi'(c) \int_{\alpha}^{\beta} f(t) d\lambda(t) \\ & + \varphi'(c) \cdot c \cdot [\lambda(\beta) - \lambda(\alpha)] - \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t) + \varphi(0)[\lambda(\beta) - \lambda(\alpha)] \geq 0. \end{aligned}$$

Now dividing the last inequality with $\lambda(\beta) - \lambda(\alpha) > 0$ we obtain (18).

(b) Since $\varphi(0) \leq 0$, from (18) it follows (20). If we set $c = \bar{x}$ in (20), we get (21). Also, since $\varphi(0) \leq 0$, from Lemma F it follows that φ is a superquadratic function.

(c) From (18) follows the first inequality in (22). By assumption the function φ is nonnegative, so the second inequality in (22) also holds.

Since $\varphi(0) = 0$, from Lemma F it follows that φ is superquadratic and then by Lemma G that φ is also convex and increasing. $\square$

3. Some results of Mercer's type

In this section we present the Mercer's type variants of Theorem H, Theorem 2.5 and some related results. We use the same technique as in [5, Theorem 1], there applied to convex functions and here applied to functions φ for which φ' is superadditive. Therefore, in many cases we get refinements of [5, Theorem 1].

In the following we denote as $(\Omega, \mathcal{A}, \mu)$ a measure space with positive measure μ . We always assume that $0 \leq m < M < b$.

Theorem 3.1.

Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be a continuously differentiable function and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be superadditive. Let $f : \Omega \rightarrow [m, M]$ be such that $f, \varphi \circ f \in L^1(\mu)$. Then

(a) the inequality

$$\begin{aligned} \varphi(c) - 3\varphi(0) + \varphi'(c) \left(m + M - \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu - c \right) + \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi(|m + M - f - c|) \, d\mu \\ \leq \varphi(m) + \varphi(M) - \frac{1}{\mu(\Omega)} \int_{\Omega} (\varphi \circ f) \, d\mu - \bar{\Delta} \end{aligned} \quad (24)$$

holds for each $c \in [m, M]$, where

$$\bar{\Delta} = \frac{2}{M - m} \frac{1}{\mu(\Omega)} \int_{\Omega} [(f - m)\varphi(M - f) + (M - f)\varphi(f - m)] \, d\mu.$$

(b) If in addition $\varphi(0) \leq 0$, then φ is superquadratic and

$$\begin{aligned} \varphi(c) + \varphi'(c) \left(m + M - \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu - c \right) + \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi(|m + M - f - c|) \, d\mu \\ \leq \varphi(m) + \varphi(M) - \frac{1}{\mu(\Omega)} \int_{\Omega} (\varphi \circ f) \, d\mu - \bar{\Delta}. \end{aligned} \quad (25)$$

(c) If in addition $\varphi \geq 0$ and $\varphi(0) = \varphi'(0) = 0$, then φ is superquadratic and convex increasing and

$$\begin{aligned} 0 \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi(|m + M - f - c|) \, d\mu \\ \leq \varphi(m) + \varphi(M) - \frac{1}{\mu(\Omega)} \int_{\Omega} (\varphi \circ f) \, d\mu - \bar{\Delta} - \varphi(c) - \varphi'(c) \left(m + M - \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu - c \right). \end{aligned} \quad (26)$$

Proof. (a) As $0 \leq m < M < b$ it follows from (8) that for every $\lambda \in [0, 1]$

$$\varphi(\lambda m + (1 - \lambda)M) \leq \lambda\varphi(m) + (1 - \lambda)\varphi(M) + \varphi(0) - \lambda\varphi((1 - \lambda)(M - m)) - (1 - \lambda)\varphi(\lambda(M - m)) \quad (27)$$

holds.

For any $f(t) \in [m, M]$, $t \in \Omega$, there exists $\lambda_t \in [0, 1]$ such that

$$f(t) = \lambda_t m + (1 - \lambda_t)M.$$

Then, using (27), we have that

$$\varphi(f(t)) \leq \lambda_t \varphi(m) + (1 - \lambda_t) \varphi(M) - \lambda_t \varphi((1 - \lambda_t)(M - m)) - (1 - \lambda_t) \varphi(\lambda_t(M - m)) + \varphi(0) \quad (28)$$

holds for all $t \in \Omega$. Now, we have

$$\begin{aligned} \varphi(m + M - f(t)) &= \varphi((1 - \lambda_t)m + \lambda_t M) \leq (1 - \lambda_t)\varphi(m) + \lambda_t \varphi(M) - (1 - \lambda_t)\varphi(\lambda_t(M - m)) - \lambda_t \varphi((1 - \lambda_t)(M - m)) + \varphi(0) \\ &= \varphi(m) + \varphi(M) - \lambda_t \varphi(m) - (1 - \lambda_t)\varphi(M) - (1 - \lambda_t)\varphi(\lambda_t(M - m)) - \lambda_t \varphi((1 - \lambda_t)(M - m)) + \varphi(0) \\ &\leq \varphi(m) + \varphi(M) - \varphi(f(t)) - 2(1 - \lambda_t)\varphi(\lambda_t(M - m)) - 2\lambda_t \varphi((1 - \lambda_t)(M - m)) + 2\varphi(0). \end{aligned}$$

Since

$$\lambda_t = \frac{M - f(t)}{M - m}$$

and

$$1 - \lambda_t = \frac{f(t) - m}{M - m}$$

we have

$$\varphi(m + M - f(t)) \leq \varphi(m) + \varphi(M) - \varphi(f(t)) - 2\frac{f(t) - m}{M - m}\varphi(M - f(t)) - 2\frac{M - f(t)}{M - m}\varphi(f(t) - m) + 2\varphi(0). \quad (29)$$

Next, we consider the function $\psi : [0, b) \rightarrow \mathbb{R}$ defined as in Lemma 2.1 by

$$\psi(y) = \varphi(y) - \varphi(z) - \varphi'(z)(y - z) - \varphi(|y - z|) + \varphi(0), \quad z \in [0, b).$$

We know from Lemma 2.1 that ψ is nonnegative on $[0, b)$, that is

$$\varphi(y) - \varphi(z) - \varphi'(z)(y - z) - \varphi(|y - z|) + \varphi(0) \geq 0, \quad z \in [0, b). \quad (30)$$

If we set $z = c$ and $y = m + M - f(t)$ in (30), we get

$$\varphi(m + M - f(t)) - \varphi(c) - \varphi'(c)(m + M - f(t) - c) - \varphi(|m + M - f(t) - c|) + \varphi(0) \geq 0,$$

that is

$$\varphi(c) - \varphi(0) + \varphi'(c)(m + M - f(t) - c) + \varphi(|m + M - f(t) - c|) \leq \varphi(m + M - f(t)). \quad (31)$$

Then, using (29), from (31) follows

$$\begin{aligned} &\varphi(c) - 3\varphi(0) + \varphi'(c)(m + M - f(t) - c) + \varphi(|m + M - f(t) - c|) \\ &\leq \varphi(m) + \varphi(M) - (\varphi \circ f)(t) - \frac{2}{M - m}[(f(t) - m)\varphi(M - f(t)) + (M - f(t))\varphi(f(t) - m)]. \end{aligned}$$

Now, integrating over Ω and dividing by $\mu(\Omega)$ we obtain (24).

(b) Since $\varphi(0) \leq 0$, from Lemma F it follows that φ is a superquadratic function and (25) follows from (24).

(c) Since $\varphi(0) = 0$ and φ is nonnegative, then (26) holds. From Lemma F it follows that φ is a superquadratic function and then from Lemma G it follows that φ is convex increasing. $\square$

If it is supposed that μ is a discrete measure on $\Omega = \{1, 2, \dots, n\}$ defined by $\mu(\{i\}) = a_i$ and $f(i) = x_i$ for $i = 1, \dots, n$, we get a discrete version of the previous theorem.

Theorem 3.2.

Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be a continuously differentiable function and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be superadditive. Let $\mathbf{x} = (x_1, \dots, x_n)$ be an n -tuple in $[m, M]^n$, $\mathbf{a} = (a_1, \dots, a_n)$ be a positive n -tuple and $A_n = \sum_{i=1}^n a_i$. Then

(a) the inequality

$$\varphi(c) - 3\varphi(0) + \varphi'(c)(m + M - \bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|m + M - x_i - c|) \leq \varphi(m) + \varphi(M) - \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \bar{\Delta} \quad (32)$$

holds for each $c \in [m, M]$, where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$ and

$$\bar{\Delta} = \frac{2}{M - m} \frac{1}{A_n} \sum_{i=1}^n a_i [(x_i - m)\varphi(M - x_i) + (M - x_i)\varphi(x_i - m)]. \quad (33)$$

(b) If in addition $\varphi(0) \leq 0$, then φ is superquadratic and

$$\varphi(c) + \varphi'(c)(m + M - \bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|m + M - x_i - c|) \leq \varphi(m) + \varphi(M) - \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \bar{\Delta}. \quad (34)$$

(c) If in addition $\varphi \geq 0$ and $\varphi(0) = \varphi'(0) = 0$, then φ is superquadratic and convex increasing and

$$0 \leq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|m + M - x_i - c|) \leq \varphi(m) + \varphi(M) - \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) - \bar{\Delta} - \varphi(c) - \varphi'(c)(m + M - \bar{x} - c). \quad (35)$$

Remark 3.3.

It is easy to see that under the conditions of Theorem 3.2,

$$\bar{\bar{x}} = m + M - \bar{x} \in [m, M]$$

holds. If to set $\bar{\bar{x}} = c$, then from (35) we get

$$0 \leq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(|\bar{\bar{x}} - x_i|) + \bar{\Delta} + \varphi(\bar{\bar{x}}) \leq \varphi(m) + \varphi(M) - \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i). \quad (36)$$

The second inequality in (36) presents a refinement of Mercer's inequality (3) in the case $\varphi \geq 0$, $\varphi(0) = \varphi'(0) = 0$ and φ' is superadditive, as e.g. for $\varphi(x) = x^p$, $x \geq 0$, $p \geq 2$.

Now we present the Mercer's type variant of Theorem H.

Theorem 3.4.

Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be a continuously differentiable function and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be superadditive. Let ξ be a monotonic k -tuple in $[m, M]^k$ and let $\mathbf{p}$ be a real k -tuple such that $0 \leq P_j = \sum_{i=1}^j P_i \leq P_k$, $j = 1, \dots, k$, $P_k > 0$. Then

(a) the inequality

$$\begin{aligned} \varphi(m) + \varphi(M) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(\xi_i) &\geq \varphi(c) - \varphi(0) + \varphi'(c)(m + M - \bar{\xi} - c) \\ &\quad + \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(|\xi_i - c|) \end{aligned} \quad (37)$$

holds for all $c \in [m, M]$, where $\bar{\xi} = \frac{1}{P_k} \sum_{i=1}^k p_i \xi_i$.

(b) If in addition $\varphi(0) \leq 0$, then φ is superquadratic and

$$\begin{aligned} \varphi(m) + \varphi(M) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(\xi_i) &\geq \varphi(c) + \varphi'(c)(m + M - \bar{\xi} - c) \\ &\quad + \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(|\xi_i - c|). \end{aligned} \quad (38)$$

(c) If in addition $\varphi \geq 0$ and $\varphi(0) = \varphi'(0) = 0$, then φ is superquadratic and convex increasing and

$$\begin{aligned} \varphi(m) + \varphi(M) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(\xi_i) &- \varphi(c) - \varphi'(c)(m + M - \bar{\xi} - c) \\ &\geq \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(|\xi_i - c|) \geq 0. \end{aligned} \quad (39)$$

Proof. Suppose that ξ is an increasing k -tuple in $[m, M]^k$. For $n = k + 2$ we define

$$\begin{aligned} x_1 &= m, & x_2 &= \xi_1, & x_3 &= \xi_2, & \dots & x_{n-1} &= \xi_k, & x_n &= M \\ a_1 &= 1, & a_2 &= -\frac{p_1}{P_k}, & a_3 &= -\frac{p_2}{P_k}, & \dots & a_{n-1} &= -\frac{p_k}{P_k}, & a_n &= 1. \end{aligned} \quad (40)$$

It is obvious that x is an increasing n -tuple in $[m, M]^n$ and that a is a real n -tuple that satisfies

$$0 \leq A_j = \sum_{i=1}^j a_i \leq A_n, \quad j = 1, \dots, n, \quad A_n > 0.$$

Now we can apply Theorem H to φ , x and a and we obtain the inequalities (37), (38) and (39). $\square$

Remark 3.5.

Note that under the conditions of Theorem 3.4, it follows

$$\bar{\bar{\xi}} = m + M - \bar{\xi} \in [m, M].$$

If we set $\bar{\bar{\xi}} = c$, then from (39) we get

$$0 \leq \varphi(|\bar{\bar{\xi}} - M|) + \varphi(|\bar{\bar{\xi}} - m|) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(|\xi_i - \bar{\bar{\xi}}|) + \varphi(\bar{\bar{\xi}}) \leq \varphi(m) + \varphi(M) - \frac{1}{P_k} \sum_{i=1}^k p_i \varphi(\xi_i). \quad (41)$$

The second inequality in (41) presents a refinement of Mercer's inequality (3) in this special case.

Now we give the integral version of Theorem 3.4, that is the Mercer's type variant of Theorem 2.5.

Theorem 3.6.

Let $f : [\alpha, \beta] \rightarrow [m, M]$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < +\infty$. Let $\varphi : [0, b) \rightarrow \mathbb{R}$ be a continuously differentiable function and $\varphi' : [0, b) \rightarrow \mathbb{R}$ be superadditive. Let $\lambda : [\alpha, \beta] \rightarrow \mathbb{R}$ be either continuous or of bounded variation satisfying (15). Then

(a) the inequality

$$\begin{aligned} \varphi(m) + \varphi(M) - \bar{y} &\geq \varphi(c) - \varphi(0) + \varphi'(c)(m + M - \bar{x} - c) \\ &\quad + \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t) \end{aligned} \quad (42)$$

holds for each $c \in [0, b)$, where $\bar{x}$ and $\bar{y}$ are defined as in (16).

(b) If in addition $\varphi(0) \leq 0$, then φ is superquadratic and

$$\begin{aligned} \varphi(m) + \varphi(M) - \bar{y} &\geq \varphi(c) + \varphi'(c)(m + M - \bar{x} - c) \\ &\quad + \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t). \end{aligned} \quad (43)$$

(c) If in addition $\varphi \geq 0$ and $\varphi(0) = \varphi'(0) = 0$, then φ is superquadratic and convex increasing and

$$\begin{aligned} \varphi(m) + \varphi(M) - \bar{y} - \varphi(c) - \varphi'(c)(m + M - \bar{x} - c) \\ \geq \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t) \geq 0. \end{aligned} \quad (44)$$

Proof. (a) The proof is analogous to the proof in [5, Theorem 3], but here we apply it to our function φ for which φ' is superadditive.

Suppose that f is nondecreasing (for f nonincreasing the proof is analogous). For $\alpha_1, \beta_1 \in \mathbb{R}$, such that $\alpha_1 < \alpha$ and $\beta_1 > \beta$, we define a function $g : [\alpha_1, \beta_1] \rightarrow [m, M]$ by

$$g(t) = \begin{cases} m + \frac{f(\alpha) - m}{\alpha - \alpha_1}(t - \alpha_1) & \text{for } t \in [\alpha_1, \alpha], \\ f(t) & \text{for } t \in [\alpha, \beta], \\ M + \frac{M - f(\beta)}{\beta_1 - \beta}(t - \beta_1) & \text{for } t \in [\beta, \beta_1]. \end{cases}$$

It is obvious that the function g is continuous and nondecreasing.

Next we define the functions $\lambda_1 : [\alpha_1, \beta_1] \rightarrow \mathbb{R}$ by

$$\lambda_1(t) = \begin{cases} 1 & \text{for } t = \alpha_1, \\ 0 & \text{for } t \in (\alpha_1, \beta_1), \\ -1 & \text{for } t = \beta_1, \end{cases}$$

and $\lambda_2 : [\alpha_1, \beta_1] \rightarrow \mathbb{R}$ by

$$\lambda_2(t) = \begin{cases} 1 & \text{for } t \in [\alpha_1, \alpha], \\ \frac{\lambda(\beta) - \lambda(t)}{\lambda(\beta) - \lambda(\alpha)} & \text{for } t \in [\alpha, \beta], \\ 0 & \text{for } t \in [\beta, \beta_1]. \end{cases}$$

If $h : [\alpha_1, \beta_1] \rightarrow \mathbb{R}$ is any function that is continuous at the points α_1 and β_1 , we get

$$\int_{\alpha_1}^{\beta_1} h(t) d\lambda_1(t) = -h(\alpha_1) - h(\beta_1). \quad (45)$$

Also we get

$$\int_{\alpha_1}^{\beta_1} h(t) d\lambda_2(t) = -\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} h(t) d\lambda(t). \quad (46)$$

Now we define the function $\tilde{\lambda} : [\alpha_1, \beta_1] \rightarrow \mathbb{R}$ by

$$\tilde{\lambda}(t) = \lambda_2(t) - \lambda_1(t).$$

Then from (45) and (46) it follows that the integral $\int_{\alpha_1}^{\beta_1} h(t) d\tilde{\lambda}(t)$ is well defined and we have

$$\int_{\alpha_1}^{\beta_1} h(t) d\tilde{\lambda}(t) = -\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} h(t) d\lambda(t) + h(\alpha_1) + h(\beta_1) \quad (47)$$

and

$$\tilde{\lambda}(\beta_1) - \tilde{\lambda}(\alpha_1) = 1.$$

If we apply Theorem 2.5 to the functions g , φ and $\tilde{\lambda}$ we obtain

$$\varphi(c) - \varphi(0) + \varphi'(c)(\tilde{x} - c) + \frac{1}{\tilde{\lambda}(\beta_1) - \tilde{\lambda}(\alpha_1)} \int_{\alpha_1}^{\beta_1} \varphi(|g(t) - c|) d\tilde{\lambda}(t) \leq \tilde{y}, \quad (48)$$

where

$$\tilde{x} = \frac{1}{\tilde{\lambda}(\beta_1) - \tilde{\lambda}(\alpha_1)} \int_{\alpha_1}^{\beta_1} g(t) d\tilde{\lambda}(t) = -\bar{x} + m + M$$

and

$$\tilde{y} = \frac{1}{\tilde{\lambda}(\beta_1) - \tilde{\lambda}(\alpha_1)} \int_{\alpha_1}^{\beta_1} \varphi(g(t)) d\tilde{\lambda}(t) = -\bar{y} + \varphi(m) + \varphi(M).$$

Since

$$\begin{aligned} \frac{1}{\tilde{\lambda}(\beta_1) - \tilde{\lambda}(\alpha_1)} \int_{\alpha_1}^{\beta_1} \varphi(|g(t) - c|) d\tilde{\lambda}(t) &= -\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t) + \varphi(|g(\alpha_1) - c|) + \varphi(|g(\beta_1) - c|) \\ &= \varphi(|m - c|) + \varphi(|M - c|) - \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} \varphi(|f(t) - c|) d\lambda(t), \end{aligned}$$

then from (48) follows the inequality (42).

(b) Since $\varphi(0) \leq 0$, from Lemma F it follows that φ is a superquadratic function and from (42) follows (43).

(c) Since $\varphi(0) = 0$, from Lemma F it follows that φ is superquadratic. From (42) follows the first inequality in (44). By our assumption in this case, the function φ is nonnegative, so the second inequality in (44) holds. $\square$

Remark 3.7.

The function φ from the previous theorem in case (c) is also convex (see Lemma E). Therefore, in this case (44) is an integral refinement of Theorem A.

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