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Descriptive properties of density preserving autohomeomorphisms of the unit interval

Research Article

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Abstract: We prove that density preserving homeomorphisms form a Π_1^1-complete subset in the Polish space $\mathbb{H}$ of all increasing autohomeomorphisms of unit interval.

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In descriptive set theory the following phenomenon is known – sets with a simple description can be extremely complex, for example they can be Π_1^1-complete. Many classical examples of such sets can be found in Kechris' monograph [4]. They appear naturally in topology, in the theory of Banach spaces, the theory of real functions, and in other branches of mathematics.

We consider the set of all density preserving homeomorphisms of the unit interval. Density preserving homeomorphisms play an important role in real analysis. They first appeared in Bruckner's paper [1] where the author studied questions related to changes of variable with respect to approximately continuous functions. Some structural properties of density preserving homeomorphisms were proved in [5]; in that paper density preserving homeomorphisms on the real plane were also considered. Ostaszewski in [6] investigated connections between homeomorphisms preserving density point and $\mathcal{D}$-continuous functions, i.e. continuous mappings with the domain and range furnished with the density topology. The Baire category analogs of density preserving homeomorphisms, namely $\mathcal{I}$-density preserving homeomorphisms, were considered in [2].

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In the following, we prove that the set of all density preserving homeomorphisms of the unit interval is Π_1^1-complete. Descriptive properties of other classes of homeomorphisms of the unit interval were investigated in [3].

This paper is organized as follows. In Section 1 we give basic definitions and facts. In Section 2 we present the main theorem which specifies descriptive set theoretical complexity of density preserving homeomorphisms.

1. Background information

We use standard set-theoretic notation. For the descriptive set-theoretical background we refer the reader to [4]. By $\mathbb{H} \subset C[0,1]$ we denote the set of all increasing autohomeomorphisms of $[0,1]$. It is easy to see that $\mathbb{H}$ is a G_δ subset of $C[0,1]$ and hence it is a Polish space.

Let μ be the Lebesgue measure on $\mathbb{R}$. For a measurable set $E \subset \mathbb{R}$ and a point $x \in \mathbb{R}$, by $d^+(x,E)$ we denote the right-hand Lebesgue density of the set E at x, i.e. the number $d^+(x,E) = \lim_{h\to 0^+} \frac{\mu([x,x+h]\cap E)}{h}$, provided this limit exists. Analogously we define $d^-(x,E)$. Finally by $d(x,E)$ we denote the density of E at x, i.e. the limit

$$d(x,E) = \lim_{h\to 0^+} \frac{\mu([x-h,x+h]\cap E)}{2h}.$$

If $d(x,E) = 1$, then we say that x is a density point of E. If $d^\pm(x,E) = 1$, then we say that x is a one-sided density point of E.

A homeomorphism $h \in \mathbb{H}$ preserves density at $x \in [0,1]$, provided, for every measurable set S, $h(x)$ is a density point of the set $h(S)$ whenever x is a density point of S. If $h \in \mathbb{H}$ preserves density at every point of $[0,1]$, then we say that h preserves density points. The set of all density preserving homeomorphisms in $\mathbb{H}$ is denoted by $DP\mathbb{H}$.

To characterize density preserving homeomorphisms we need the notion of an interval set. A set S is called an interval set at a point x if there exist sequences (x_n) and (y_n) such that $x_n \to x$ and $y_1 < x_1 < y_2 < x_2 < \dots < x$ or $x_1 > y_1 > x_2 > y_2 > \dots > x$ such that $S = \bigcup_{n\in\mathbb{N}}[y_n, x_n]$.

It can be easily seen that if $S = \bigcup_{n\in\mathbb{N}}[y_n, x_n]$ and $x_1 > y_1 > x_2 > y_2 > \dots > x$, then

$$d^+(x,S) = 1 \quad \text{if and only if} \quad \frac{\sum_{k=n+1}^{\infty}(x_k - y_k)}{y_n - x} \to 1$$

and if $S = \bigcup_{n\in\mathbb{N}}[y_n, x_n]$ and $y_1 < x_1 < y_2 < x_2 < \dots < x$, then

$$d^-(x,S) = 1 \quad \text{if and only if} \quad \frac{\sum_{k=n+1}^{\infty}(x_k - y_k)}{x - x_n} \to 1.$$

We will need the following facts regarding density preserving homeomorphisms taken from [1].

Theorem 1.1.

If h is a homeomorphism of $[0,1]$ onto itself which preserves density points, then h is absolutely continuous.

Theorem 1.2.

Let h be an absolutely continuous homeomorphism of $[0,1]$ onto itself. A necessary and sufficient condition for h to preserve density points is that h preserves one-sided density points of every interval set.

Theorem 1.3.

If h is a continuously differentiable homeomorphism of $[0,1]$ onto itself and the derivative h' never vanishes, then h preserves density points.

Let X be a Polish space. A subset A of X is called *analytic* if it is the projection of a Borel subset B of $X \times X$. A subset C of X is called *coanalytic* if $X \setminus C$ is analytic. The pointclasses of analytic and coanalytic sets are denoted by $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, respectively. A set $C \subset X$ is called $\mathbf{\Pi}^1_1$*-hard* if for every zero-dimensional Polish space Y and every coanalytic set $B \subset Y$ there is a continuous function $f : Y \to X$ such that $f^{-1}(C) = B$. A set is called $\mathbf{\Pi}^1_1$*-complete* if it is $\mathbf{\Pi}^1_1$-hard and coanalytic.

Let A be any set and let $\mathbb{N}$ stand for the set of all nonnegative integers. By $A^{<\mathbb{N}}$ we denote the set of all finite sequences of elements from A. For a sequence $s = (s(0), s(1), ..., s(k-1)) \in A^{<\mathbb{N}}$ and $m \in A$ let $|s| = k$ be the length of s, and let $s\hat{\ }m = (s(0), s(1), ..., s(k-1), m)$ denote the concatenation of s and m; in a similar way we define the concatenation of two finite sequences. For a sequence $\alpha \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\alpha|n = (\alpha(0), \alpha(1), ..., \alpha(n-1)) \in A^{<\mathbb{N}}$. Similarly for $s \in A^{<\mathbb{N}}$ and $n \leq |s|$, let $s|n = (s(0), s(1), ..., s(n-1))$ (additionally $s|0 = \emptyset$, where $\emptyset$ is the empty sequence). A set $T \subset A^{<\mathbb{N}}$ is called a *tree* if for every $s \in T$ and every $n \leq |s|$ we have $s|n \in T$, in particular each tree contains the empty sequence $\emptyset$. We will use $\emptyset$ to denote the empty set and the empty sequence, but it will never lead to confusion. For any tree T define its body by $[T] = \{\alpha \in A^{\mathbb{N}} : \forall n\ \alpha|n \in T\}$. By PTr_2 we denote the set of all pruned trees on $\{0,1\}$ (a tree T on A is *pruned* if for every $s \in T$ there is $m \in A$ with $s\hat{\ }m \in T$). Let $WF^*_2 = \{T \in PTr_2 : [T] \cap N = \emptyset\}$, $IF^*_2 = PTr_2 \setminus WF^*_2$, where $N = \left\{\alpha \in \{0,1\}^{\mathbb{N}} : \exists^\infty_n \alpha(n) = 1\right\}$ (where for brevity $\exists^\infty_n$ denotes "infinitely many n" and $\forall^\infty_n$ denotes "for all but finitely many n"). It is well known (cf. [4]) that WF^*_2 is $\mathbf{\Pi}^1_1$-complete.

Let A be subset of a Polish space X and let C, D be disjoint subsets of a Polish space Y. By $A \leq_W (C, D)$ we mean that there is a continuous map $f : X \to Y$ with $f^{-1}(C) = A$ and $f^{-1}(D) = X \setminus A$. Clearly, if A is $\mathbf{\Pi}^1_1$-complete and $A \leq_W (C, D)$, then C is $\mathbf{\Pi}^1_1$-hard.

2. Density preserving homeomorphisms

Fix two decreasing sequences (α_n) and (β_n) of positive real numbers tending to 0 with $\alpha_1 < 1/4$ and $\beta_n/\alpha_n \to 0$.

We define Cantor schemes of closed intervals $\left\{I_s : s \in \{0,1\}^{<\mathbb{N}}\right\}$, $\left\{IL_s : s \in \{0,1\}^{<\mathbb{N}} \setminus \{\emptyset\}\right\}$, $\left\{IR_s : s \in \{0,1\}^{<\mathbb{N}} \setminus \{\emptyset\}\right\}$ by recursion with respect to length $n = |s|$ of s as follows:

(i) $I_\emptyset = [0,1]$;

(ii) Let $I_s = [a_s, b_s]$. Then $I_{s\hat{\ }0}$ and $I_{s\hat{\ }1}$ have the length $\frac{1}{2}\alpha_{n+1}|I_s|$ and they have the common centers with the left and the right halves of I_s, respectively;

(iii) Let $I_s = [a_s, b_s]$. Then $IL_s = [c_s, a_s]$ and $IR_s = [b_s, d_s]$ are such that $|IL_s| = |IR_s| = \alpha_n|I_s|$.

Note that $\bigcup_{\gamma\in\{0,1\}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} I_{\gamma|n} = \bigcap_{n=1}^\infty \bigcup_{|s|=n} I_s$ is a perfect Lebesgue null subset of $[0,1]$.

Now, for every $T \in PTr_2$ we will define a sequence of continuous functions (f^T_n). For this purpose fix $T \in PTr_2$. Let f^T_1 be a continuous function with

$$f^T_1(0) = f^T_1(c_{(0)}) = f^T_1(d_{(0)}) = f^T_1(c_{(1)}) = f^T_1(d_{(1)}) = f^T_1(1) = 1,$$
$$f^T_1(x) = \beta_1 \quad \text{for} \quad x \in I_{(0)} \cup I_{(1)},$$

and piece-wise linear elsewhere on $[0,1]$. To define f^T_{n+1} for general n, we modify f^T_n on each interval I_s with $s \in T$, $|s| = n$ and $s(n-1) = 1$. On $I_s = [a_s, b_s]$ we can then define a continuous function f^T_{n+1} with

$$f^T_{n+1}(a_s) = f^T_{n+1}(c_{s\hat{\ }0}) = f^T_{n+1}(d_{s\hat{\ }0}) = f^T_{n+1}(c_{s\hat{\ }1}) = f^T_{n+1}(d_{s\hat{\ }1}) = f^T_{n+1}(b_s) = f^T_n(b_s),$$
$$f^T_{n+1}(x) = \beta_{n+1} f^T_n(b_s) \quad \text{for} \quad x \in I_{s\hat{\ }0} \cup I_{s\hat{\ }1},$$

and piece-wise linear elsewhere on I_s. On the rest of $[0,1]$, a function f^T_{n+1} remains unchanged, i.e. $f^T_{n+1}(x) = f^T_n(x)$ for every point $x \in [0,1] \setminus \bigcup\left\{I_s : s \in T, |s| = n, s(n-1) = 1\right\}$. Since for every $x \in [0,1]$, the sequence $(f^T_n(x))$ is nonincreasing, the sequence (f^T_n) is pointwise and monotonically convergent to some function f^T.

Now if $f:[0,1]\to\mathbb{R}$ is Lebesgue integrable, let $\|f\|_{L_1}$ denote $\int_0^1|f(t)|dt$. Recall that $N=\{\gamma\in\{0,1\}^{\mathbb{N}}:$ γ has infinitely many 1's$\}$.

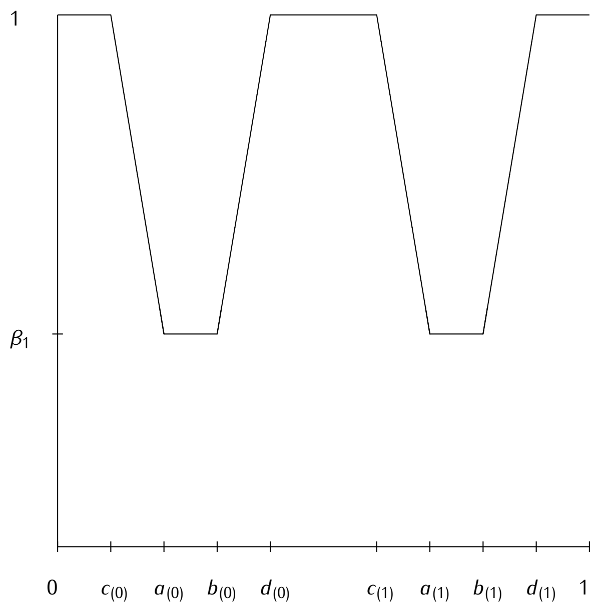

Figure 1. Graph of f_1^T.

Lemma 2.1.

The following statements hold:

(i) for every x, $f^T(x)=0$ if and only if $x\in\bigcup_{\gamma\in[T]\cap N}\bigcap I_{\gamma|n}$;

(ii) f^T is Lebesgue integrable;

(iii) $\lim\limits_{n\to\infty}\left\|f_n^T-f^T\right\|_{L_1}=0$ uniformly on PTr_2;

(iv) the mapping $T\mapsto\left\|f^T\right\|_{L_1}$ is continuous.

Proof. Parts (i) and (ii) follow directly from the construction. For (iii), if $T\in PTr_2$ and $n\in\mathbb{N}$, then f^T and f_n^T can differ only on the set $\bigcup_{|s|=n}I_s$. Since $\lim_{n\to\infty}\sum_{|s|=n}|I_s|=0$, the result follows. For (iv), if $S,T\in PTr_2$ are such that $\{s\in S:|s|<n\}=\{s\in T:|s|<n\}$, then f^T and f^S can differ only on the set $\bigcup_{|s|=n}I_s$. Thus we get (iv). $\square$

Now, for every $T\in PTr_2$ and $x\in[0,1]$, we put

$$g^T(x)=\frac{1}{\|f^T\|_{L_1}}\int_0^x f^T(t)dt.$$

By Lemma 2.1(ii), g^T is absolutely continuous. Moreover, by Lemma 2.1(i), g^T is strictly increasing, and hence $g^T\in\mathbb{H}$.

Lemma 2.2.

The mapping $T\mapsto g^T$ is continuous.

Proof. For every $S \in PTr_2$, $n \in \mathbb{N}$ and $x \in [0,1]$, we define

$$g_n^T(x) = \frac{1}{\|f^T\|_{L_1}} \int_0^x f_n^T(t)dt.$$

Fix any $\varepsilon > 0$ and $T \in PTr_2$. By Lemma 2.1(iii) and since there exists $A > 0$ such that $\|f^S\|_{L_1} \geq A$ for every $S \in PTr_2$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $S \in PTr_2$,

$$\|g^S - g_n^S\|_{\sup} < \varepsilon/3$$

($\|\cdot\|_{\sup}$ denotes the supremum norm). Moreover, by Lemma 2.1(iv), the mapping $T \mapsto \frac{1}{\|f^T\|_{L_1}}$ is continuous and hence there exists $n_1 \in \mathbb{N}$ such that for every $S \in PTr_2$ and $n \geq n_1$, if $T \cap \{s : |s| < n\} = S \cap \{s : |s| < n\}$, then

$$\left| \frac{1}{\|f^T\|_{L_1}} - \frac{1}{\|f^S\|_{L_1}} \right| < \frac{\varepsilon}{3}.$$

Set $n' = \max\{n_0, n_1\} + 1$. Then for every $S \in PTr_2$ with $T \cap \{s : |s| < n'\} = S \cap \{s : |s| < n'\}$ we have that $f_{n'}^T = f_{n'}^S$, and therefore

$$\|g^T - g^S\|_{\sup} \leq \|g^T - g_{n'}^T\|_{\sup} + \|g_{n'}^T - g_{n'}^S\|_{\sup} + \|g_{n'}^S - g^S\|_{\sup} \leq \frac{\varepsilon}{3} + \|f_{n'}^T\|_{L_1} \left| \frac{1}{\|f^T\|_{L_1}} - \frac{1}{\|f^S\|_{L_1}} \right| + \frac{\varepsilon}{3} < \varepsilon.$$

□

Theorem 2.3.
$WF_2^* \leq_W (DP\mathbb{H}, \mathbb{H} \setminus DP\mathbb{H})$ and hence $DP\mathbb{H}$ is Π_1^1-hard.

Proof. It suffices to prove that for every $T \in PTr_2$,

$$T \in WF_2^* \quad \text{if and only if} \quad g^T \in DP\mathbb{H}.$$

Let $T \in PTr_2$ and $x \in [0,1]$. If $x \notin \bigcup_{\gamma \in \{0,1\}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} I_{\gamma|n}$ or $x \in \bigcup_{\gamma \in \{0,1\}^{\mathbb{N}} \setminus [T]} \bigcap_{n \in \mathbb{N}} I_{\gamma|n}$, the construction of the sequence (f_n^T) stops at some neighborhood U of x. Hence g^T is continuously differentiable in U and $(g^T)'(y) = \frac{f^T(y)}{\|f^T\|_{L_1}} = \frac{f_n^T(y)}{\|f^T\|_{L_1}} > 0$ for every $y \in U$ and some $n \in \mathbb{N}$. By Theorem 1.3, $g^T_{|U}$ preserves density points. Since $x \in U$ and U is open, g^T preserves density at x.

Now, assume that $x \in \bigcup_{\gamma \in ([T] \setminus N)} \bigcap_{n \in \mathbb{N}} I_{\gamma|n}$. We will show that g^T preserves density at x for every interval set. Let $\gamma \in ([T] \setminus N)$ be such that $x \in \bigcap_{n \in \mathbb{N}} I_{\gamma|n}$ and let $n_0 \in \mathbb{N}$ be such that $\gamma(n-1) = 0$ for every $n \geq n_0$. It is easy to see that there exists $\beta > 0$ such that $f^T \equiv \beta$ on the set

$$I_{\gamma|n_0} \setminus \bigcup_{n \geq n_0} \left(IL_{(\gamma|n)^\frown 1^\frown 0} \cup I_{(\gamma|n)^\frown 1^\frown 0} \cup IR_{(\gamma|n)^\frown 1^\frown 0} \cup IL_{(\gamma|n)^\frown 1^\frown 1} \cup I_{(\gamma|n)^\frown 1^\frown 1} \cup IR_{(\gamma|n)^\frown 1^\frown 1} \right).$$

Now let $M = \bigcup_{n \in \mathbb{N}} [y_n, x_n]$ be any interval set at x. We need to consider two cases:

Case 1. $y_1 < x_1 < x_2 < y_2 < \ldots < x$ and $d^-(x, M) = 1$. Then f^T is constant on $[a_{\gamma|n_0}, x)$, which easily implies $d^-(g^T(x), g^T(M)) = 1$.

Case 2. $x_1 > y_1 > x_2 > y_2 > \ldots > x$ and $d^+(x, M) = 1$. Since g^T is increasing, we only have to prove that

$$\lim_{n \to \infty} \frac{\sum_{k \geq n+1} (g^T(x_k) - g^T(y_k))}{g^T(y_n) - g^T(x)} = 1.$$

For every $n \in \mathbb{N}$, let k_n be such that $y_n \in I_{\gamma|k_n} \setminus I_{\gamma|(k_n+1)}$. Then there exists $n_1 \in \mathbb{N}$ such that $k_n \geq n_0$ for every $n \geq n_1$. Fix $n \geq n_1$. Then f^T equals β on the set

$$I_{\gamma|k_n} \setminus \bigcup_{i=0,1} \bigcup_{j=0,1} \left(IL_{(\gamma|k_n)\hat{}i\hat{}j} \cup I_{(\gamma|k_n)\hat{}i\hat{}j} \cup IR_{(\gamma|k_n)\hat{}i\hat{}j} \right).$$

Hence

$$\begin{aligned}\sum_{k=n+1}^{\infty} \left(g^T(x_k) - g^T(y_k) \right) &\geq \beta \left(\sum_{k=n+1}^{\infty} (x_k - y_k) - 4\left|I_{(\gamma|k_n)\hat{}0\hat{}0}\right| - 8\left|IL_{(\gamma|k_n)\hat{}0\hat{}0}\right| \right) \\ &\geq \beta \left(\sum_{k=n+1}^{\infty} (x_k - y_k) - 12\left|I_{(\gamma|k_n)\hat{}0\hat{}0}\right| \right) \geq \beta \left(\sum_{k=n+1}^{\infty} (x_k - y_k) - 6\alpha_{k_n+2}\left|I_{(\gamma|k_n)\hat{}0}\right| \right)\end{aligned}$$

and

$$g^T(y_n) - g^T(x) \leq \beta\,(y_n - x)\,.$$

We also have that

$$y_n - x \geq \left(b_{\gamma|(k_n+1)} - b_{\gamma|(k_n+2)} \right) \geq \frac{1}{2}\left|I_{\gamma|(k_n+1)}\right|.$$

Therefore we obtain

$$\begin{aligned}\frac{\sum_{k=n+1}^{\infty}(g^T(x_k) - g^T(y_k))}{g^T(y_n) - g^T(x)} &\geq \frac{\sum_{k=n+1}^{\infty}(x_k - y_k) - 6\alpha_{k_n+2}|I_{(\gamma|k_n)\hat{}0}|}{y_n - x} \\ &= \frac{\sum_{k=n+1}^{\infty}(x_k - y_k)}{(y_n - x)} - 6\frac{\alpha_{k_n+2}|I_{(\gamma|k_n)\hat{}0}|}{(y_n - x)} \geq \frac{\sum_{k\geq n+1}(x_k - y_k)}{(y_n - x)} - 6\frac{\alpha_{k_n+2}|I_{(\gamma|k_n)\hat{}0}|}{\frac{1}{2}|I_{(\gamma|k_n)\hat{}0}|}.\end{aligned}$$

Since $k_n \to \infty$, if $n \to \infty$, then

$$\lim_{n\to\infty} \frac{\sum_{k\geq n+1}(g^T(x_k) - g^T(y_k))}{g^T(y_n) - g^T(x)} = 1.$$

This shows that if $T \in WF_2^*$, then g^T preserves density points for interval sets, hence and by Theorem 1.2, g^T preserves density points.

Now, let $T \in IF_2^*$ and let $\gamma \in [T]$ be a sequence with infinitely many 1's. Define sequences (x_k) and (y_k) in the following way: $x_k = b_{\gamma|k}$ and $y_k = d_{\gamma|(k+1)}$ for $k = 1, 2, \ldots$, and let x be the unique element of $\bigcap_{n\in\mathbb{N}} I_{\gamma|n}$. Clearly, $x_1 > y_1 > x_2 > y_2 > \ldots > x$ and $x_n \to x$. It is enough to show that

(i) x is a right-sided density point of $\bigcup_{k\in\mathbb{N}}[y_k, x_k]$;

(ii) $g^T(x)$ is not a right-sided density point of $\bigcup_{k\in\mathbb{N}}\left[g^T(y_k), g^T(x_k)\right]$.

To prove (i) it is enough to show that

$$\frac{x_k - y_k}{y_{k-1} - x} \to 1.$$

Let $k \geq 2$. If $\gamma(k) = 1$, then

$$\begin{aligned}x_k - y_k = b_{\gamma|k} - d_{(\gamma|k)\hat{}1} &= \frac{1}{4}\left(\left|I_{\gamma|k}\right| - 2\left|I_{\gamma|(k+1)}\right| - 4\left|IL_{\gamma|(k+1)}\right|\right) \\ &= \frac{1}{4}\left(\left|I_{\gamma|k}\right| - 2\frac{1}{2}\alpha_{k+1}\left|I_{\gamma|k}\right| - 4\frac{1}{2}\alpha_{k+1}^2\left|I_{\gamma|k}\right|\right) = \frac{1}{4}\left|I_{\gamma|k}\right|\left(1 - \alpha_{k+1} - 2\alpha_{k+1}^2\right)\end{aligned}$$

and

$$\begin{aligned} y_{k-1} - x \le y_{k-1} - a_{(\gamma|k)^\frown 1} &= d_{\gamma|k} - a_{(\gamma|k)^\frown 1} \\ &= d_{\gamma|k} - b_{\gamma|k} + b_{\gamma|k} - d_{(\gamma|k)^\frown 1} + d_{(\gamma|k)^\frown 1} - b_{(\gamma|k)^\frown 1} + b_{(\gamma|k)^\frown 1} - a_{(\gamma|k)^\frown 1} \\ &= \alpha_k |I_{\gamma|k}| + \frac{1}{4}|I_{\gamma|k}|\left(1 - \alpha_{k+1} - 2\alpha_{k+1}^2\right) + \frac{1}{2}\alpha_{k+1}^2 |I_{\gamma|k}| + \frac{1}{2}\alpha_{k+1}|I_{\gamma|k}| = |I_{\gamma|k}|\left(\alpha_k + \frac{1}{4}\alpha_{k+1} + \frac{1}{4}\right). \end{aligned}$$

Hence

$$\frac{x_k - y_k}{y_{k-1} - x} \ge \frac{|I_{\gamma|k}|\left(\frac{1}{4} - \frac{1}{4}\alpha_{k+1} - \frac{1}{2}\alpha_{k+1}^2\right)}{|I_{\gamma|k}|\left(\frac{1}{4} + \frac{1}{4}\alpha_{k+1} + \alpha_k\right)} = \frac{1 - \alpha_{k+1} - 2\alpha_{k+1}^2}{1 + \alpha_{k+1} + 4\alpha_k}.$$

If $\gamma(k) = 0$, then

$$\begin{aligned} x_k - y_k = b_{\gamma|k} - d_{(\gamma|k)^\frown 0} &= b_{\gamma|k} - \frac{a_{\gamma|k} + b_{\gamma|k}}{2} + \frac{a_{\gamma|k} + b_{\gamma|k}}{2} - d_{(\gamma|k)^\frown 0} \\ &= \frac{1}{2}|I_{\gamma|k}| + \frac{1}{4}|I_{\gamma|k}|\left(1 - \alpha_{k+1} - 2\alpha_{k+1}^2\right) = \frac{1}{4}|I_{\gamma|k}|\left(3 - \alpha_{k+1} - 2\alpha_{k+1}^2\right) \end{aligned}$$

and

$$\begin{aligned} y_{k-1} - x \le d_{\gamma|k} - a_{(\gamma|k)^\frown 0} &= d_{\gamma|k} - a_{(\gamma|k)^\frown 1} + a_{(\gamma|k)^\frown 1} - c_{(\gamma|k)^\frown 1} + c_{(\gamma|k)^\frown 1} - d_{(\gamma|k)^\frown 0} + d_{(\gamma|k)^\frown 0} - b_{(\gamma|k)^\frown 0} + b_{(\gamma|k)^\frown 0} - a_{(\gamma|k)^\frown 0} \\ &= \frac{1}{4}|I_{\gamma|k}|(1 + \alpha_{k+1} + 4\alpha_k) + \frac{1}{2}|I_{\gamma|k}|\alpha_{k+1}^2 + 2\frac{1}{4}|I_{\gamma|k}|\left(1 - \alpha_{k+1} - 2\alpha_{k+1}^2\right) + \frac{1}{2}|I_{\gamma|k}|\alpha_{k+1}^2 + \frac{1}{2}|I_{\gamma|k}|\alpha_{k+1} \\ &= \frac{1}{4}|I_{\gamma|k}|(3 + \alpha_{k+1} + 4\alpha_k). \end{aligned}$$

Hence

$$\frac{x_k - y_k}{y_{k-1} - x} \ge \frac{3 - \alpha_{k+1} - 2\alpha_{k+1}^2}{3 + \alpha_{k+1} + 4\alpha_k}.$$

Since $\alpha_n \to 0$, then $\frac{x_k - y_k}{y_{k-1} - x} \to 1$.

To prove (ii) fix an increasing sequence (n_k) of natural numbers with $\gamma(n_k - 1) = 1$, $k \in \mathbb{N}$, and let $h_k = g^T(y_{n_k}) - g^T(x)$, $k = 1, 2, \dots$ Then

$$\frac{\mu\left(\bigcup_{n\in\mathbb{N}}[y_n, x_n] \cap [g^T(x), g^T(x) + h_k]\right)}{h_k} = \frac{\sum_{l \ge n_k+1}\left(g^T(x_l) - g^T(y_l)\right)}{g^T(y_{n_k}) - g^T(x)}.$$

Moreover,

$$\frac{\sum_{l \ge n_k+1}\left(g^T(x_l) - g^T(y_l)\right)}{g^T(y_{n_k}) - g^T(x)} + \frac{g^T(y_{n_k}) - g^T(x_{n_k+1})}{g^T(y_{n_k}) - g^T(x)} \le 1,$$

so it suffices to show that

$$\lim_{k\to\infty} \frac{g^T(y_{n_k}) - g^T(x_{n_k+1})}{g^T(y_{n_k}) - g^T(x)} = 1.$$

Let $k \in \mathbb{N}$. We have

$$\frac{g^T(y_{n_k}) - g^T(x_{n_k+1})}{g^T(y_{n_k}) - g^T(x)} = \frac{\int_0^{y_{n_k}} f^T(t)dt - \int_0^{x_{n_k+1}} f^T(t)dt}{\int_0^{y_{n_k}} f^T(t)dt - \int_0^x f^T(t)dt} = \frac{\int_{x_{n_k+1}}^{y_{n_k}} f^T(t)dt}{\int_x^{y_{n_k}} f^T(t)dt} \ge \frac{\int_{x_{n_k+1}}^{y_{n_k}} f^T(t)dt}{\int_{a_{\gamma|(n_k+1)}}^{y_{n_k}} f^T(t)dt}.$$

Note that $\left[x_{n_k+1}, y_{n_k}\right] = IR_{\gamma|(n_k+1)}$ and f^T on $IR_{\gamma|(n_k+1)}$ is linear with $f^T\left(x_{n_k+1}\right) = \beta_{n_k+1} f^T\left(y_{n_k}\right)$. Note also that $\left[a_{\gamma|(n_k+1)}, x_{n_k+1}\right] = I_{\gamma|(n_k+1)}$ and f^T on $I_{\gamma|(n_k+1)}$ is less than or equal to $f^T\left(x_{n_k+1}\right)$. Using this we obtain

$$\begin{aligned} \int_{a_{\gamma|(n_k+1)}}^{y_{n_k}} f^T(t)dt &= \int_{a_{\gamma|(n_k+1)}}^{x_{n_k+1}} f^T(t)dt + \int_{x_{n_k+1}}^{y_{n_k}} f^T(t)dt \le f^T\left(x_{n_k+1}\right)\left|I_{\gamma|(n_k+1)}\right| + \frac{1}{2}\left(f^T\left(x_{n_k+1}\right) + f^T\left(y_{n_k}\right)\right)\left|IR_{\gamma|(n_k+1)}\right| \\ &= \beta_{n_k+1} f^T\left(y_{n_k}\right)\left|I_{\gamma|(n_k+1)}\right| + \frac{1}{2}\left(\beta_{n_k+1} f^T\left(y_{n_k}\right) + f^T\left(y_{n_k}\right)\right)\alpha_{n_k+1}\left|I_{\gamma|(n_k+1)}\right| \end{aligned}$$

and

$$\int_{x_{n_k+1}}^{y_{n_k}} f^T(t)dt \geq \frac{1}{2} f^T(y_{n_k}) |IR_{y|(n_k+1)}| = \frac{1}{2} f^T(y_{n_k}) \alpha_{n_k+1} |I_{y|(n_k+1)}|.$$

Hence for any $k \in \mathbb{N}$

$$\frac{g^T(y_{n_k}) - g^T(x_{n_k+1})}{g^T(y_{n_k}) - g^T(x)} \geq \frac{\alpha_{n_k+1}}{\alpha_{n_k+1}(1+\beta_{n_k+1}) + 2\beta_{n_k+1}} = \frac{1}{1+\beta_{n_k+1} + 2\frac{\beta_{n_k+1}}{\alpha_{n_k+1}}}.$$

Since $\frac{\beta_{n_k+1}}{\alpha_{n_k+1}} \to 0$, we get

$$\frac{g^T(y_{n_k}) - g^T(x_{n_k+1})}{g^T(y_{n_k}) - g^T(x)} \to 1.$$

□

Corollary 2.4.

$DP\mathbb{H}$ is $\mathbf{\Pi}^1_1$-complete.

Proof. By Theorem 2.3 it is enough to prove that $DP\mathbb{H}$ is coanalytic. As usual c_0 is the Banach space of all sequences tending to 0 with supremum norm. Let c_0^+ denote the set of all strictly decreasing sequences from c_0, and let c_0^- denote the set of all strictly increasing sequences from c_0. Then c_0^+ and c_0^-, as G_δ subsets of c_0, are Polish spaces. By Theorem 1.1 and Theorem 1.2 it follows that $f \in DP\mathbb{H}$ if and only if

$$\forall (a_n) \in c_0^+ \ \forall x \in [0,1) \ \left\{ d^+\left(x, \bigcup_{n\in\mathbb{N}}[x+a_{2n}, x+a_{2n-1}]\right) = 1 \ \Rightarrow \ d^+\left(f(x), f\left(\bigcup_{n\in\mathbb{N}}[x+a_{2n}, x+a_{2n-1}]\right) = 1\right)\right\},$$

$$\forall (a_n) \in c_0^- \ \forall x \in (0,1] \ \left\{ d^-\left(x, \bigcup_{n\in\mathbb{N}}[x+a_{2n-1}, x+a_{2n}]\right) = 1 \ \Rightarrow \ d^-\left(f(x), f\left(\bigcup_{n\in\mathbb{N}}[x+a_{2n-1}, x+a_{2n}]\right) = 1\right)\right\},$$

and f is absolutely continuous. Note that

$$d^+\left(0, \bigcup [a_{2n}, a_{2n-1}]\right) = 1 \iff \lim_{h\to 0} \frac{\lambda(\bigcup[a_{2n}, a_{2n-1}] \cap [0,h])}{h} = 1 \iff \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{a_{2k+1} - a_{2k+2}}{a_{2n}} = 1 \iff$$

$$\iff \forall t \in \mathbb{N} \ \exists n_0 \ \forall n \geq n_0 \quad \sum_{k=n}^{\infty} \frac{a_{2k+1} - a_{2k+2}}{a_{2n}} > 1 - \frac{1}{t} \iff$$

$$\iff \forall t \in \mathbb{N} \ \exists n_0 \ \forall n \geq n_0 \ \forall p \in \mathbb{N} \ \exists m_0 \ \forall m \geq m_0 \quad \sum_{k=n}^{m} \frac{a_{2k+1} - a_{2k+2}}{a_{2n}} > 1 - \frac{1}{t} - \frac{1}{p}.$$

From this we obtain that the set $\{(a_n) \in c_0^+ : d^+(\bigcup_{n\in\mathbb{N}}[a_{2n}, a_{2n+1}], 0) = 1\}$ is Borel. Note that if $f \in \mathbb{H}$ then

$$d^+\left(f(x), f\left(\bigcup_{n\in\mathbb{N}}[x+a_{2n}, x+a_{2n-1}]\right)\right) = 1 \iff$$

$$\iff \forall t \in \mathbb{N} \ \exists n_0 \ \forall n \geq n_0 \ \forall p \in \mathbb{N} \ \exists m_0 \ \forall m \geq m_0 \quad \sum_{k=n}^{m} \frac{f(x+a_{2k+1}) - f(x+a_{2k+2})}{f(x+a_{2n}) - f(x)} > 1 - \frac{1}{t} - \frac{1}{p}.$$

This shows that the set

$$\left\{(f,(a_n),x) \in \mathbb{H} \times c_0^+ \times [0,1) \, : \, d^+\left(f(x), \bigcup_{n\in\mathbb{N}}[f(x+a_{2n}), f(x+a_{2n+1})]\right) = 1\right\}$$

is Borel. Since $\{f \in \mathbb{H} : f \text{ is absolutely continuous}\}$ is a Borel subset of $\mathbb{H}$ (this is an easy observation), then we obtain that $DP\mathbb{H}$ is a coanalytic subset of $\mathbb{H}$. The result follows. □

It would be interesting to verify whether the same fact holds for $\mathcal{I}$-density preserving homeomorphisms in $\mathbb{H}$.

References

[1] Bruckner A.M., Density-preserving homeomorphisms and a theorem of Maximoff, Quart. J. Math. Oxford, 1970, 21(3), 337–347
[2] Ciesielski K., Larson L., Ostaszewski K., $\mathcal{I}$-Density Continuous Functions, Mem. Amer. Math. Soc., 1994, 107(515)
[3] Głąb S., Descriptive properties of families of autohomeomorphisms of the unit interval, J. Math. Anal. Appl., 2008, 343(2), 835–841
[4] Kechris A.S., Classical Descriptive Set Theory, Graduate Texts in Mathematics, 156, Springer, New York, 1995
[5] Niewiarowski J., Density-preserving homeomorphisms, Fund. Math., 1980, 106 (2), 77–87
[6] Ostaszewski K., Continuity in the density topology II, Rend. Circ. Mat. Palermo, 1983, 32(3), 398–414