

On equivalent strictly G -convex renormings of Banach spaces

Research Article

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Abstract: We study strictly G -convex renormings and extensions of strictly G -convex norms on Banach spaces. We prove that $\ell_\omega(\Gamma)$ space cannot be strictly G -convex renormed given Γ is uncountable and G is bounded and separable.

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1. Introduction

In this paper X stands for a Banach space (real or complex) and $L(X)$ for the algebra of bounded operators on X . By $\ell_\infty(\Gamma)$ we denote the sup-normed Banach space of all scalar-valued bounded functions on the set Γ . $\ell_\omega(\Gamma)$ is the subspace of $\ell_\infty(\Gamma)$ formed by elements with countable supports. For the basic definitions and standard facts from Banach space geometry we refer to the Diestel's book [4], and for Functional Analysis terminology we refer to [3] and [5].

The concept of strict G -convexity is a natural generalization of classical concepts of strict convexity (rotundity) and complex strict convexity (rotundity). The idea of this generalization consists of substituting the choice of sign (or of modulus-one scalar) in the corresponding definitions by the choice of operator from a given subset $G \subset L(X)$. Strict G -convexity and the related concept of uniform G -convexity were introduced in [1] and were studied also in [2].

Let $G \subset L(X)$. X is said to be *strictly G -convex* if $\sup_{T \in G} \|x + \varepsilon Ty\| > 1$ for all $x, y \in S_X$ and $\varepsilon > 0$.

For $G = \{I, -I\}$ and $G = \{I, -I, iI, -iI\}$ one gets the classical concepts of strict convexity and complex strict convexity, respectively. Since in these cases we deal with finite groups of isometries which are symmetric about 0, finite groups G with

$$\sum_{T \in G} T = 0 \tag{1}$$

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are of the major interest for us. Nevertheless we consider more general classes of G as well. Note that by imposing the condition (1) we ensure that at least $\sup_{T \in G} \|x + \varepsilon Ty\| \geq 1$ for all $x, y \in S_X$ and $\varepsilon > 0$. So the strict G -convexity just means that the last inequality is strict. In [2, Lemma 2.1] one can find more motivation to impose (1).

There are a number of well known results about strictly convex spaces, i.e. about strictly G -convex spaces for $G = \{I, -I\}$. The basic question for us is what properties of G are responsible for these results. In our paper we study two theorems of this kind. Day's theorem [4] states that for uncountable Γ , $\ell_\infty(\Gamma)$ does not have an equivalent strictly convex norm. We show that separability of G enables to prove the same result in the general case (Theorem 4.4), and that there is a group G of continuum cardinality for which this result is false. Another result is the Wee-Kee Tang extension theorem [6]. We check that his construction of a strictly convex norm extension is applicable to arbitrary finite groups G with $\sum_{T \in G} T = 0$ (Theorem 3.2).

Recall that in our previous papers [1] and [2], we developed analogous program for uniform G -convexity. Namely, for series in Banach spaces, we studied the connection between G -convergence and unconditional convergence in ordinary sense, as well as obtained the analogues of the theory of cotype and the famous M. Kadets's theorem about unconditionally convergent series in uniformly convex space. We also studied the inheritance of the uniform G -convexity of the space X by the space of X -valued functions $L_p(\mu, X)$, $p \in [1, \infty)$ in the (most interesting for us) case when G is a regular finite group. We showed that for a fixed $p \in [1, \infty)$ the uniform G -convexity of $L_p(\mu, X)$ for all measure spaces (Ω, Σ, μ) is equivalent to a stronger property of X , i.e. to the uniform G -convexity in terms of p -average. We demonstrated that for $p \in (1, \infty)$ the last property is equivalent to uniform G -convexity of X , but for $p = 1$ this equivalence is no longer true.

2. General remarks

Theorem 2.1.

Let G be a finite group, and let X be a strictly G -convex space. Then one can find an equivalent norm on X , for which X endowed with this norm is strictly G -convex and the group G consists of isometries.

Proof. Let us define a new norm by $\|x\| := \max_{T \in G} \|Tx\|$. For the norm defined in such a way the following inequalities hold: $\|x\| \geq \|x\|$, because G is a group and therefore it contains the identity operator; and $\|x\| \leq \max_{T \in G} \|T\| \|x\|$, the maximum of operator norms is finite because the group G is finite. These two inequalities imply the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|$.

For every $T \in G$

$$\|T\| = \sup_{x \in S_{(X, \|\cdot\|)}} \max_{U \in G} \|UTx\| = \sup_{x \in S_{(X, \|\cdot\|)}} \max_{R \in G} \|Rx\| = 1.$$

Since G is a group this means also that $\|T^{-1}\| = 1$ and this implies that T is an isometry.

Let us prove now that the space $(X, \|\cdot\|)$ is strictly G -convex. For every positive real t and for two arbitrary elements $x, y \in S_{(X, \|\cdot\|)}$ we have

$$\begin{aligned} \sup_{T \in G} \|x + tTy\| &= \sup_{T \in G} \sup_{U \in G} \|Ux + tUTy\| = \sup_{U \in G} \sup_{T \in G} \|Ux + tUTy\| \\ &= \sup_{U \in G} \left(\|Ux\| \sup_{T \in G} \left\| \frac{Ux}{\|Ux\|} + \frac{t\|y\|}{\|Ux\|} T \left(\frac{y}{\|y\|} \right) \right\| \right) > \sup_{U \in G} \|Ux\| = 1. \end{aligned} \quad \square$$

Note that in the last inequality we used strongly the finiteness of G – otherwise there would be no reason to say that this inequality is strict.

Theorem 2.2.

Let $G = \{T_1, \dots, T_n\}$ be a finite group, $\sum_{T \in G} T = 0$, X be a strictly convex space. Then X is strictly G -convex.

Proof. Suppose that X is not strictly G -convex, then there exist elements $x_1, y_1 \in S_X$ and $\varepsilon > 0$ such that for all $T \in G$ the following inequality holds: $\|x_1 + \varepsilon T y_1\| \leq 1$. Then by convexity of the unit ball $\|x_1 + t T y_1\| \leq 1$ for all $t \in (0, \varepsilon)$. Since X is strictly convex then $\|x_1 - t T y_1\| > 1$ for all $t \in (0, \varepsilon)$. So

$$1 < \left\| x_1 - \frac{\varepsilon}{n-1} T_1 y_1 \right\| = \left\| x_1 + \frac{\varepsilon}{n-1} \sum_{i=2}^n T_i y_1 \right\| \leq \frac{1}{n-1} \sum_{i=2}^n \|x_1 + \varepsilon T_i y_1\| \leq 1.$$

This is a contradiction. □

It is known that for the space ℓ_∞ one can find an equivalent strictly convex norm. Then the last theorem implies the following corollary, which supplements naturally our considerations from Section 3 below:

Corollary 2.3.

For every finite group $G \subset L(\ell_\infty)$ with $\sum_{T \in G} T = 0$ there is an equivalent strictly G -convex norm on ℓ_∞ .

Note also that strict G -convexity of a space does not imply neither ordinary strict convexity nor complex strict convexity. One can easily see this if one considers $X = \ell_\infty^{(n)}$ (the space of vectors $x = (x_1, \dots, x_n)$ with $\|x\| := \max_k |x_k|$), and G being the (finite and symmetric) group of all isometries of X .

3. The extension theorem

Lemma 3.1.

Let $p : B \rightarrow \mathbb{R}^+$ be a convex function defined on a convex set B , and for some collection $\{x_k\}_{k=1}^n \subset B$ suppose the inequality

$$\frac{1}{n} \sum_{k=1}^n p^2(x_k) - p^2 \left(\frac{1}{n} \sum_{k=1}^n x_k \right) \leq 0$$

holds true. Then $p(x_i) = p(x_j) = p\left(\frac{1}{n} \sum_{k=1}^n x_k\right)$ for every i and j .

Proof. By convexity $p\left(\frac{1}{n} \sum_{k=1}^n x_k\right) \leq \frac{1}{n} \sum_{k=1}^n p(x_k)$. Then

$$0 \geq \frac{1}{n} \sum_{k=1}^n p^2(x_k) - p^2 \left(\frac{1}{n} \sum_{k=1}^n x_k \right) \geq \frac{1}{n} \sum_{k=1}^n p^2(x_k) - \frac{1}{n^2} \left(\sum_{k=1}^n p(x_k) \right)^2 = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (p(x_i) - p(x_j))^2.$$

Since all the summands of the last sum are non-negative and their sum is bounded by 0, then each summand is equal to 0. □

Recall that the Minkowski functional of a convex absorbing rounded subset B of a linear space ([3, p. 106] or [5, section 5.4.2]) is

$$\phi_B(x) = \inf \{t > 0 : x \in tB\}.$$

Theorem 3.2.

Let $G = \{T_1, \dots, T_n\} \subset L(X)$ be a finite group, $\sum_{i=1}^n T_i = 0$, and suppose X has a strictly G -convex norm $\|\cdot\|$. Then every equivalent strictly G -convex norm $|\cdot|$, defined on a subspace Y invariant with respect to G , can be extended to an equivalent strictly G -convex norm, which is defined on the entire space X .

Proof. Without loss of generality we may assume that $\sqrt{2}\|\cdot\| \leq |\cdot| \leq r\|\cdot\|$ on Y for some $r > 0$. Let us extend the norm $|\cdot|$ from Y to the entire space X in the standard way, taking the closed convex hull of $B_{(Y,|\cdot|)} \cup B_{(X,r\|\cdot\|)}$ as the unit ball of the extended norm. Further we will use $|\cdot|$ for this extension as well. The inequality $\sqrt{2}\|\cdot\| \leq |\cdot| \leq r\|\cdot\|$ is valid on the whole X after this extension. Denote as $\text{dist}(x, Y)$ the distance from x to Y in the metric $\|\cdot\|$.

Let us define the function $p(\cdot)$ as

$$p^2(x) = q_1(x) + q_2(x) + q_3(x),$$

where $q_1(x) = |x|^2$, $q_2(x) = \text{dist}(x, Y)^2 e^{\|x\|^2}$, and $q_3(x) = \text{dist}(x, Y)^2$. Note that the functions q_1 and q_3 are convex on the whole X , and q_2 is convex on $B_{(X,|\cdot|)}$. In order to prove the latter it is enough to see that the Hessian of the function $f(x, y) = y^2 e^{x^2}$, where $(x, y) \in \mathbb{R}^2$ is positively defined on $(0, \frac{1}{\sqrt{2}}) \times (0, \frac{1}{\sqrt{2}})$. Therefore, p^2 is a convex function on $B_{(X,|\cdot|)}$. Define $B = \{x \in X : p(x) \leq 1\}$. Since $B \subset B_{(X,|\cdot|)}$, we have the boundedness of B and by convexity of p^2 we have the convexity of B as well. Moreover, for all $x \in B_{(X,|\cdot|)}$ we have $p^2(x) \leq 2 + e$. By convexity of p^2 we have $p^2(tx) \leq tp^2(x)$ for $x \in B_{(X,|\cdot|)}$, $t \in (0, 1)$. This means that $B \supset \frac{1}{2+e} B_{(X,|\cdot|)}$. Then the Minkowski functional of B defines an equivalent norm $p_1(\cdot)$ on X .

Let X be not strictly G -convex in the norm $p_1(\cdot)$. Then there exist elements $x, y \in S_{(X,p_1)} \subset B_{(X,|\cdot|)}$ and $t > 0$ such that for all operators T_k from the group G the inequality $p_1(x + tT_k y) \leq 1$ holds. Then,

$$\frac{1}{n} \sum_{k=1}^n p^2(x + tT_k y) - p^2(x) \leq 0.$$

Recall that $p^2 = q_1(x) + q_2(x) + q_3(x)$ is a sum of convex functions, and by convexity each summand satisfies the inverse inequality:

$$\frac{1}{n} \sum_{k=1}^n q_j(x + tT_k y) - q_j(x) \geq 0, \quad j = 1, 2, 3$$

(at this point we use the condition $\sum_{k=1}^n T_k = 0$). So in fact all these inequalities are equations:

$$\frac{1}{n} \sum_{k=1}^n q_j(x + tT_k y) - q_j(x) = 0, \quad j = 1, 2, 3.$$

In particular

$$\frac{1}{n} \sum_{k=1}^n \text{dist}(x + tT_k y, Y)^2 - \text{dist}(x, Y)^2 = 0.$$

So according to the Lemma 3.1, for all $k \in \{1, 2, \dots, n\}$

$$\text{dist}(x + tT_k y, Y) = \text{dist}(x, Y). \quad (2)$$

Consider two cases: when x belongs to Y , and when it is out of this subspace. In the first case (2) implies that $x + tT_k y$ also belong to Y for all the operators $T_k \in G$. Therefore, $p^2(x + tT_k y) = |x + tT_k y|^2$, and the assumption that $p_1(x + tT_k y) \leq 1$ for all k contradicts strict G -convexity of the norm $|\cdot|$ defined on Y . In the second case (2) together with $\frac{1}{n} \sum_{k=1}^n q_2(x + tT_k y) - q_2(x) = 0$ implies that

$$\frac{1}{n} \sum_{k=1}^n e^{\|x + tT_k y\|^2} - e^{\|x\|^2} = 0.$$

Thus,

$$\sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{1}{n} \sum_{k=1}^n \|x + tT_k y\|^{2j} - \|x\|^{2j} \right) = 0.$$

By the same convexity argument as above this means that

$$\frac{1}{n} \sum_{k=1}^n \|x + tT_k y\|^{2i} - \|x\|^{2i} = 0$$

for all $i \in \mathbb{N}$. But by the Lemma 3.1 already in the case of $i = 1$,

$$\frac{1}{n} \sum_{k=1}^n \|x + tT_k y\|^2 - \|x\|^2$$

implies that $\|x + tT_k y\| = \|x\|$ for all k , which contradicts strict G -convexity of the norm $\|\cdot\|$. Thereby both cases lead to a contradiction and this proves that our assumption was incorrect. Thus, the norm p_1 is strictly G -convex on the entire space X . $\square$

If the group G equals $\{I, -I, iI, -iI\}$, then G -convexity is the same as complex convexity. Therefore we get the following corollary:

Corollary 3.3.

Let X have a complex strictly convex norm $\|\cdot\|$. Then every equivalent complex strictly convex norm $|\cdot|$, defined on a subspace Y can be extended to an equivalent complex strictly convex norm defined on the entire space X .

4. Renormings of $\ell_\infty(\Gamma)$

It is natural to study the existence of Banach spaces which cannot be strictly G -convex renormed for any finite group of isometries G . As it is shown below, the space $\ell_\infty(\Gamma)$ where Γ is uncountable is an example of such a space.

From now on Γ is an uncountable set. For every $A \subset B$ we consider $\ell_\omega(B \setminus A)$ as the subspace of $\ell_\omega(B)$ consisting of A -supported elements of $\ell_\omega(B)$.

Lemma 4.1.

For every functional $x^* \in \ell_\omega(\Gamma)^*$ there exists an at most countable set $A \subset \Gamma$ such that $x^*(y) = 0$ for all $y \in \ell_\omega(\Gamma \setminus A)$.

Proof. For every $A \subset \Gamma$ denote $u(x^*, A) = \sup_{x \in S_{\ell_\omega(\Gamma \setminus A)}} |x^*(x)|$. Let us consider two cases: $\inf_{|A| \leq \omega} u(x^*, A) = 0$ and $\inf_{|A| \leq \omega} u(x^*, A) \neq 0$.

In the first case for every n there exists an at most countable set A_n with $u(x^*, A_n) < \frac{1}{n}$. This way the set $A = \bigcup_{n=1}^\infty A_n$ is at most countable. $A \supset A_n$, $u(x^*, A) \leq u(x^*, A_n) < \frac{1}{n}$, hence $u(x^*, A) = 0$. So the lemma is proved for this case.

In the second case, let us fix a positive $\alpha < \inf_{|A| \leq \omega} u(x^*, A)$. There exists an element $x_1 \in S_{\ell_\omega(\Gamma)}: |x^*(x_1)| \geq \alpha$, $A_1 = \text{supp } x_1$. One can also find an element $x_2 \in S_{\ell_\omega(\Gamma \setminus A_1)}: |x^*(x_2)| \geq \alpha$, $A_2 = \text{supp } x_2 \cup \text{supp } x_1$ etc. We obtain a disjoint sequence (x_k) of norm 1 elements with $|x^*(x_k)| \geq \alpha$. Denote $\lambda_k = \text{sign } x^*(x_k)$. Then $\|\sum_{k=1}^n \lambda_k x_k\| = 1$ and $\|x^*\| \geq x^*(\sum_{k=1}^n \lambda_k x_k) \geq \alpha n \rightarrow +\infty$ as $n \rightarrow \infty$. This is a contradiction. So the first case is the only possible case. $\square$

Lemma 4.2.

Let G be a countable bounded family consisting of operators $T_k : \ell_\omega(\Gamma) \rightarrow \ell_\omega(\Gamma)$, $k \in \mathbb{N}$ and let $A \subset \Gamma$ be at most countable. Then one can find an at most countable set $B \supset A$ such that for every element $x \in \ell_\omega(\Gamma \setminus B)$ and every operator T from G the element $T(x)$ belongs to $\ell_\omega(\Gamma \setminus A)$.

Proof. Denote by $e_y^* \in \ell_\omega(\Gamma)^*$ the corresponding coordinate functional: $e_y^*(x) = x(y)$. Consider functionals $x_{k,a}^*(x) := e_a^*(T_k(x))$, $a \in A$, $k \in \mathbb{N}$. Applying the previous lemma for all $a \in A$, $k \in \mathbb{N}$ we can find an at most countable set $A_{k,a} \subset \Gamma$ such that $x_{k,a}^*|_{\ell_\omega(\Gamma \setminus A_{k,a})} = 0$. Then $B_k := \bigcup_{a \in A} A_{k,a}$ is countable and has the following property: for every $x \in \ell_\omega(\Gamma \setminus B_k)$ the value of $T_k(x)$ belongs to $\ell_\omega(\Gamma \setminus A)$. The set B that we need can be defined as $B := \bigcup_{k=1}^\infty B_k$. $\square$

Lemma 4.3.

Let p be an arbitrary equivalent norm on $\ell_\omega(\Gamma)$. Then there exists a non-zero element $x \in \ell_\omega(\Gamma)$ and exists a countable set $A \subset \Gamma$ such that for every $y \in \ell_\omega(\Gamma \setminus A)$, $\|y\| \leq 1$, the equality $p(x + y) = p(x)$ holds.

Proof. For every element $x \in S_{\ell_\omega(\Gamma)}$ define $F_x = \{y \in S_{\ell_\omega(\Gamma)} : y|_{\text{supp}(x)} = x|_{\text{supp}(x)}\}$. Without loss of generality we can assume that $\|\cdot\|_\infty \leq p(\cdot)$.

Define $M_x = \sup\{p(y) : y \in F_x\}$, $m_x = \inf\{p(y) : y \in F_x\}$. It is obvious that $2x - F_x = F_x$, hence $p(2x - y) \leq M_x$, so $2p(x) \leq M_x + p(y)$. Moreover for every $\varepsilon > 0$ there exists a $y \in F_x$, such that $p(y) \leq m_x + \varepsilon$. This way we obtain that for every $\varepsilon > 0$ the inequality $2p(x) \leq m_x + M_x + \varepsilon$ holds. Thus the following inequality holds as well:

$$2p(x) \leq m_x + M_x. \quad (3)$$

Define $K = \sup\{p(y) : y \in S_{\ell_\omega(\Gamma)}\}$. It is obvious that $K \geq 1$ and let us choose an $x_1 \in S_{\ell_\omega(\Gamma)}$ such that $(3K + 1)/4 \leq p(x_1)$. The inequality (3) implies that $(3K + 1)/2 \leq m_{x_1} + M_{x_1}$, then $M_{x_1} - m_{x_1} \leq (K - 1)/2$. Choose an $x_2 \in F_{x_1}$ such that $(3M_{x_1} + p(x_1))/4 \leq p(x_2)$. Thus, $(M_{x_1} + p(x_1))/2 \leq m_{x_2}$. It is obvious that $M_{x_2} \leq M_{x_1}$, because $x_2 \in F_{x_1}$ and $F_{x_2} \subset F_{x_1}$. Hence

$$M_{x_2} - m_{x_2} \leq (M_{x_1} - p(x_1))/2 \leq (M_{x_1} - m_{x_2})/2 \leq \frac{K - 1}{4}.$$

Inductively let us choose $x_n \in F_{x_{n-1}}$ and we have $M_{x_n} - m_{x_n} \leq (K - 1)/2^n$, $M_{x_{k+1}} \leq M_{x_k}$, $m_{x_{k+1}} \geq m_{x_k}$. Therefore there exists $\lim_{k \rightarrow \infty} M_{x_k} = \lim_{k \rightarrow \infty} m_{x_k} = \mu$. It is obvious that $p(y) = \mu$ for every element $y \in \bigcap_n F_{x_n}$.

Choose an element $x \in \ell_\omega(\Gamma)$, such that $x(v) = x_n(v)$ for all $n \in \mathbb{N}$ and $v \in \text{supp}(x_n)$, and $x(v) = 0$ outside $A := \bigcup_{n \in \mathbb{N}} \text{supp}(x_n)$. For this x the corresponding set F_x consists of elements w with $p(w) = p(x)$. Since every element of the form $x + y$: $y \in \ell_\omega(\Gamma \setminus A)$, $\|y\| \leq 1$, belongs to F_x , we are done. $\square$

Theorem 4.4.

If Γ is uncountable and G is a separable bounded operator family, then $\ell_\omega(\Gamma)$ cannot be equivalently renormed in strictly G -convex way.

Proof. Let $p(x)$ be an arbitrary equivalent norm on $\ell_\omega(\Gamma)$. By Lemma 4.3 there exist an $x \in \ell_\omega(\Gamma) \setminus \{0\}$ and a countable set $A \subset \Gamma$, such that for every $y \in \ell_\omega(\Gamma \setminus A)$, $\|y\| \leq 1$, the equality $p(x + y) = p(x)$ holds. Since the group G is separable, then it has a countable dense subset G_1 . According to Lemma 4.2, for a given set A we can find a set $B \supset A$, such that for all y from $B_{\ell_\omega(\Gamma \setminus B)}$ and for all $T \in G_1$ the elements Ty belong to $\ell_\omega(\Gamma \setminus A)$. Since the group G is bounded there exists a $t > 0$, such that $\|tTy\| \leq 1$ for all $\|y\| \leq 1$ and all $T \in G$. Then, for all the elements $y \in B_{\ell_\omega(\Gamma \setminus B)}$, the following holds: $\sup_{T \in G_1} p(x + tTy) = p(x)$. By density of G_1 in G and continuity of norm the same is true for G : $\sup_{T \in G} p(x + tTy) = p(x)$. This means that $(\ell_\omega(\Gamma), p)$ is not strictly G -convex. $\square$

Let us remark that the conditions of boundness and separability of G cannot be removed. At first, in every space X if one takes $G \supset \{I, 2I, 3I, \dots\}$ then X is strictly G -convex. By this reason only bounded collections of operators are considered. Further, if one considers the group G on $\ell_\omega(\Gamma)$ generated by I , $-I$ and operators of pairwise rearrangements of coordinates, then G has the same cardinality as Γ and so it can, for example, be equal to the first uncountable cardinal. On the other hand $\ell_\omega(\Gamma)$ in its original norm is strictly G -convex (and even uniformly G -convex) with respect to this group of isometries.

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