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Galois realizability of groups of order 64

Research Article

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Abstract: This article examines the realizability of groups of order 64 as Galois groups over arbitrary fields. Specifically, we provide necessary and sufficient conditions for the realizability of 134 of the 200 noncyclic groups of order 64 that are not direct products of smaller groups.

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1. Introduction

Given a field K and a group G, the inverse problem of Galois theory asks whether or not there exists a Galois field extension L of K such that the Galois group of L over K is isomorphic to G. Extensive research has focused on small 2-groups, asking, for each group, what are necessary and sufficient conditions on a field K for the group to be realizable. Realizability conditions for groups of order 64 that are direct products of smaller groups are easily derived from those for the component groups, but complete conditions have been found for very few other groups of order 64. In this work, we provide necessary and sufficient conditions for the realizability of 134 groups of order 64. Combining these with the known conditions for the cyclic group and those that are direct products of smaller groups leaves unsolved only 66 of the 267 groups of order 64.

Compilations of conditions for the realizability of groups of order 2, 4, 8, and 16 can be found in [6] and [10]. Conditions for groups of order 32 can be found or derived from results in [2, 4, 5, 7, 11–13].

For fields of characteristic 2, a classic result of Witt [14] solves the realizability problem for finite 2-groups in general. Therefore, all fields in this paper are assumed to have characteristic other than 2.

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Throughout, we use C_n and D_n to denote the cyclic and dihedral groups of order n, respectively. Although there is no universally accepted notation for small 2-groups, in general, two common numbering schemes can be found in [3] and in [8]. In this paper, we use $G_{(r,s)}$ to denote the group of order 64 corresponding to r in Hall and Senior's list [8] and to $[64, s]$ in GAP's small groups library [3]. Group presentations for these groups can be found in Carson's index of 2-groups [1].

2. Methods

We begin by defining the Galois embedding problem and explaining its relationship to the Galois realizability problem. Given a Galois extension L/K and a short exact sequence

$$1 \longrightarrow N \underset{\iota}{\longrightarrow} E \underset{\psi}{\longrightarrow} \mathrm{Gal}\,(L/K) \longrightarrow 1, \tag{1}$$

the Galois embedding problem of L/K and (1) asks if there exist a Galois field extension M/K with $L \subset M$ and a homomorphism $\varphi\colon\mathrm{Gal}\,(M/K) \to E$ such that $\psi\varphi\colon\mathrm{Gal}\,(M/K) \to \mathrm{Gal}\,(L/K)$ is the natural restriction of Galois groups. The pair $(M/K, \varphi)$ is called a proper solution if φ is surjective. We can assume, without loss of generality, that φ is injective. Further, when the short exact sequence has kernel C_2 and is not split, φ is necessarily surjective.

Let G be a group with normal subgroup $N \cong C_2$, such that the short exact sequence

$$1 \longrightarrow C_2 \underset{\iota}{\longrightarrow} G \underset{\psi}{\longrightarrow} G/N \longrightarrow 1 \tag{2}$$

is not split. If the group G is realizable over K, so there exists a Galois extension M of K with $\mathrm{Gal}\,(M/K) \cong G$, then by basic Galois theory, there exists an intermediate field L with $\mathrm{Gal}\,(L/K) \cong G/N$. This in turn implies that the Galois embedding problem given by L/K and (2) is solvable. Conversely, if the Galois embedding problem given by L/K and the non-split extension (2) is solvable, then there exists a Galois field extension M/K with an isomorphism $\varphi\colon\mathrm{Gal}\,(M/K) \to G$, and thus G is realizable over K. So the realizability of G over K is equivalent to the realizability of G/N over K, say by a field L, and the solvability of the Galois embedding problem given by L/K and (2).

We use a method of Ledet [10] to determine the obstructions for 34 non-split embedding problems with kernel C_2 and quotient $(C_2)^r \times (C_4)^s \times (D_8)^t$. As described above, this allows us to solve the realizability problem for these groups. Specifically we use Theorem 2.2, below, a detailed proof of which can be found in [7].

We extend this method to one applying to an additional 100 groups, as follows. Let N_1 and N_2 be disjoint normal subgroups of a group G, each of order two, and consider the short exact sequence

$$1 \longrightarrow N_1N_2 \underset{\iota}{\longrightarrow} G \underset{\psi}{\longrightarrow} G/N_1N_2 \longrightarrow 1. \tag{3}$$

It is easy to see that the group G is realizable over K if and only if G/N_1N_2 is realizable over K, say by the field L, and the Galois embedding problem given by L/K and (3) is properly solvable.

Let L be an extension of K with $\mathrm{Gal}\,(L/K) \cong G/(N_1N_2)$. By the following theorem [9], it suffices to consider embedding problems associated to the short exact sequences

$$1 \longrightarrow N_1N_2/N_1 \underset{\iota}{\longrightarrow} G/N_1 \underset{\pi_2}{\longrightarrow} G/N_1N_2 \longrightarrow 1, \tag{4}$$

and

$$1 \longrightarrow N_1N_2/N_2 \underset{\iota}{\longrightarrow} G/N_2 \underset{\pi_1}{\longrightarrow} G/N_1N_2 \longrightarrow 1. \tag{5}$$

Theorem 2.1.
The Galois embedding problem given by L/K and (3) is solvable if and only if the Galois embedding problems given by L/K and (4) and by L/K and (5) are both solvable.

To state Theorem 2.2 [7, Theorem 1], we need the following notation and well-known characterizations of certain extensions. Let L/K be a $(C_2)^r \times (C_4)^s \times (D_8)^t$-extension, so L is the composite of a $(C_2)^r$-extension, a $(C_4)^s$-extension, and a $(D_8)^t$-extension.

Any $(C_2)^r$-extension of K can be written in the form $K\left(\sqrt{a_1}, \ldots, \sqrt{a_r}\right)$, where the a_i are quadratically independent elements of $\dot{K} = K - \{0\}$. The Galois group of the extension is generated by $\sigma_1, \ldots, \sigma_r$, where $\sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}}\sqrt{a_j}$ for all $i, j \leq r$.

Any $(C_4)^s$-extension can be written as

$$K\left(\sqrt{q_{r+1}(a_{r+1} + \sqrt{a_{r+1}})}, \ldots, \sqrt{q_{r+s}(a_{r+s} + \sqrt{a_{r+s}})}\right),$$

where the a_i are quadratically independent elements of $\dot{K}$, $q_i \in \dot{K}$, and for each i, $r < i \leq s$, there exists $\varepsilon_i \in \dot{K}$ such that $a_i = \varepsilon_i^2 + 1$ or, equivalently, $(a_i, -1) = 1$ in $\mathrm{Br}(K)$. The Galois group of the extension is generated by $\sigma_{r+1}, \ldots, \sigma_{r+s}$, where $\sigma_i\left(\sqrt{q_i(a_i + \sqrt{a_i})}\right) = \frac{q_i\varepsilon_i\sqrt{a_i}}{\sqrt{q_i(a_i+\sqrt{a_i})}}$ and $\sigma_i\left(\sqrt{q_j(a_j + \sqrt{a_j})}\right) = \sqrt{q_j(a_j + \sqrt{a_j})}$ for all $i \neq j$.

Finally, any $(D_8)^t$-extension can be written as

$$K\left(\sqrt{q_{r+s+1}(\alpha_{r+s+1} + \beta_{r+s+1}\sqrt{a_{r+s+1}})}, \sqrt{b_{r+s+1}}, \ldots, \sqrt{q_{r+s+t}(\alpha_{r+s+t} + \beta_{r+s+t}\sqrt{a_{r+s+t}})}, \sqrt{b_{r+s+t}}\right),$$

where $a_i, b_j \in \dot{K}$ are quadratically independent, $a_ib_i = \alpha_i^2 - a_i\beta_i^2$, $\alpha_i \in K$, and $\beta_i, q_i \in \dot{K}$ for all i, $r+s < i \leq r+s+t$. Such α_i and $\beta_i \in \dot{K}$ exist if and only if $(a, -b) = 1$ in $\mathrm{Br}(K)$. The Galois group of the extension is generated by the automorphisms $\{\sigma_i, \tau_i | r+s < i \leq r+s+t\}$ where

- $\sigma_i\left(\sqrt{q_i(\alpha_i + \beta_i\sqrt{a_i})}\right) = \frac{q_i\sqrt{a_ib_i}}{\sqrt{q_i(\alpha_i+\beta_i\sqrt{a_i})}}$ for all i;
- $\sigma_i\left(\sqrt{q_j(\alpha_j + \beta_j\sqrt{a_j})}\right) = \sqrt{q_j(\alpha_j + \beta_j\sqrt{a_j})}$ for all $i \neq j$;
- $\sigma_i\left(\sqrt{b_j}\right) = \sqrt{b_j}$ for all j;
- $\tau_i\left(\sqrt{q_j(\alpha_j + \beta_j\sqrt{a_j})}\right) = \sqrt{q_j(\alpha_j + \beta_j\sqrt{a_j})}$ for all i, j;
- $\tau_i(\sqrt{b_j}) = (-1)^{\delta_{ij}}\sqrt{b_j}$ for all i, j.

It follows that the Galois group of L/K is generated by $\{\sigma_i, \tau_j | 1 \leq i \leq r+s+t, r+s < j \leq r+s+t\}$, where the σ_i and τ_i that generate each factor are defined as above, with the additional requirement that σ_i and τ_i fix the elements corresponding to the j^{th} factor for all $i \neq j$.

Theorem 2.2.

Let K be a field and let L be a $(C_2)^r \times (C_4)^s \times (D_8)^t$-extension of K. Let $\{\sigma_1, \ldots, \sigma_{r+s+t}, \tau_{r+s+1}, \ldots, \tau_{r+s+t}\}$ be a minimal generating set of $\mathrm{Gal}(L/K)$ such that $|\sigma_i| = 2$ for $i \leq r$, $|\sigma_i| = 4$ for $r < i \leq r+s$, $\sigma_i^4 = \tau_i^2 = (\sigma_i\tau_i)^2 = 1$ for $i > r+s$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ and $\tau_i\tau_j = \tau_j\tau_i$ for all i, j, and $\sigma_i\tau_j = \tau_j\sigma_i$ for all $i \neq j$. Let

$$1 \longrightarrow C_2 \longrightarrow E \longrightarrow (C_2)^r \times (C_4)^s \times (D_8)^t \longrightarrow 1 \tag{6}$$

be a non-split extension of groups, and choose pre-images $g_1, \ldots, g_{r+s+t} \in E$ of $\sigma_1, \ldots, \sigma_{r+s+t}$ and $h_{r+s+1}, \ldots, h_{r+s+t} \in E$ of $\tau_{r+s+1}, \ldots, \tau_{r+s+t}$. Where appropriate, let -1 denote the image of -1 in E. Then the obstruction to the embedding problem given by L/K and (6) is

$$\prod_{i=1}^{r}(a_i, -1)^{d_i} \times \prod_{i=r+1}^{r+s}[(a_i, 2)(-1, q_i)]^{d_i} \times \prod_{i=r+s+1}^{r+s+t}\left[[(a_i, -2)(-b_i, 2\alpha_iq_i)]^{d_i}(b_i, -1)^{e_i}(a_i, -1)^{f_i}\right]$$
$$\times \prod_{i<j}(a_i, a_j)^{d_{ij}} \times \prod_{i<j}(a_i, b_j)^{e_{ij}} \times \prod_{i<j}(b_i, b_j)^{f_{ij}},$$

where $g_i^2 = (-1)^{d_i}$ for $i \le r$; $g_i^4 = (-1)^{d_i}$ for $r < i \le r+s+t$; $h_i^2 = (-1)^{e_i}$ for $r+s < i \le r+s+t$; $h_i g_i = (-1)^{f_i} g_i^3 h_i$ for $r+s < i \le r+s+t$; $g_i g_j = (-1)^{d_{ij}} g_j g_i$ for all i, j; $g_i h_j = (-1)^{e_{ij}} h_j g_i$ for all $1 \le i < j$, with $r+s < j \le r+s+t$; and $h_i h_j = (-1)^{f_{ij}} h_j h_i$ for $r+s < i < j \le r+s+t$. (If $\alpha_i = 0$, then $-b_i$ is a square in $\dot{K}$ and we set $(-b_i, 2\alpha_i q_i) = 1$ in $\mathrm{Br}(K)$.)

3. Results and sample proofs

Our results are given in Tables 1–8, which give necessary and sufficient conditions for the realizability of the group over a field K in terms of the group of square classes in $\dot{K}$ and the Brauer group of K. The tables are organized by the kernels and quotients of the related embedding problems; the first three tables corresponding to C_2-kernels, the rest to $C_2 \times C_2$-kernels. The quotients are indicated in the table captions.

In each table, the first column indicates the group, using the notation described in Section 1. The second column gives labels for the elements of $\dot{K}$ that are required to be quadratically independent. The third column lists the elements of $\mathrm{Br}(K)$ required to be trivial, expressed in terms of the elements in the previous column and, for some groups, unrestricted elements q and q' in $\dot{K}$.

We now give detailed proofs for two representative groups, the first using a C_2-kernel, the second using a $C_2 \times C_2$-kernel. Proofs for the other groups follow the same general outline. We first consider the group

$$G_{(58,124)} = \langle x, y, z \mid x^2, y^8, y^4 z^4, zxz^5 x, xyxy^7, yzy^7 z^7 \rangle.$$

Theorem 3.1.
The group $G_{(58,124)}$ is realizable as a Galois group over a field K if and only if there exist quadratically independent elements a, b, and $c \in \dot{K}$ and $q, q' \in \dot{K}$ such that $(a, -1) = (b, -c) = (ab, -2)(q, -1)(q', -c) = 1$ in $\mathrm{Br}(K)$.

Proof. The group $C_4 \times D_8$ is realizable over K if and only if there exist quadratically independent elements a_1, a_2, and $b_2 \in \dot{K}$ such that $(a_1, -1) = (a_2, -b_2) = 1$. Let L/K be a $C_4 \times D_8$-extension, if one exists.

Using the notation of Theorem 2.2 for the generators of $C_4 \times D_8$, consider the exact sequence

$$1 \longrightarrow C_2 \xrightarrow[-1 \mapsto y^4]{} G_{(58,124)} \xrightarrow[\substack{x \mapsto \tau_2 \\ y \mapsto \sigma_1 \\ z \mapsto \sigma_2}]{} C_4 \times D_8 \longrightarrow 1.$$

Let $g_1 = y$, $g_2 = z$, and $h_2 = x$ in $G_{(58,124)}$. Then $d_1 = d_2 = 1$ and, since x is of order 2, $e_2 = 0$. Now $xz = z^3 x$ implies that $f_3 = 0$ and, since y is central, $d_{12} = e_{12} = 0$. Therefore, in the notation introduced before Theorem 2.2, the obstruction to the embedding problem given by L/K and (3) is $(a_1, 2)(-1, q_1)(a_2, -2)(-b_2, 2\alpha_2 q_2)$.

Thus $G_{(58,124)}$ is realizable over K if and only if there exist quadratically independent elements a_1, a_2, and $b_2 \in \dot{K}$ such that $(a_1, -1) = (a_2, -b_2) = (a_1 a_2, -2)(q, -1)(q', -b_2) = 1$ in $\mathrm{Br}(K)$ where $q = q_1$ and $q' = 2\alpha_2 q_2$. $\square$

For an example with kernel $C_2 \times C_2$, we consider the group

$$G_{(53,97)} = \langle w, x, y, z \mid w^4 = x^2 = y^4 = y^2 z^2 = 1, xw = wxz^3, zw = wz^3, zx = xz^3, y \in Z \rangle.$$

Theorem 3.2.
The group $G_{(53,97)}$ is realizable as a Galois group over a field K if and only if there exist quadratically independent elements $a, b, c \in \dot{K}$ such that $(a, -b) = (a, -1) = (a, 2)(c, -1)(-b, q) = 1$ in $\mathrm{Br}(K)$.

Proof. The group $C_2 \times D_8$ is realizable over K if and only if there exist three quadratically independent elements $a_1, a_2, b_2 \in \dot{K}$ such that $(a_2, -b_2) = 1$. Let L/K be a $C_2 \times D_8$-extension, if one exists.

Using the notation of Theorem 2.2 for the generators of $C_2 \times D_8$, consider the exact sequence

$$1 \longrightarrow C_2 \times C_2 \xrightarrow[\substack{(-1,1)\mapsto w^2 \\ (1,-1)\mapsto y^2}]{} G_{(53,97)} \xrightarrow[\substack{w\mapsto \sigma_2\tau_2 \\ x\mapsto \tau_2 \\ y\mapsto \sigma_1 \\ z\mapsto \sigma_2^2}]{} C_2 \times D_8 \longrightarrow 1. \tag{7}$$

By Theorem 2.1, the embedding problem given by L/K and (7) is solvable if and only if the embedding problems given by L/K and each of the sequences

$$1 \longrightarrow C_2 \xrightarrow[-1\mapsto y^2]{} G_{(53,97)}/\langle w^2 \rangle \xrightarrow[\substack{w\mapsto \sigma_2\tau_2 \\ x\mapsto \tau_2 \\ y\mapsto \sigma_1 \\ z\mapsto \sigma_2^2}]{} C_2 \times D_8 \longrightarrow 1 \tag{8}$$

and

$$1 \longrightarrow C_2 \xrightarrow[-1\mapsto w^2]{} G_{(53,97)}/\langle y^2 \rangle \xrightarrow[\substack{w\mapsto \sigma_2\tau_2 \\ x\mapsto \tau_2 \\ y\mapsto \sigma_1 \\ z\mapsto \sigma_2^2}]{} C_2 \times D_8 \longrightarrow 1 \tag{9}$$

are solvable. In the notation of Theorem 2.2, let $g_1 = y$, $g_2 = wx$, and $h_2 = x$. Then for (8), $d_1 = d_2 = f_2 = 1$ are the only nontrivial exponents and so the obstruction is $(a_1, -1)(a_2, -2)(-b_2, 2\alpha_2 q_2)(a_2, -1)$. For (9), $f_2 = 1$ is the only nontrivial exponent, yielding the obstruction $(a_2, -1)$. The theorem follows. $\square$

Table 1. $C_2 \times (C_4)^2$, $(C_2)^3 \times C_4$, and $(C_2)^5$ Quotients

Group	Quad ind	Trivial elements in Br (K)
$G_{(30,57)}$	a, b, c	$(a, -1), (b, -1), (a, b)(c, -1)$
$G_{(32,86)}$	a, b, c	$(a, -1), (b, -1), (ac, b)(c, 2)(q, -1)$
$G_{(106,199)}$	a, b, c, d	$(a, -1), (a, b)(c, -d)$
$G_{(107,201)}$	a, b, c, d	$(a, -1), (-a, b)(c, bd)$
$G_{(108,200)}$	a, b, c, d	$(a, -1), (a, b)(c, d)(cd, -1)$
$G_{(109,249)}$	a, b, c, d	$(a, -1), (a, 2b)(c, d)(q, -1)$
$G_{(105,266)}$	a, b, c, d, e	$(a, -1)(b, -c)(d, e)$

Table 2. $(C_2)^2 \times D_8$ Quotients

Group	Quad ind	Trivial elements in Br (K)
$G_{(77,206)}$	a, b, c, d	$(c, -d), (ab, -1)(a, c)$
$G_{(78,213)}$	a, b, c, d	$(c, -d), (bd, -1)(a, d)$
$G_{(112,256)}$	a, b, c, d	$(c, -d), (a, c)(bd, -1)(c, 2)(-d, q)$
$G_{(157,227)}$	a, b, c, d	$(c, -d), (a, bc)(c, -1)$
$G_{(158,231)}$	a, b, c, d	$(c, -d), (a, -b)(b, -d)$
$G_{(159,229)}$	a, b, c, d	$(c, -d), (a, -bc)(b, -1)$
$G_{(160,228)}$	a, b, c, d	$(c, -d), (a, b)(ad, -1)$
$G_{(169,215)}$	a, b, c, d	$(c, -d), (a, d)(b, cd)$
$G_{(170,216)}$	a, b, c, d	$(c, -d), (a, c)(b, -d)$
$G_{(171,218)}$	a, b, c, d	$(c, -d), (a, -c)(b, -cd)$
$G_{(172,217)}$	a, b, c, d	$(c, -d), (a, cd)(b, -c)(c, -1)$
$G_{(241,257)}$	a, b, c, d	$(c, -d), (-a, b)(c, 2)(-d, q)$
$G_{(242,258)}$	a, b, c, d	$(c, -d), (2a, c)(ad, b)(-d, q)$
$G_{(243,259)}$	a, b, c, d	$(c, -d), (a, b)(-b, d)(c, 2)(-d, q)$

Table 3. $C_4 \times D_8$ Quotients

Group	Quad ind	Trivial elements in Br(K)
$G_{(58,124)}$	a, b, c	$(a,-1), (b,-c), (ab,-2)(q,-1)(q',-c)$
$G_{(84,67)}$	a, b, c	$(a,-1), (b,-c), (a,b)$
$G_{(85,71)}$	a, b, c	$(a,-1), (b,-c), (a,c)$
$G_{(86,66)}$	a, b, c	$(a,-1), (b,-c), (-a,b)$
$G_{(88,70)}$	a, b, c	$(a,-1), (b,-c), (-a,c)$
$G_{(91,69)}$	a, b, c	$(a,-1), (b,-c), (a,bc)(b,-1)$
$G_{(92,68)}$	a, b, c	$(a,-1), (b,-c), (a,bc)(c,-1)$
$G_{(99,116)}$	a, b, c	$(a,-1), (b,-c), (a,2b)(q,-1)$
$G_{(100,117)}$	a, b, c	$(a,-1), (b,-c), (a,2c)(q,-1)$
$G_{(117,123)}$	a, b, c	$(a,-1), (b,-c), (2a,b)(q,-c)$
$G_{(118,121)}$	a, b, c	$(a,-1), (b,-c), (-2a,b)(q,-c)$
$G_{(119,122)}$	a, b, c	$(a,-1), (b,-c), (2a,b)(c,-1)(q,-c)$
$G_{(122,125)}$	a, b, c	$(a,-1), (b,-c), (ab,2)(q,-1)(q',-c)$

Table 4. $(C_2)^2 \times C_4$ Quotients

Group	Quad ind	Trivial elements in Br(K)
$G_{(33,112)}$	a, b, c	$(a,-1), (a,2)(q,-1), (a,bc)(b,-1)$
$G_{(81,60)}$	a, b, c	$(a,-1), (a,b), (a,c)$
$G_{(82,65)}$	a, b, c	$(a,-1), (-a,b), (-a,c)$
$G_{(83,61)}$	a, b, c	$(a,-1), (a,b), (-a,c)$
$G_{(87,72)}$	a, b, c	$(a,-1), (a,b), (b,-1)(-b,c)$
$G_{(89,62)}$	a, b, c	$(a,-1), (a,b), (a,c)(b,-1)$
$G_{(90,63)}$	a, b, c	$(a,-1), (-a,c), (a,bc)(b,-1)$
$G_{(93,64)}$	a, b, c	$(a,-1), (a,b)(c,-1), (a,bc)(b,-1)$
$G_{(94,88)}$	a, b, c	$(a,-1), (a,b), (a,2c)(q,-1)$
$G_{(95,104)}$	a, b, c	$(a,-1), (-a,b), (a,2c)(q,-1)$
$G_{(96,89)}$	a, b, c	$(a,-1), (a,b), (a,2)(b,c)(q,-1)$
$G_{(97,105)}$	a, b, c	$(a,-1), (-a,b), (a,2c)(b,c)(q,-1)$
$G_{(98,113)}$	a, b, c	$(a,-1), (-a,c)(b,-1), (a,2b)(q,-1)$
$G_{(101,127)}$	a, b, c	$(a,-1), (-ab,c)(b,-1), (a,2b)(q,-1)$
$G_{(102,114)}$	a, b, c	$(a,-1), (a,c)(b,-1), (a,2c)(b,c)(q,-1)$

Table 5. $(C_2)^4$ Quotients

Group	Quad ind	Trivial elements in $\mathrm{Br}(K)$
$G_{(79,214)}$	a, b, c, d	$(a, bc)(c, -1), (a, -d)(d, -1)$
$G_{(80,210)}$	a, b, c, d	$(a, -1)(b, c), (bd, -1)(c, d)$
$G_{(161,235)}$	a, b, c, d	$(-a, d)(b, -1), (b, -c)(c, -1)$
$G_{(162,238)}$	a, b, c, d	$(-a, d)(c, -d), (a, -b)(b, -1)$
$G_{(163,232)}$	a, b, c, d	$(a, d)(b, -d), (b, c)(-a, d)$
$G_{(164,234)}$	a, b, c, d	$(a, -1)(b, c), (a, -cd)(d, -1)$
$G_{(165,240)}$	a, b, c, d	$(abc, -1)(b, c), (a, d)(abd, -1)$
$G_{(166,236)}$	a, b, c, d	$(a, -1)(-b, c), (a, bd)(d, -1)$
$G_{(167,233)}$	a, b, c, d	$(a, -1)(-b, c), (a, d)(b, -1)$
$G_{(168,237)}$	a, b, c, d	$(a, -1)(-b, c), (a, -bd)(d, -1)$
$G_{(173,224)}$	a, b, c, d	$(a, c)(b, d), (a, -d)(d, -1)$
$G_{(174,225)}$	a, b, c, d	$(a, b)(-c, d), (b, -c)(c, -1)$
$G_{(175,219)}$	a, b, c, d	$(a, b)(-b, d), (a, bd)(b, c)$
$G_{(176,221)}$	a, b, c, d	$(a, -b)(c, -1), (-a, d)(b, c)$
$G_{(177,220)}$	a, b, c, d	$(a, -1)(-c, d), (a, d)(b, c)$
$G_{(178,223)}$	a, b, c, d	$(a, d)(b, -d), (a, d)(b, -1)(-ab, c)$
$G_{(179,222)}$	a, b, c, d	$(a, b)(-b, c), (a, c)(b, -d)(d, -1)$
$G_{(183,242)}$	a, b, c, d	$(a, d)(b, c), (a, bd)(c, d)$
$G_{(184,241)}$	a, b, c, d	$(-a, c)(b, d), (-ab, d)(b, c)$
$G_{(185,243)}$	a, b, c, d	$(a, c)(bc, d), (a, c)(b, -c)(ac, d)$
$G_{(186,244)}$	a, b, c, d	$(a, c)(a, d)(b, c), (a, c)(ac, -1)(b, d)$
$G_{(187,245)}$	a, b, c, d	$(a, cd)(b, -d)(d, -1), (acd, -1)(a, d)(b, c)$

Table 6. $(C_4)^2$ Quotients

Group	Quad ind	Trivial elements in $\mathrm{Br}(K)$
$G_{(37,17)}$	a, b	$(a, -1), (b, -1), (a, b), (ab, 2)(q, -1)$
$G_{(38,3)}$	a, b	$(a, -1), (b, -1), (a, 2)(q, -1), (2a, b)(q', -1)$

Table 7. $C_2 \times D_8$ Quotients: Part I

Group	Quad ind	Trivial elements in $\mathrm{Br}(K)$
$G_{(53,97)}$	a, b, c	$(a, -b), (a, -1), (a, 2)(c, -1)(-b, q)$
$G_{(54,108)}$	a, b, c	$(a, -b), (b, -1), (a, 2)(c, -1)(-b, q)$
$G_{(113,99)}$	a, b, c	$(a, -b), (a, -1), (a, -2c)(-b, q)$
$G_{(114,98)}$	a, b, c	$(a, -b), (a, -1), (a, 2c)(b, c)(-b, q)$
$G_{(115,100)}$	a, b, c	$(a, -b), (a, -1), (a, 2c)(b, -1)(-b, q)$
$G_{(116,109)}$	a, b, c	$(a, -b), (b, -1), (a, -2c)(-b, q)$
$G_{(144,73)}$	a, b, c	$(a, -b), (a, c), (b, c)$
$G_{(145,76)}$	a, b, c	$(a, -b), (a, c), (b, -c)(c, -1)$
$G_{(146,75)}$	a, b, c	$(a, -b), (a, c), (b, -c)$
$G_{(147,74)}$	a, b, c	$(a, -b), (a, -c)(c, -1), (ab, c)$
$G_{(148,80)}$	a, b, c	$(a, -b), (a, -c)(c, -1), (-b, c)$
$G_{(149,79)}$	a, b, c	$(a, -b), (-a, c)(b, -1), (b, c)$
$G_{(150,77)}$	a, b, c	$(a, -b), (a, -c)(c, -1), (b, c)$

Table 8. $C_2 \times D_8$ Quotients: Part II

Group	Quad ind	Trivial elements in $\mathrm{Br}(K)$
$G_{(151,78)}$	a,b,c	$(a,-b),(a,c)(b,-1),(b,c)$
$G_{(152,81)}$	a,b,c	$(a,-b),(a,c)(b,-c),(a,-c)(c,-1)$
$G_{(188,174)}$	a,b,c	$(a,-b),(-b,c),(a,2)(-b,q)$
$G_{(189,173)}$	a,b,c	$(a,-b),(-b,c),(a,-2)(-b,q)$
$G_{(190,181)}$	a,b,c	$(a,-b),(b,-c)(c,-1),(a,2)(-b,q)$
$G_{(191,179)}$	a,b,c	$(a,-b),(b,-c)(c,-1),(a,-2)(-b,q)$
$G_{(192,175)}$	a,b,c	$(a,-b),(-b,c),(a,2)(b,-1)(-b,q)$
$G_{(193,167)}$	a,b,c	$(a,-b),(a,-1)(-b,c),(a,-2)(-b,q)$
$G_{(194,168)}$	a,b,c	$(a,-b),(a,-1)(-b,c),(a,-2)(b,-1)(-b,q)$
$G_{(195,147)}$	a,b,c	$(a,-b),(b,-c),(a,2)(-b,q)$
$G_{(196,146)}$	a,b,c	$(a,-b),(b,c),(a,-2)(-b,q)$
$G_{(197,148)}$	a,b,c	$(a,-b),(b,c),(a,2)(b,-1)(-b,q)$
$G_{(198,176)}$	a,b,c	$(a,-b),(-b,c),(a,-2)(bc,-1)(-b,q)$
$G_{(199,180)}$	a,b,c	$(a,-b),(b,-c)(c,-1),(a,-2)(b,c)(-b,q)$
$G_{(200,169)}$	a,b,c	$(a,-b),(a,-1)(-b,c),(a,-2)(b,c)(-b,q)$
$G_{(201,128)}$	a,b,c	$(a,-b),(a,-c),(a,2)(-b,q)$
$G_{(202,131)}$	a,b,c	$(a,-b),(a,c),(a,-2)(-b,q)$
$G_{(203,129)}$	a,b,c	$(a,-b),(ab,c),(a,-2)(-b,q)$
$G_{(204,132)}$	a,b,c	$(a,-b),(a,c),(a,2)(b,-1)(-b,q)$
$G_{(205,140)}$	a,b,c	$(a,-b),(-a,c),(a,2)(-b,q)$
$G_{(206,141)}$	a,b,c	$(a,-b),(-a,c),(a,-2)(b,-1)(-b,q)$
$G_{(207,155)}$	a,b,c	$(a,-b),(a,-c)(c,-1),(a,2)(-b,q)$
$G_{(208,142)}$	a,b,c	$(a,-b),(-a,c),(a,-2)(-b,q)$
$G_{(209,157)}$	a,b,c	$(a,-b),(a,-c)(c,-1),(a,-2)(-b,q)$
$G_{(210,156)}$	a,b,c	$(a,-b),(a,-c)(c,-1),(a,-2)(b,-1)(-b,q)$
$G_{(211,143)}$	a,b,c	$(a,-b),(-a,c),(a,2)(b,-1)(-b,q)$
$G_{(212,158)}$	a,b,c	$(a,-b),(a,-c)(c,-1),(a,2)(b,-1)(-b,q)$
$G_{(213,161)}$	a,b,c	$(a,-b),(ab,c)(b,-1),(a,2)(-b,q)$
$G_{(214,162)}$	a,b,c	$(a,-b),(a,c)(b,-c),(a,-2)(-b,q)$
$G_{(215,164)}$	a,b,c	$(a,-b),(a,-c)(b,-1),(a,-2)(-b,q)$
$G_{(216,165)}$	a,b,c	$(a,-b),(a,c)(b,-1),(a,2c)(-b,q)$
$G_{(217,130)}$	a,b,c	$(a,-b),(ab,c),(a,2)(b,c)(-b,q)$
$G_{(218,133)}$	a,b,c	$(a,-b),(a,c),(a,2)(b,-c)(-b,q)$
$G_{(219,144)}$	a,b,c	$(a,-b),(-a,c),(a,2)(c,-1)(-b,q)$
$G_{(220,145)}$	a,b,c	$(a,-b),(-ab,c),(a,-2)(c,-1)(-b,q)$
$G_{(221,159)}$	a,b,c	$(a,-b),(a,-c)(c,-1),(a,-2c)(-b,q)$
$G_{(222,160)}$	a,b,c	$(a,-b),(a,-c)(c,-1),(a,2c)(b,-1)(-b,q)$
$G_{(223,163)}$	a,b,c	$(a,-b),(a,-c)(b,c),(a,2)(c,-1)(-b,q)$
$G_{(224,166)}$	a,b,c	$(a,-b),(ab,c)(b,-1),(a,2)(b,c)(-b,q)$
$G_{(225,177)}$	a,b,c	$(a,-b),(-b,c),(a,2c)(-b,q)$
$G_{(226,178)}$	a,b,c	$(a,-b),(-b,c),(a,-2c)(c,-1)(-b,q)$
$G_{(227,182)}$	a,b,c	$(a,-b),(b,-c)(c,-1),(a,2c)(b,-1)(-b,q)$
$G_{(228,150)}$	a,b,c	$(a,-b),(b,c),(a,2c)(-b,q)$
$G_{(229,149)}$	a,b,c	$(a,-b),(b,-c),(a,2c)(-b,q)$
$G_{(230,151)}$	a,b,c	$(a,-b),(b,c),(a,2c)(b,-1)(-b,q)$
$G_{(231,171)}$	a,b,c	$(a,-b),(a,-1)(-b,c),(a,-2c)(-b,q)$
$G_{(232,170)}$	a,b,c	$(a,-b),(a,-1)(-b,c),(a,2c)(b,c)(-b,q)$
$G_{(233,172)}$	a,b,c	$(a,-b),(a,-1)(-b,c),(a,-2c)(b,-1)(-b,q)$

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